28-07-2020

terça-feira, 28 de julho de 2020 18:52

32) $\int x \,\mathrm{arctg}\, x \, dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$\begin{cases} u = \mathrm{arctg}\, x \Rightarrow du = \dfrac{1}{1+x^2}\,dx \\ dv = x\,dx \Rightarrow v = \dfrac{x^2}{2} \end{cases}$$

$$\int x\,\mathrm{arctg}\,x\,dx = \frac{x^2}{2}\mathrm{arctg}\,x - \int \frac{x^2}{2}\,\frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\mathrm{arctg}\,x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$\int \dfrac{x^2}{1+x^2}\,dx$, $\mathrm{grau}(x^2) = \mathrm{grau}(1+x^2) = 2$

$$\begin{array}{r|l} x^2 & 1+x^2 \\ \hline -1-x^2 & 1 \\ \hline -1 & \end{array} \qquad x^2 = (1+x^2)\cdot 1 - 1$$

Algoritmo da divisão para polinômios

Sejam p e g polinômios com grau $g \geq 1$. Então existem polinômios q e r tais que

$$p = gq + r, \text{ onde } r = 0 \text{ ou } 0 \leq \mathrm{grau}\, r < \mathrm{grau}\, g$$

q e r são únicos, a menos de uma multiplicação por um número real

$\mathrm{grau}(-1) = 0 < \mathrm{grau}(1+x^2) = 2$

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \int\left(1 - \frac{1}{1+x^2}\right)dx = \int dx - \int \frac{1}{1+x^2}\,dx = x - \mathrm{arctg}\,x + k$$

$$\int x\,\mathrm{arctg}\,x\,dx = \frac{x^2}{2}\mathrm{arctg}\,x - \frac{1}{2}\left[x - \mathrm{arctg}\,x\right] + k$$

$$= \frac{x^2}{2}\mathrm{arctg}\,x - \frac{x}{2} + \frac{1}{2}\mathrm{arctg}\,x + k$$

verificação: $\left(\dfrac{x^2}{2}\mathrm{arctg}\,x - \dfrac{x}{2} + \dfrac{1}{2}\mathrm{arctg}\,x\right)' = x\,\mathrm{arctg}\,x$

$$\left(\frac{x^2}{2}\mathrm{arctg}\,x - \frac{x}{2} + \frac{1}{2}\mathrm{arctg}\,x\right)' = \frac{2x}{2}\mathrm{arctg}\,x + \frac{x^2}{2}\,\frac{1}{1+x^2}$$

$$- \frac{1}{2} + \frac{1}{2}\,\frac{1}{1+x^2} = x\,\text{arctg}\,x + \left(\frac{x^2}{2}+\frac{1}{2}\right)\frac{1}{1+x^2} - \frac{1}{2}$$

$$= x\,\text{arctg}\,x + \frac{\cancel{x^2+1}}{2}\cdot\frac{1}{\cancel{1+x^2}} - \frac{1}{2}$$

$$= x\,\text{arctg}\,x + \frac{1}{2} - \frac{1}{2} = x\,\text{arctg}\,x$$

INTEGRAÇÃO DE FUNÇÕES RACIONAIS

Sejam p e q duas funções polinomiais, quero resolver

$$\boxed{\int \frac{p(x)}{q(x)}\,dx}$$

1) grau $p(x) \geq$ grau $q(x)$, usaremos o algoritmo da divisão

Existem r e s tais que

$$p(x) = s(x)\,q(x) + r(x), \text{ onde } r(x) = 0 \text{ ou } 0 \leq \text{grau } r(x) < \text{grau } q(x)$$

$$\frac{p(x)}{q(x)} = \frac{s(x)\,q(x)}{q(x)} + \frac{r(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

$$\Longrightarrow \int \frac{p(x)}{q(x)}\,dx = \int s(x)\,dx + \boxed{\int \frac{r(x)}{q(x)}\,dx}$$

onde $r(x) = 0$ ou $\boxed{0 \leq \text{grau } r(x) < \text{grau } q(x)}$

Exercícios: 39) $\displaystyle\int \frac{3x^2+4x+5}{(x-1)(x-2)(x-3)}\,dx$

grau $(3x^2+4x+5) = 2 < 3 =$ grau$\big((x-1)(x-2)(x-3)\big)$

$$\boxed{\frac{3x^2+4x+5}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}\;;\; A, B, C \in \mathbb{R}}$$

$$\Longrightarrow \frac{3x^2+4x+5}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Longrightarrow 3x^2+4x+5 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2),\quad x \neq 1,\ x \neq 2,\ x \neq 3$$

$$\lim (3x^2+4x+5) = \lim\left[A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)\right]$$

$$\lim_{\substack{x\to 1\\ x\neq 1}} (3x^2+4x+5) = \lim_{x\to 1}\left[A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)\right] \quad x\neq 1,\ x\neq 2,\ x\neq 3$$

$$\Rightarrow 3.1^2+4.1+5 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$= 2A \Rightarrow \boxed{A=6}$$

$\boxed{x=2}$

$$\Rightarrow 3.2^2+4.2+5 = B.1.(-1) = -B \Rightarrow \boxed{B=-25}$$

$\boxed{x=3}$

$$3.9+4.3+5 = -2C \Rightarrow 44 = 2C \Rightarrow \boxed{C=22}$$

$$\frac{3x^2+4x+5}{(x-1)(x-2)(x-3)} = \frac{6}{x-1} - \frac{25}{x-2} + \frac{22}{x-3}$$

$$\int \frac{3x^2+4x+5}{(x-1)(x-2)(x-3)} = 6\int \frac{1}{x-1}dx - 25\int \frac{1}{x-2}dx + 22\int \frac{dx}{x-3}$$

$$= 6\ln|x-1| - 25\ln|x-2| + 22\ln|x-3| + k$$

$$\int \frac{1}{x}dx = \ln|x| + k$$

$$\int \frac{1}{x-1}dx = \int \frac{1}{u}du = \ln|u| + k = \ln|x-1| + k$$

$$\begin{cases} u = x-1 \\ du = dx \end{cases}$$

$$\boxed{\int \frac{1}{x-a}dx = \ln|x-a| + k}$$

41) $\displaystyle\int \frac{3x^2+4x+5}{(x-1)^2(x-2)}dx$, grau $(3x^2+4x+5) = 2 < 3 = \text{grau}\left((x-1)^2(x-2)\right)$

$$\boxed{\frac{3x^2+4x+5}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}}$$

$$= \frac{A(x-1)(x-2) + B(x-2) + C(x-1)^2}{(x-1)^2(x-2)}$$

$$\Rightarrow 3x^2+4x+5 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$x=1 \Rightarrow 3+4+5 = -B \Rightarrow \boxed{B=-12}$

$x=2 \Rightarrow 12+8+5 = C \Rightarrow \boxed{C=25}$

$x=0 \Rightarrow 5 = 2A-2B+C = 2A+24+25 = 2A+49$

$\Rightarrow 2A = -44 \Rightarrow \boxed{A=-22}$

$$\int \frac{3x^2+4x+5}{(x-1)^2(x-1)}\,dx = -22\int\frac{dx}{x-1} - 12\int\frac{dx}{(x-1)^2} + 25\int\frac{dx}{x-2}$$

$$= -22\ln|x-1| - 12\int\frac{dx}{(x-1)^2} + 25\ln|x-2| + k$$

$$\int\frac{dx}{(x-1)^2} = \int(x-1)^{-2}dx = \int u^{-2}du = \frac{u^{-2+1}}{-2+1}+k_1 = -u^{-1}+k_1 = \frac{-1}{u}+k_1$$

$u = x-1 \Rightarrow \boxed{du = dx}$

$$= \frac{-1}{x-1}+k_1$$

$$\int\frac{dx}{(x-1)^2} = \int(x-1)^{-2}dx = \frac{(x-1)^{-1}}{-1}+k = -\frac{1}{x-1}+k_1$$

$$\int\frac{3x^2+4x+5}{(x-1)^2(x-1)}\,dx = -22\ln|x-1| + \frac{12}{x-1} + 25\ln|x-2| + k$$

42) $\displaystyle\int\frac{x^5+x+1}{x^3-8}\,dx$, $\quad \text{grau}(x^5+x+1) = 5 > 3 = \text{grau}(x^3-8)$

$$\begin{array}{l|l} x^5+x+1 & x^3-8 \\ \hline -x^5+8x^2 & x^2 \\ \hline 8x^2+x+1 & \end{array}$$

$x^5+x+1 = (x^3-8)x^2 + 8x^2+x+1$

$$\frac{x^5+x+1}{x^3-8} = \frac{(x^3-8)x^2}{x^3-8} + \frac{8x^2+x+1}{x^3-8} = x^2 + \frac{8x^2+x+1}{x^3-8}$$

$$\Rightarrow \int\frac{x^5+x+1}{x^3-8}\,dx = \underbrace{\int x^2dx}_{=\frac{x^3}{3}+k_0} + \int\frac{8x^2+x+1}{x^3-8}\,dx$$

$$\frac{8x^2+x+1}{x^3-8} = \frac{8x^2+x+1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{B(x^2+2x+4)'+C}{x^2+2x+4}$$

$\Delta = 4-8 = -4 < 0$

$$= \frac{A}{x-2} + \frac{B(2x+2)+C}{x^2+2x+4}$$

$$\Rightarrow \frac{8x^2+x+1}{x^3-8} = \frac{A(x^2+2x+4)+[B(2x+2)+C](x-2)}{(x-2)(x^2+2x+4)}$$

$$\Rightarrow \boxed{8x^2+x+1 = A(x^2+2x+4)+[B(2x+2)+C](x-2)}$$

$$x=2 \Rightarrow 32+2+1 = A(4+4+4) \Rightarrow A = \frac{35}{12}$$

$$x=-1 \Rightarrow 8-1+1 = A(1-2+4)+C(-3) \Rightarrow 8 = 3A-3C$$

$$\Rightarrow 8-3A = -3C \Rightarrow 8-3\cdot\frac{35}{12} = -3C$$

$$\Rightarrow 8-\frac{35}{4} = -3C \Rightarrow -\frac{3}{4} = -3C \Rightarrow C = \frac{1}{4}$$

$$x=0 \Rightarrow 1 = 4A+(2B+C)(-2) = 4\cdot\frac{35}{12}-2\left(2B+\frac{1}{4}\right)$$

$$= \frac{35}{3}-4B-\frac{1}{2} \Rightarrow 4B = \frac{35}{3}-\frac{1}{2}-1 = \frac{70-3-6}{6} = \frac{61}{6}$$

$$\Rightarrow B = \frac{61}{24}$$

$$\int \frac{8x^2+x+1}{x^3-8}\,dx = \frac{35}{12}\int\frac{dx}{x-2}+\frac{61}{24}\int\frac{2x+2}{x^2+2x+4}\,dx+\frac{1}{4}\int\frac{dx}{x^2+2x+4}$$

$$= \frac{35}{12}\ln|x-2|+\frac{61}{24}\ln|x^2+2x+4|+\frac{1}{4}\int\frac{dx}{x^2+2x+4}$$

$$\int\frac{2x+2}{x^2+2x+4}\,dx = \int\frac{du}{u} = \ln|u|+k_1 = \ln|x^2+2x+4|+k_1$$

$$\begin{cases} u = x^2+2x+4 \\ du = (2x+2)\,dx \end{cases}$$

$$\frac{1}{x^2+2x+4} = \frac{1}{(x+1)^2-1+4} = \frac{1}{(x+1)^2+3} =$$

$$= \frac{1}{3\left[\frac{(x+1)^2}{3}+1\right]} = \frac{1}{3\left[\left(\frac{x+1}{\sqrt{3}}\right)^2+1\right]}$$

$$\int\frac{dx}{x^2+2x+4} = \frac{1}{3}\int\frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1} = \frac{\sqrt{3}}{3}\int\frac{du}{u^2+1} = \frac{\sqrt{3}}{3}\operatorname{arctg} u+k_2$$

$$= \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{x+1}{\sqrt{3}}\right)+k_2$$

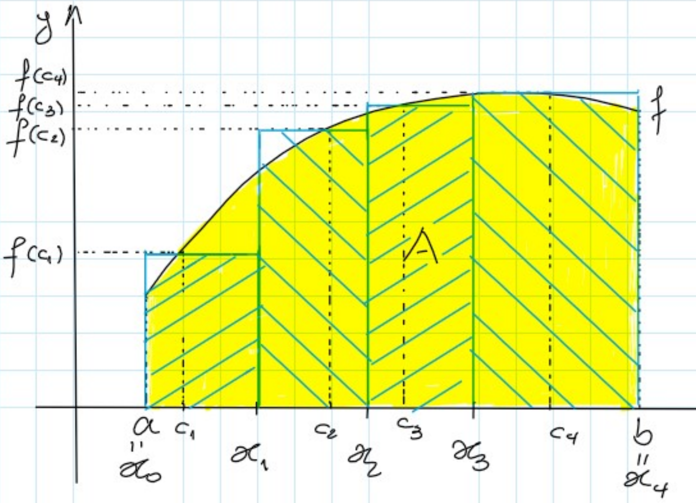
$$u = \frac{x+1}{\sqrt{3}} \Rightarrow du = \frac{dx}{\sqrt{3}} \Rightarrow dx = \sqrt{3}\,du$$

$$u = \frac{x+1}{\sqrt{3}} \Rightarrow du = \frac{dx}{\sqrt{3}} \Rightarrow dx = \sqrt{3}\, du \qquad = \frac{\sqrt{3}}{3}\, \text{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + k_2$$

$$\int \frac{x^5 + x + 1}{x^3 - 8}\, dx = \frac{x^3}{3} + \frac{35}{12} \ln|x-2| + \frac{61}{24} \ln(x^2 + 2x + 4)$$

$$+ \frac{1}{4}\, \frac{\sqrt{3}}{3}\, \text{arctg}\left(\frac{x+1}{\sqrt{3}}\right) + k$$

INTEGRAL DEFINIDA



Queremos calcular a área A

$A \cong (x_1 - x_0) f(c_1) + (x_2 - x_1) f(c_2)$

$+ (x_3 - x_2) f(c_3) + (x_4 - x_3) f(c_4)$

$=$ áreas dos retângulos

quantos mais retângulos tivermos melhor será a aproximação

Chamando $\Delta x_i = x_i - x_{i-1}$, $1 \leq i \leq 4$, temos que

$$A \cong f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + f(c_4)\Delta x_4 = \sum_{i=1}^{4} f(c_i)\Delta x_i$$

Σ é uma letra grega que é o sigma maiúsculo, o minúsculo é σ, que utilizamos como somatória

DEF: Dizemos que $P = \{x_0, x_1, \ldots, x_n\}$ é uma partição do intervalo $[a,b]$ se

$$x_0 = a < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

DEF: Seja f uma função definida no intervalo $[a,b]$. Dizemos que f é integrável em $[a,b]$ se o limite

$\lim \sum^{n} f(c_i)\Delta x_i$ existe e é um número real

$$\lim_{n\to+\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$ existe e é um número real

para toda partição $P=\{x_0, x_1, \ldots, x_n\}$ e para todo $c_i \in [x_{i-1}, x_i]$

Notação: $$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

chamada integral definida da f no intervalo $[a,b]$

ou integral de Riemann da f no intervalo $[a,b]$