Terceira Prova

1. Um modelo clássico para um metal é considerar a condução elétrica devido a uma densidade de portadores N, ~~de mesma carga~~ por unidade de volume, com massa m e carga q, se movendo com velocidade $\vec{v}$ em relação a um fundo que mantém a neutralidade de carga. Considerando as colisões dos portadores com as cargas do fundo neutralizante, a equação de movimento para cada portador é

$$m\frac{d\vec{v}}{dt} + \frac{m}{\tau}\vec{v} = q\vec{E}$$

a) Justifique porque τ corresponde ao "tempo de relaxação".

b) Sabendo que a densidade de corrente é dada por $\vec{J} = Nq\vec{v}$ e também por $\vec{J} = \sigma\vec{E}$, onde σ é a condutividade do meio, mostre que, para um campo harmônico $\vec{E} = \vec{E}_0 e^{-i\omega t}$, a expressão para a condutividade é

$$\sigma = \frac{Nq^2\tau}{m}\,\frac{1}{1 - i\omega\tau}$$

c) Considerando que a relação $\vec{J} = \sigma\vec{E}$ é causal, porque só há corrente após aplicação do campo, ~~mostre que~~ escreva as relações de Kramers - Kronig para σ.

d) Mostre que a expressão para a potência média dissipada por unidade de volume no meio é dada por

$$\langle P \rangle = \frac{1}{2} E_0^2 \, \mathrm{Re}(\sigma)$$

e) Considerando o limite $\omega \to \infty$ nas relações de Kramers-Kronig, mostre que

$$\int_0^\infty d\omega \, \mathrm{Re}[\sigma(\omega)] = \frac{\pi N e^2}{2m}$$

Portanto, medindo a potência dissipada, por unidade de volume, em altas frequências, é possível estimar a densidade de portadores.

Solução:

a) Se $\vec{E} \to 0$, a solução da equação é $\vec{v}(t) = \vec{v}(0) e^{-t/\tau}$, levando ao decaimento da velocidade

b) $$m \frac{d\vec{v}}{dt} + \frac{m}{\tau} \vec{v} = q\vec{E}_0 e^{-i\omega t} \Rightarrow -i\omega m \vec{v} + \frac{m}{\tau} \vec{v} = q\vec{E}$$

$$\therefore \vec{v} = \frac{\tau q}{m(1 - i\omega\tau)} \vec{E} \quad \therefore \quad \sigma = \frac{N q^2 \tau}{m(1 - i\omega\tau)}$$

c) $$\mathrm{Re}[\sigma(\omega)] = \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega' \, \mathrm{Im}[\sigma(\omega')]}{\omega'^2 - \omega^2} d\omega'$$

$$\mathrm{Im}[\sigma(\omega)] = -\frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega \, \mathrm{Re}[\sigma(\omega')]}{\omega'^2 - \omega^2} d\omega'$$

d) $$P(t) = \frac{1}{4}[\vec{E} + \vec{E}^*] \cdot [\vec{J} + \vec{J}^*] = \frac{1}{4}\left[\vec{E}_0 \cos(\omega t)\right] \cdot \left[\vec{E}_0 \mathrm{Re}(\sigma) \cos(\omega t) + \vec{E}_0 \mathrm{Im}(\sigma) \sin(\omega t)\right]$$

$$\therefore P(t) = \frac{1}{4} E_0^2 \mathrm{Re}(\sigma) \cos^2(\omega t) + \frac{1}{4} E_0^2 \mathrm{Im}(\sigma) \cos(\omega t) \sin(\omega t)$$

$$\therefore \langle P \rangle = \frac{1}{2} \mathrm{Re}(\sigma) E_0^2$$

e) $\omega \to \infty \Rightarrow \mathrm{Im}[\sigma(\omega)] = \frac{2}{\pi}\frac{1}{\omega}\int_0^\infty \mathrm{Re}[\sigma(\omega')]\,d\omega'$

$$\therefore \lim_{\omega\to\infty} \omega \frac{Ne^2}{m}\frac{\omega\tau^2}{1+\omega^2\tau^2} = \frac{2}{\pi}\int_0^\infty \mathrm{Re}[\sigma(\omega')]\,d\omega'$$

$$\therefore \int_0^\infty d\omega'\,\mathrm{Re}[\sigma(\omega')] = \frac{\pi N e^2}{2m}$$

2. No caso de radiação por uma carga com aceleração paralela à velocidade, obtivemos que a potência radiada é dada por

$$\frac{dP}{d\Omega} = \frac{dU}{dt'} = \frac{\mu_0 q^2 \dot{v}_q^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

a) Faça a integral desta expressão sobre o ângulo sólido $d\Omega = \sin\theta \, d\theta \, d\varphi$, sem utilizar tabelas, mas empregando adequadamente integração por partes, e obtenha a expressão 6.163 do livro do Bo Thidé

$$P = \int \frac{dU}{dt'} d\Omega = \frac{\mu_0 q^2 \dot{v}_q^2}{6\pi c} \frac{1}{(1-\beta^2)^3}$$

b) Determine a expressão para o ângulo θ_{max}, correspondente à direção de máxima intensidade de radiação

Solução

a)

$$P = \frac{\mu_0 q^2 \dot{v}_q^2}{16\pi^2 c} \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{\sin^3\theta}{(1-\beta\cos\theta)^5} d\theta = \frac{\mu_0 q^2 \dot{v}_q^2}{8\pi c} \int_0^{\pi} \frac{\sin^3\theta \, d\theta}{(1-\beta\cos\theta)^5}$$

$$\int_0^{\pi} \frac{\sin^3\theta \, d\theta}{(1-\beta\cos\theta)^5} = \frac{-1}{4\beta} \int_0^{\pi} \sin^2\theta \left[\frac{d}{d\theta}(1-\beta\cos\theta)^{-4}\right] d\theta$$

$$= \frac{-1}{4\beta}\left[\sin^2\theta (1-\beta\cos\theta)^{-4}\right]_0^{\pi} + \frac{1}{2\beta}\int_0^{\pi} \frac{\sin\theta\cos\theta}{(1-\beta\cos\theta)^4} d\theta$$

$$= \frac{-1}{6\beta^2}\left[\cos\theta (1-\beta\cos\theta)^{-3}\right]_0^{\pi} - \frac{1}{6\beta^2}\int_0^{\pi} \frac{\sin\theta \, d\theta}{(1-\beta\cos\theta)^3}$$

$$= \frac{1}{6\beta^2} \frac{(1+\beta)^3 + (1-\beta)^3}{(1-\beta^2)^3} + \frac{1}{12\beta^3} \frac{(1-\beta)^2 - (1+\beta)^2}{(1-\beta^2)^2}$$

$$\therefore \int_0^{\pi} \frac{\sin^3\theta \, d\theta}{(1-\beta\cos\theta)^5} = \frac{4}{3} \frac{1}{(1-\beta^2)^3}$$

$$\therefore\ P = \frac{\mu_0 q^2 \dot{v}_q^2}{16\pi^2 c} \cdot 2\pi \cdot \frac{4}{3} \frac{1}{(1-\beta^2)^3} \therefore\ P = \frac{\mu_0 q^2 \dot{v}_q^2}{6\pi c} \frac{1}{(1-\beta^2)^3}$$

b) $$\frac{d}{d\theta} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = 0 \therefore\ 2\cos\theta(1-\beta\cos\theta) - 5\beta\sin^2\theta = 0$$

$$\therefore\ 3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0 \therefore\ \cos\theta_{max} = \frac{1}{3\beta}\left[\sqrt{1+15\beta^2} - 1\right]$$

$$\beta \to 1 \Rightarrow \theta_{max} \to \frac{1}{2\gamma}$$

Solução Questão 3

a) $[\rho(\vec{r}',t')]_{ret} = \rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \simeq \rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{cr} + \dots\right)$

$\therefore [\rho(\vec{r}',t')]_{ret} \simeq \rho(\vec{r}',u) + \frac{1}{c}(\hat{e}_r\cdot\vec{r}')\frac{\partial\rho}{\partial u} + \dots \quad u = t - \frac{r}{c}$

b) $\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{[\rho(\vec{r}',t')]_{ret}}{|\vec{r}-\vec{r}'|}dV'$

$|\vec{r}-\vec{r}'|^{-1} \simeq \frac{1}{r} \Rightarrow \phi(\vec{r},t) \simeq \frac{1}{4\pi\epsilon_0 r}\left[\int \rho(\vec{r}',u)dV' + \frac{1}{c}\hat{e}_r\cdot\frac{\partial}{\partial u}\int \vec{r}'\rho(\vec{r}',u)dV'\right]$

$\int \rho(\vec{r}',u)dV' = Q \,; \quad \int \vec{r}'\rho(\vec{r}',u)dV' = \vec{p}(u)$

$\therefore \quad \phi(\vec{r},t) \simeq \frac{Q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0}\frac{\hat{e}_r\cdot\dot{\vec{p}}(u)}{cr} \Rightarrow \phi_{rad}(\vec{r},t) \simeq \frac{1}{4\pi\epsilon_0}\frac{\hat{e}_r\cdot\dot{\vec{p}}(u)}{cr}$

c) $[\vec{j}(\vec{r}',t')]_{ret} = \vec{j}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right) \simeq \vec{j}(\vec{r}',u) + \frac{1}{c}(\hat{e}_r\cdot\vec{r}')\frac{\partial\vec{j}}{\partial u} + \dots$

$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{[\vec{j}(\vec{r}',t')]_{ret}}{|\vec{r}-\vec{r}'|}dV'$

$|\vec{r}-\vec{r}'|^{-1} \simeq \frac{1}{r} \Rightarrow \vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}\left[\int \vec{j}(\vec{r}',u)dV' + \frac{1}{c}\int \frac{\partial\vec{j}}{\partial u}(\hat{e}_r\cdot\vec{r}')dV'\right]$

Como primeira integral não é nula, para $\vec{j}$ dependente do tempo,

$$\vec{A}_{rad}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}\int \vec{j}(\vec{r}',u)dV'$$

$$\nabla\cdot(x_i\vec{j}) = x_i(\nabla\cdot\vec{j}) + \vec{j}\cdot(\nabla x_i) = -x_i\frac{\partial\rho}{\partial t} + j_i$$

$\therefore \int \nabla\cdot(x_i\vec{j})dV' = -\int x_i\frac{\partial\rho}{\partial t}dV' + \int j_i dV$

$\therefore \int x_i\vec{j}\cdot d\vec{S}' = -\frac{d}{dt}\int x_i'\rho\, dV' + \int j_i dV'$

Tomando a superfície S englobando toda o volume da fonte, mas fora dele, $\vec{J}=0$ na superfície e

$$\int J_i \, dV' = \frac{d}{dt}\int x_i' \rho \, dV' \therefore \int \vec{J}(\vec{r}',u)\,dV' = -\frac{d}{dt}\int \vec{r}'\rho\, dV'$$

$$\therefore \int \vec{J}(\vec{r}',u)\,dV' = \dot{\vec{p}}(u) \Rightarrow \vec{A}_{rad}(\vec{r},t) \simeq \frac{\mu}{4\pi}\frac{\dot{\vec{p}}(u)}{r}$$

c) $$\nabla[\hat{e}_r\cdot\dot{\vec{p}}] = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}\left[\frac{\vec{r}}{r}\cdot\dot{\vec{p}}(u)\right]; \frac{\partial}{\partial x_i}\left[\frac{\vec{r}}{r}\cdot\dot{\vec{p}}(u)\right] = \frac{\vec{r}}{r}\cdot\frac{\partial\dot{\vec{p}}(u)}{\partial x_i} + \dot{\vec{p}}(u)\cdot\frac{\partial}{\partial x_i}\left(\frac{\vec{r}}{r}\right)$$

$$\therefore \frac{\partial}{\partial x_j}\left[\frac{\vec{r}}{r}\cdot\dot{\vec{p}}(u)\right] = \frac{\vec{r}}{r}\cdot\frac{d\dot{\vec{p}}(u)}{du}\frac{\partial u}{\partial x_j} + \dot{\vec{p}}(u)\cdot\left[\frac{\hat{e}_j}{r} - \vec{r}\,\frac{x_j}{r^3}\right]$$

$$\therefore \nabla[\hat{e}_r\cdot\dot{\vec{p}}] = -\frac{\hat{e}_r}{c}\cdot\ddot{\vec{p}}(u)\sum_i \hat{e}_i\frac{x_i}{r} + \cancel{\dot{\vec{p}}(u)\left[\frac{1}{r}\right]}\, \frac{1}{r}\sum_i\left[\dot{p}_i\hat{e}_i - (\dot{\vec{p}}\cdot\vec{r})\frac{\hat{e}_i x_i}{r^2}\right]$$

$$\therefore \nabla[\hat{e}_r\cdot\dot{\vec{p}}] = -\frac{1}{c}[\hat{e}_r\cdot\ddot{\vec{p}}(u)]\hat{e}_r + \frac{1}{r}[\dot{\vec{p}} - (\dot{\vec{p}}\cdot\hat{e}_r)\hat{e}_r]$$

$$\therefore \nabla[\hat{e}_r\cdot\dot{\vec{p}}] = \frac{-1}{c}[\hat{e}_r\cdot\ddot{\vec{p}}(u)]\hat{e}_r + \mathcal{O}\left(\frac{1}{r}\right)$$

$$\vec{E}_{rad}(\vec{r},t) = -\nabla\phi_{rad} - \frac{\partial \vec{A}_{rad}}{\partial t} = \frac{-1}{4\pi\epsilon_0}\frac{1}{c}\nabla\left[\frac{\hat{e}_r\cdot\dot{\vec{p}}}{r}\right] - \frac{\mu}{4\pi}\frac{\partial}{\partial t}\left[\frac{\dot{\vec{p}}(u)}{r}\right]$$

$$= \frac{-1}{4\pi\epsilon_0 c}\left[\frac{1}{r}\nabla(\hat{e}_r\cdot\dot{\vec{p}}) - (\hat{e}_r\cdot\dot{\vec{p}})\frac{1}{r^2}\nabla r\right] - \frac{\mu}{4\pi r}\ddot{\vec{p}}(u)\frac{\partial u}{\partial t}$$

$$= \frac{1}{4\pi\epsilon_0 c^2}\frac{(\hat{e}_r\cdot\ddot{\vec{p}})\hat{e}_r}{r} - \frac{\mu_0}{4\pi r}\ddot{\vec{p}} + \frac{1}{4\pi\epsilon_0 c}\frac{(\hat{e}_r\cdot\dot{\vec{p}})\hat{e}_r}{r^2}$$

$$\therefore \vec{E}_{rad}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r}\left[-\ddot{\vec{p}}(u) + \frac{1}{\mu_0\epsilon_0 c^2}(\hat{e}_r\cdot\ddot{\vec{p}})\hat{e}_r\right] + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\therefore \vec{E}_{rad}(\vec{r},t) \simeq \frac{-\mu_0}{4\pi}\frac{[\hat{e}_r\times\ddot{\vec{p}}(u)]\times\hat{e}_r}{r}$$

e) $\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\nabla\times\left[\frac{\dot{\vec{p}}(u)}{r}\right]$

$$\nabla\times\left[\frac{\dot{\vec{p}}(u)}{r}\right] = \frac{1}{r}\nabla\times\dot{\vec{p}}(u) + \nabla\left(\frac{1}{r}\right)\times\dot{\vec{p}}(u)$$

$$= \frac{-1}{rc}\left[\hat{e}_x\frac{\partial r}{\partial x} + \hat{e}_y\frac{\partial r}{\partial y} + \hat{e}_z\frac{\partial r}{\partial z}\right]\times\ddot{\vec{p}}(u) - \frac{\vec{r}}{r^3}\times\dot{\vec{p}}(u)$$

$$= \frac{-1}{rc}\left[\frac{x\hat{e}_x}{r} + \frac{y\hat{e}_y}{r} + \frac{z\hat{e}_x}{r}\right]\times\ddot{\vec{p}}(u) - \frac{\vec{r}}{r^3}\times\dot{\vec{p}}(u)$$

$$\therefore \vec{B}(\vec{r},t) = \frac{-\mu_0}{4\pi c}\,\frac{\hat{e}_r\times\ddot{\vec{p}}(u)}{r}$$

f) $\vec{S}_{rad} = \frac{1}{\mu_0}\vec{E}_{rad}\times\vec{B}_{rad} = \frac{1}{\mu_0}c\left[(\vec{B}_{rad}\times\hat{e}_r)\times\vec{B}_{rad}\right]$

$$= \frac{c}{\mu_0}\left[-(\vec{B}_{rad}\cdot\hat{e}_r)\vec{B}_{rad} + |\vec{B}_{rad}|^2\hat{e}_r\right] \quad ; \quad \vec{B}_{rad}\cdot\hat{e}_r = 0$$

$$\therefore \vec{S}_{rad} = \frac{c}{\mu_0}|\vec{B}_{rad}|^2\hat{e}_r$$

$$\vec{B}_{rad} = \frac{-\mu_0}{4\pi\epsilon_0}\,\frac{\hat{e}_r\times\ddot{\vec{p}}(u)}{r} = \frac{-\mu_0}{4\pi\epsilon_0}\,\frac{\ddot{p}}{r}\left[\sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z\right]\times\hat{e}_p$$

$$\therefore \vec{B}_{rad} = \frac{-\mu_0}{4\pi c}\,\frac{\ddot{p}}{r}\left[\sin\theta\cos\varphi\,\hat{e}_y - \sin\theta\sin\varphi\,\hat{e}_x\right]$$

$$\therefore |B_{rad}|^2 = \left(\frac{\mu_0\ddot{p}}{4\pi c r}\right)^2\left[\sin^2\theta\,(\cos^2\varphi + \sin^2\varphi)\right] = \left(\frac{\mu_0\ddot{p}}{4\pi c r}\right)\sin^2\theta$$

$$P = \int\vec{S}_{rad}\cdot d\vec{a} = \frac{c}{\mu_0}\left(\frac{\mu_0\ddot{p}}{4\pi c}\right)^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\sin^3\theta\,\partial$$

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin\theta \, d\theta - \int_0^\pi \sin\theta \cos^2\theta \, d\theta$$

$$= -\left[\cos\theta\right]_0^\pi + \left[\frac{1}{3}\cos^3\theta\right]_0^\pi = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\therefore \quad P = \frac{c}{\mu_0} \frac{\mu_0^2}{16\pi^2 c^2} |\ddot{p}(u)|^2 \times 2\pi \times \frac{4}{3} \quad \therefore \quad P = \frac{\mu_0}{6\pi c} |\ddot{p}(u)|^2$$

Para ondas monocromáticas: $\vec{p}(t) = \vec{p}\, e^{-i\omega u} \quad \therefore \ddot{p}(u) = -\omega^2 \vec{p}(u)$

$$[\ddot{p}(u)]^2 = \frac{\omega^4}{4}\left[\vec{p}\, e^{-i\omega u} + \vec{p}^{*} e^{i\omega u}\right] \cdot \left[\vec{p}\, e^{-i\omega u} + \vec{p}^{*} e^{i\omega u}\right]$$

$$= \frac{\omega^4}{4}\left[2p^2 + (\vec{p}\cdot\vec{p})\, e^{-2i\omega u} + (\vec{p}^{*}\cdot\vec{p}^{*})\, e^{2i\omega u}\right]$$

$$\therefore \langle |\ddot{p}(u)|^2 \rangle = \frac{\omega^4}{2} p^2$$

$$\therefore \langle P \rangle = \frac{\mu_0}{6\pi c} \frac{\omega^4}{2} p^2 = \frac{\mu_0}{12\pi c} c^4 k^4 p^2 = Z_0 \frac{c^2 k^4}{12\pi} p^2$$

que é o resultado derivado em aula.

g) $\vec{p}(t) = -e\vec{a}$; $\vec{F} = \frac{-e^2}{4\pi\epsilon_0} \frac{\vec{a}}{a^3} = m\ddot{\vec{a}}$

$$\therefore \ddot{\vec{p}}(t) = \frac{1}{4\pi\epsilon_0} \frac{e}{m} \frac{e^2}{a^2} \hat{a}(t)$$

h) $\frac{dE_{orb}}{dt} = -P$; $E_{orb} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a} \Rightarrow \frac{dE_{orb}}{dt} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{2a^2}\frac{da}{dt}$

$$\therefore \frac{1}{4\pi\epsilon_0}\frac{e^2}{2a^2}\frac{da}{dt} = -\frac{\mu_0}{6\pi c}\left(\frac{e^3}{4\pi\epsilon_0 m}\right)^2 \frac{1}{a^4}$$

$$\therefore \frac{da}{dt} = -\frac{8\pi\epsilon_0}{e^2}\frac{\mu_0}{6\pi c}\frac{e^6}{(4\pi\epsilon_0)^2 m^2}\frac{1}{a^2} = \frac{-4}{3}\frac{1}{(4\pi\epsilon_0)^2}\frac{e^4}{m^2 c^3}\frac{1}{a^2}$$

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin\theta \, d\theta - \int_0^\pi \sin\theta \cos^2\theta \, d\theta$$

$$= -\left[\cos\theta\right]_0^\pi + \left[\frac{1}{3}\cos^3\theta\right]_0^\pi = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\therefore \quad P = \frac{c}{\mu_0} \frac{\mu_0^2}{16\pi^2 c^2} |\ddot{p}(u)|^2 \times 2\pi \times \frac{4}{3} \quad \therefore \quad P = \frac{\mu_0}{6\pi c} |\ddot{p}(u)|^2$$

Para ondas monocromáticas: $\vec{p}(t) = \vec{p}\, e^{-i\omega u} \quad \therefore \ddot{p}(u) = -\omega^2 \vec{p}(u)$

$$[\ddot{p}(u)]^2 = \frac{\omega^4}{4}\left[\vec{p}\, e^{-i\omega u} + \vec{p}^{\,*} e^{i\omega u}\right] \cdot \left[\vec{p}\, e^{-i\omega u} + \vec{p}^{\,*} e^{i\omega u}\right]$$

$$= \frac{\omega^4}{4}\left[2p^2 + (\vec{p}\cdot\vec{p})\, e^{-2i\omega u} + (\vec{p}^{\,*}\cdot\vec{p}^{\,*})\, e^{2i\omega u}\right]$$

$$\therefore \langle |\ddot{p}(u)|^2 \rangle = \frac{\omega^4}{2} p^2$$

$$\therefore \langle P \rangle = \frac{\mu_0}{6\pi c} \frac{\omega^4}{2} p^2 = \frac{\mu_0}{12\pi c} c^4 k^4 p^2 = Z_0 \frac{c^2 k^4}{12\pi} p^2$$

que é o resultado derivado em aula.

g) $\ddot{\vec{p}}(t) = -e\ddot{\vec{a}} \; ; \; \vec{F} = \frac{-e^2}{4\pi\epsilon_0} \frac{\vec{a}}{a^3} = m\ddot{\vec{a}}$

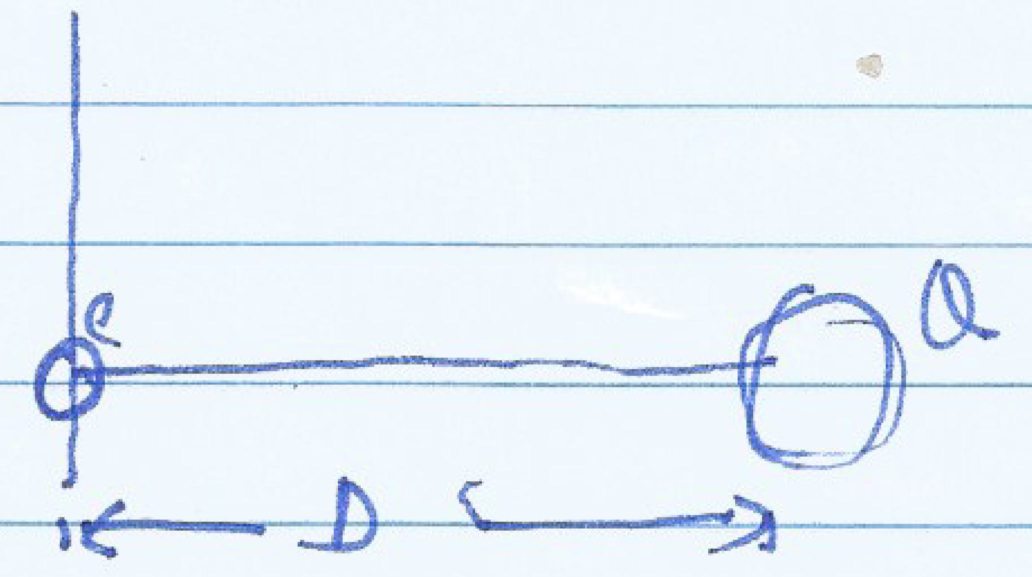
$$\therefore \ddot{\vec{p}}(t) = \frac{1}{4\pi\epsilon_0} \frac{e}{m} \frac{e^2}{a^2} \hat{a}(t)$$

h) $\frac{dE_{orb}}{dt} = -P \; ; \; E_{orb} = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{2a} \Rightarrow \frac{dE_{orb}}{dt} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{2a^2} \frac{da}{dt}$

$$\therefore \frac{1}{4\pi\epsilon_0} \frac{e^2}{2a^2} \frac{da}{dt} = -\frac{\mu_0}{6\pi c}\left(\frac{e^3}{4\pi\epsilon_0 m}\right)^2 \frac{1}{a^4}$$

$$\therefore \frac{da}{dt} = -\frac{8\pi\epsilon_0}{e^2} \frac{\mu_0}{6\pi c} \frac{e^6}{(4\pi\epsilon_0)^2 m^2} \frac{1}{a^2} = -\frac{4}{3} \frac{1}{(4\pi\epsilon_0)^2} \frac{e^4}{m^2 c^3} \frac{1}{a^2}$$

4. Um elétron se encontra no campo de uma carga positiva Q, que está fixa no ponto $x = D$. No instante $t = 0$ o elétron é solto, se deslocando na direção de Q.



a) Calcule a força que o elétron exerce sobre a carga positiva em função de sua posição $[\vec{R}]_{ret}$ e velocidade normalizada $[\vec{\beta}]_{ret}$

b) O princípio de ação e reação de Newton se verifica neste caso? Justifique fisicamente tua resposta

c) Calcule a potência total radiada pelo elétron quando ele começa a ser acelerado, isto é, em $x = 0$, em função de sua massa m, carga e e Q.

Solução

a) $\vec{E}(\vec{r},t) = \vec{E}_v(\vec{r},t) + \vec{E}_a(\vec{r},t)$; $\vec{E}_v(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{s^3}(\vec{R} - R\vec{\beta})(1-\beta^2)\right]_{ret}$

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{1}{s^3}\vec{R}\times[(\vec{R}-R\vec{\beta})\times\vec{a}]\right]$$

$[\vec{R}]_{ret} = [R]_{ret}\hat{e}_x$; $[\vec{\beta}]_{ret} = [\beta]_{ret}\hat{e}_x$; $[\vec{a}]_{ret} = [a]_{ret}\hat{e}_x$; $s = R(1-\beta)$

$$\therefore \vec{E}_v(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{R}{s^3}(1-\beta)(1-\beta^2)\right]_{ret}\hat{e}_x = \frac{q}{4\pi\epsilon_0}\frac{1}{[R^2]_{ret}}\left[\frac{1+\beta}{1-\beta}\right]_{ret}\hat{e}_x$$

$$\vec{E}_a(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\left[\frac{1}{s^3}\vec{R}\times R(1-\beta)\,\overbrace{\hat{e}_x\times\vec{a}}^{0}\right] = 0$$

$$\therefore \vec{F} = Q\vec{E}_v \Rightarrow \vec{F} = -\frac{Qe}{4\pi\epsilon_0}\left[\frac{1}{R^2}\frac{1+\beta}{1-\beta}\right]_{ret}\hat{e}_x$$

b) A força que a carga positiva exerce no elétron é

$$\vec{F}_Q = \frac{Q|e|}{4\pi\epsilon_0}\left[\frac{1}{R^2}\right]_{ret}\hat{e}_x$$

Portanto, a terceira lei de Newton não é satisfeita. Isto ocorre porque o campo produzido pelo elétron não atua "instantaneamente" na carga Q. Mas esta questão será revista em Electro II.

c) $$P = \frac{\mu_0 q^2 a^2}{6\pi c}\frac{1}{(1-\beta^2)^3} \; ; \; a = \frac{Q|e|}{4\pi\epsilon_0[R]^2 m}$$

$$\beta \approx 0 \Rightarrow [R] \doteq D \quad \therefore \quad P = \frac{\mu_0}{96\pi^3 c\,\epsilon_0^2}\frac{Q^2 e^4}{m^2 D^4} = \frac{1}{96\pi^3\epsilon_0^3 c^3}\frac{Q^2 e^4}{m^2 D^4}$$

Solução Questão 5

a) A expressão para o potencial vetor de um dipolo derivada em aula é

$$\vec{A}(\vec{r},t) = -\frac{i\omega\mu_0}{4\pi}\vec{p}\,\frac{e^{i(kr-\omega t)}}{r}$$

Como temos o efeito do dipolo, $\vec{p} = p e^{-i\omega t}\hat{e}_z$ e de sua imagem, $\vec{p}\,' = -p e^{-i\omega t}\hat{e}_z$, o potencial vetor resultante é

$$\vec{A}(\vec{r},t) = \frac{-i\omega\mu_0 p}{4\pi}\left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right)e^{-i\omega t}\hat{e}_z$$

b) $$r_1 = [\vec{r}_1\cdot\vec{r}_1]^{1/2} = \left[\left(\vec{r}-\frac{a}{2}\hat{e}_x\right)\left(\vec{r}-\frac{a}{2}\hat{e}_x\right)\right]^{1/2}$$

$$= \left[r^2 - ar\,\hat{e}_r\cdot\hat{e}_x + \frac{a^2}{4}\right]^{1/2} = r\left[1 - \frac{a}{r}\hat{e}_x\cdot\hat{e}_r + \frac{a^2}{4}\right]^{1/2}$$

Considerando $\frac{a}{r} \ll 1$, temos

$$r_1 \simeq r - \frac{a}{2}\hat{e}_x\cdot\hat{e}_r$$

e, da mesma forma, $r_2 \simeq r + \frac{a}{2}\hat{e}_x\cdot\hat{e}_r$

Como $\hat{e}_r = \sin\theta\cos\varphi\,\hat{e}_x + \sin\theta\sin\varphi\,\hat{e}_y + \cos\theta\,\hat{e}_z$, obtemos

$$r_1 \simeq r - \frac{a}{2}\sin\theta\cos\varphi\;;\quad r_2 \simeq r + \frac{a}{2}\sin\theta\cos\varphi$$

c) Usando este resultado e tomando no denominador $\frac{1}{r_1} \simeq \frac{1}{r_2} \simeq \frac{1}{r}$, temos

$$\vec{A}(\vec{r},t) \simeq i\frac{\mu_0\omega p}{4\pi r}\left[e^{i\frac{ka}{2}\sin\theta\cos\varphi} - e^{-i\frac{ka}{2}\sin\theta\cos\varphi}\right]e^{i(kr-\omega t)}\hat{e}_z$$

Portanto

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 \omega p}{2\pi r} e^{i(kr-\omega t)} \operatorname{sen}\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right)\hat{e}_y$$

d) $$\hat{e}_z = \cos\theta\,\hat{e}_r - \operatorname{sen}\theta\,\hat{e}_\theta$$

$$\vec{B}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t) = -\frac{1}{r\operatorname{sen}\theta}\frac{\partial A_\theta}{\partial\varphi}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\hat{e}_\varphi$$

Na zona de radiação, a componente $B_r \sim \frac{B_\varphi}{r}$, de forma que a segunda predomina. Então

$$\vec{B}(\vec{r},t) \simeq -\frac{\hat{e}_\varphi}{r}\frac{\partial}{\partial r}\left[r\operatorname{sen}\theta\, A(\vec{r},t)\right] = \frac{i\mu_0 k\omega p}{2\pi r}\cancel{\operatorname{sen}\theta}\, e^{i(kr-\omega t)} \times$$

$$\times \operatorname{sen}\theta\operatorname{sen}\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right)\hat{e}_\varphi$$

Tomando $k = \frac{\omega}{c}$ e $c^2 = \frac{1}{\mu_0\epsilon_0}$, obtemos

$$\vec{B}(\vec{r},t) \simeq i\frac{\omega^2 p\, e^{i(kr-\omega t)}}{2\pi\epsilon_0 c^3 r}\operatorname{sen}\theta\operatorname{sen}\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right)\hat{e}_\varphi$$

e) $$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{c}{\mu_0}(\vec{B}\times\hat{e}_r)\times\hat{e}_r = \frac{c}{\mu_0}B^2\hat{e}_r$$

Como B varia harmonicamente com o tempo

$$\langle\vec{S}\rangle = \frac{c}{2\mu_0}|B|^2\hat{e}_r = \frac{c}{2\mu_0}\frac{\omega^4 p^2}{4\pi^2\epsilon_0^2 c^6 r^2}\operatorname{sen}^2\theta\operatorname{sen}^2\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right)$$

$$\frac{d\langle P\rangle}{d\Omega} = r^2\langle\vec{S}\rangle\cdot\hat{e}_r \quad\therefore\quad \frac{d\langle P\rangle}{d\Omega} = \frac{\omega^4 p^2\operatorname{sen}^2\theta}{8\pi^2\epsilon_0 c^3}\operatorname{sen}^2\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right)$$

f) $$ka \ll 1 \Rightarrow \operatorname{sen}\left(\frac{ka}{2}\operatorname{sen}\theta\cos\varphi\right) \approx \frac{ka}{2}\operatorname{sen}\theta\cos\varphi$$

portanto

$$\frac{d\langle P\rangle}{d\Omega} \simeq \frac{\omega^6 p^2 a^2}{32\pi^2 \epsilon_0 c^5}\, \mathrm{sen}^4\theta \cos^2\varphi$$

$\therefore$

$$\langle P\rangle = \frac{\omega^6 p^2 a^2}{32\pi^2 \epsilon_0 c^5} \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^{\pi} \mathrm{sen}^5\theta \, d\theta$$

$$= \frac{\omega^6 p^2 a^2}{32\pi^2 \epsilon_0 c^5} \times \pi \int_0^{\pi} \mathrm{sen}^5\theta \, d\theta$$

$$\int_0^{\pi} \mathrm{sen}^5\theta \, d\theta = \int_0^{\pi} \mathrm{sen}\,\theta (1 - \cos^2\theta)^2 d\theta = \int_{-1}^{1} (1 - x^2)^2 dx$$

$$= \left[\int_{-1}^{1} dx - 2\int_{-1}^{1} x^2 dx + \int_{-1}^{1} x^4 dx \right] = \frac{16}{15}$$

$$\therefore \quad \langle P\rangle = \frac{\omega^6 p^2 a^2}{\cancel{32}_{2}\,\cancel{\pi^2} \epsilon_0 c^5} \times \cancel{\pi} \times \frac{\cancel{16}}{15} \qquad \therefore \langle P\rangle = \frac{\omega^6 p^2 a^2}{30\pi \epsilon_0 c^5}$$