PGF 5003 – Eletrodinâmica Clássica I

Prova III

<u>Descrição</u>: prova para ser feita individualmente em casa, com consulta livre a livros e apontamentos

<u>Data de disponibilização</u>: 14 Julho 2020 – 9hs

<u>Prazo máximo para recebimento</u>: 16 Julho 2020 – 19hs

1. Um modelo clássico para um metal é considerar a condução elétrica devido a uma densidade de volume, com massa m e carga q, se movendo com velocidade $\vec{v}$ em relação a um fundo que mantém a neutralidade de carga. Considerando as colisões dos portadores com as cargas do fundo neutralizante, a equação de movimento para cada portador é

$$m\frac{d\vec{v}}{dt} + \frac{m}{\tau}\vec{v} = q\vec{E}.$$

a) Justifique porque τ corresponde ao "tempo de relaxação".

b) Sabendo que a densidade de corrente é dada por $\vec{j} = Nq\vec{v}$, e também por $\vec{j} = \sigma\vec{v}$, onde σ é a condutividade do meio, mostre que, para um campo harmônico $\vec{E} = \vec{E}_0 e^{-i\omega t}$, a expressão da condutividade é

$$\sigma = \frac{Nq^2\tau}{m}\frac{1}{1 - i\omega\tau}.$$

c) Considerando que a relação $\vec{j} = \sigma\vec{E}$ é causal, pois só há corrente após aplicação do campo elétrico, escreva as relações de Kramers-Kronig para σ.

d) Mostre que a expressão para a potência média dissipada por unidade de volume no meio é dada por

$$\langle P \rangle = \frac{1}{2}E_0^2 Re(\sigma).$$

e) Considerando o limite $\omega \to \infty$ nas relações de Kramers-Kronig, mostre que

$$\int_0^\infty Re[\sigma(\omega)]\, d\omega = \frac{\pi N e^2}{2m}.$$

Portanto, medindo a potência dissipada, por unidade de volume, em altas frequências, é possível estimar a densidade de portadores.

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2. No caso de radiação por uma carga com aceleração paralela à velocidade, obtivemos que a potência radiada é dada por

$$\frac{dP}{d\Omega} = \frac{dU}{dt'} = \frac{\mu_0 q^2 \dot{v}_q^2}{16\pi^2 c}\frac{(\sin\theta)^2}{(1 - \beta\cos\theta)^5}.$$

a) Faça a integral dessa expressão sobre o ângulo sólido $d\Omega = \sin\theta \, d\theta d\varphi$, <u>sem utilizar</u> resultados dados em tabelas, mas empregando adequadamente integração por partes, e obtenha a expressão 6.163 do livro do Bo Thidé,

$$P = \int \frac{dU}{dt'} d\Omega = \frac{\mu_0 q^2 \dot{v}_q^2}{6\pi c} \frac{1}{(1-\beta^2)^3}.$$

b) Determine a expressão para o ângulo θ_{max} correspondente à direção de intensidade da radiação.

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3. Em aula discutimos o desenvolvimento do campo de radiação em multipolos considerando fontes monocromáticas. Neste problema vamos estender a teoria para incluir fontes com variação temporal arbitrária, mas lenta. Por isso, queremos dizer que, se τ_c for o tempo característico de variação da fonte e d sua maior dimensão, então $d/\tau_c \ll c$. Além disso, vamos considerar apenas a região de radiação, ou seja,

$$\frac{|\vec{r}'|}{r} \ll 1 \quad \Longrightarrow \quad |\vec{r} - \vec{r}'| \cong r - \frac{\vec{r} \cdot \vec{r}'}{r}.$$

Com esta condição, podemos aproximar a expressão para o tempo retardado por

$$t'_{ret} = t - \frac{|\vec{r} - \vec{r}'|}{c} \cong u + \frac{\vec{r} \cdot \vec{r}'}{rc}; \;\; u = t - \frac{r}{c}; \;\; \frac{\vec{r} \cdot \vec{r}'}{rc} \ll u.$$

Nessa aproximação, a variável u passa a ser um tempo retardado com dependência explícita de (r, t), e não implícita envolvendo $\vec{r}'(t')$.

a) Mostre que, até primeira ordem em $\vec{r} \cdot \vec{r}'/rc$, a expressão para a densidade de carga no tempo retardado pode ser aproximada como

$$[\rho(\vec{r}', t')]_{ret} \approx \rho(\vec{r}', u) + \frac{\vec{r} \cdot \vec{r}'}{rc} \dot{\rho}(\vec{r}', u) + \cdots; \;\; \dot{\rho}(\vec{r}', u) = \frac{\partial}{\partial u} \rho(\vec{r}', u)$$

b) Usando esse resultado e aproximando $|\vec{r} - \vec{r}'|$, no denominador da expressão para o potencial escalar, só pelo primeiro termo de seu desenvolvimento, mostre que

$$\phi(\vec{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{Q}{r} + \frac{1}{4\pi\epsilon_0} \frac{\hat{e}_r \cdot \dot{\vec{p}}(u)}{cr}; \;\; \hat{e}_r = \frac{\vec{r}}{r};$$

$$Q = \int \rho(\vec{r}', u) \, dV'; \;\; \vec{p}(u) = \int \vec{r}' \rho(\vec{r}', u) dV'.$$

Como Q independe de u, porque a carga total do sistema é fixa, o primeiro termo é o potencial eletrostático. Portanto, somente o segundo corresponde ao campo de radiação,

$$\phi_{rad}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\hat{e}_r \cdot \dot{\vec{p}}(u)}{cr}.$$

c) Faça o mesmo desenvolvimento para a densidade de corrente no instante retardado e mostre que

$$\vec{A}(\vec{r},t) \approx \frac{\mu_0}{4\pi}\frac{1}{r}\left[\int \vec{j}(\vec{r}',u)dV' + \frac{1}{c}\int \frac{\partial \vec{j}}{\partial u}(\hat{e}_r \cdot \vec{r}')dV'\right].$$

No caso estático, vimos que $\int \vec{j}(\vec{r}')dV' = 0$. Mas quando a densidade de corrente varia com o tempo, sua integral não é nula, como será visto a seguir. Mostre que a relação entre o segundo termo e o primeiro é da ordem de $(d/c\tau_c) \ll 1$, de modo que pode ser desprezado na aproximação de mais baixa ordem. Então,

$$\vec{A}_{rad}(\vec{r},t) \approx \frac{\mu_0}{4\pi r}\int \vec{j}(\vec{r}',u)dV'.$$

Empregando o mesmo método utilizado no cálculo do quadrupolo elétrico, use a identidade vetorial $\nabla \cdot (x_i\vec{j}) = x_i(\nabla \cdot \vec{j}) + \vec{j} \cdot \nabla x_i$ e a equação da continuidade para mostrar que

$$\int \vec{j}(\vec{r}',u)dV' = \frac{d\vec{p}}{du} = \dot{\vec{p}}(u) \Longrightarrow \vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(u)}{r}.$$

d) Considerando que $u = t - r/c$, mostre que

$$\nabla \cdot [\hat{e}_r \cdot \dot{\vec{p}}(u)] = -\frac{1}{c}(\hat{e}_r.\ddot{\vec{p}})\hat{e}_r + \mathcal{O}\left(\frac{1}{r}\right).$$

Usando esse resultado, obtenha a expressão para o campo elétrico,

$$\vec{E}_{rad}(\vec{r},t) = -\frac{\mu_0}{4\pi}\frac{\ddot{\vec{p}}}{r} + \frac{1}{4\pi\epsilon_0}\frac{(\hat{e}_r \cdot \ddot{\vec{p}})\hat{e}_r}{c^2 r} = -\frac{\mu_0}{4\pi}\frac{[\hat{e}_r \times \ddot{\vec{p}}(u)] \times \hat{e}_r}{r}$$

e) Para calcular o campo magnético, justifique porque o segundo termo da relação vetorial

$$\nabla \times \left[\frac{\dot{\vec{p}}(u)}{r}\right] = \frac{1}{r}\nabla \times \dot{\vec{p}} + \nabla\left(\frac{1}{r}\right) \times \dot{\vec{p}}(u)$$

pode ser desprezado em comparação com o primeiro e obtenha

$$\vec{B}_{rad}(\vec{r},t) = -\frac{\mu_0}{4\pi}\frac{\hat{e}_r \times \ddot{\vec{p}}(u)}{cr}.$$

f) Obtenha a expressão para o vetor de Poynting,

$$\vec{S}_{rad} = \frac{c}{\mu_0}\left|\vec{B}_{rad}\right|^2 \hat{e}_r.$$

Integre o vetor de Poynting sobre o elemento de área $d\vec{a} = r^2 \sin\theta\, d\theta d\varphi \hat{e}_r$, supondo $\vec{p} = p\hat{e}_z$, e obtenha a expressão para a potência total radiada

$$P = \frac{\mu_0}{6\pi c}|\ddot{p}(u)|^2.$$

Verifique se esta expressão reproduz a obtida para ondas monocromáticas, no caso $\vec{p}(t) = \vec{p}e^{-i\omega t}$.

g) Vamos empregar esse resultado para calcular o tempo de decaimento do átomo de hidrogênio, na descrição clássica. Considerando átomo com um núcleo fixo, em torno do qual orbita o elétron, temos que seu momento de dipolo é dado por

$$\vec{p}(t) = -e\vec{a}(t),$$

onde $\vec{a}(t)$ é o vetor posição do elétron na órbita circular. Sabendo que o elétron é mantido em órbita pelo campo elétrico do núcleo, mostre que

$$\ddot{\vec{p}}(t) = \frac{1}{4\pi\epsilon_0}\frac{e^3}{ma^2}\hat{a}(t); \;\; \hat{a} = \frac{\vec{a}}{a}.$$

h) Substituindo esse resultado na potência radiada pelo elétron, e considerando que ela é compensada pela diminuição da energia orbital do elétron, isto é,

$$\frac{dE_{orb}}{dt} = -P; \quad E_{orb} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a},$$

Mostre que o raio da órbita do elétron decresce segundo a equação

$$\frac{da}{dt} = -\frac{4}{3}\frac{1}{(4\pi\epsilon_0)^2}\frac{e^4}{m^2c^3a^2}.$$

i) Supondo que o elétron esteja numa órbita inicial com raio $a = a_0$, calcule o tempo característico τ para ele cair sobre o núcleo, integrando a equação de $a = a_0$ até $a = 0$ (note que esta integral é feita facilmente por quadratura), obtendo

$$\tau = \frac{1}{4e^2}(4\pi\epsilon_0)^2 m^2 c^3 a_0^3.$$

Misturando agora um pouco de física moderna, tome para o raio inicial o raio de Bohr, $a_0 = 4\pi\epsilon_0\hbar^2/me^2$. Mostre então que o tempo de decaimento pode ser escrito como

$$\tau = \frac{1}{8\pi\alpha^5}\frac{\lambda_C}{c},$$

onde $\alpha = e^2/4\pi\epsilon_0\hbar c \cong 1/137$ é a constante de estrutura fina e $\lambda_C = 2\pi\hbar/mc \cong 2{,}43 \times 10^{-12} m$ é o comprimento de Compton do elétron. Usando os valores dessas constantes, obtemos $\tau \cong 1{,}6 \times 10^{-11} s$!

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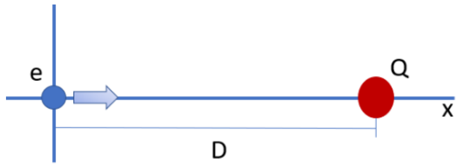
4. Um elétron se encontra no campo de uma carga positiva Q, que está fixa no ponto $x = D$. No instante $t = 0$ o elétron é solto, se deslocando na direção de Q.



a) Calcule a força que o elétron exerce sobre a carga positiva em função de sua posição $[R]_{ret}$ e velocidade normalizada $[\beta]_{ret}$.

b) O princípio de ação e reação de Newton se verifica neste caso? Justifique sua resposta.

c) Calcule a potência total radiada pelo elétron quando ele começa a ser acelerado, isto é, em $x = 0$, em função de sua massa m, carga e, D e Q.

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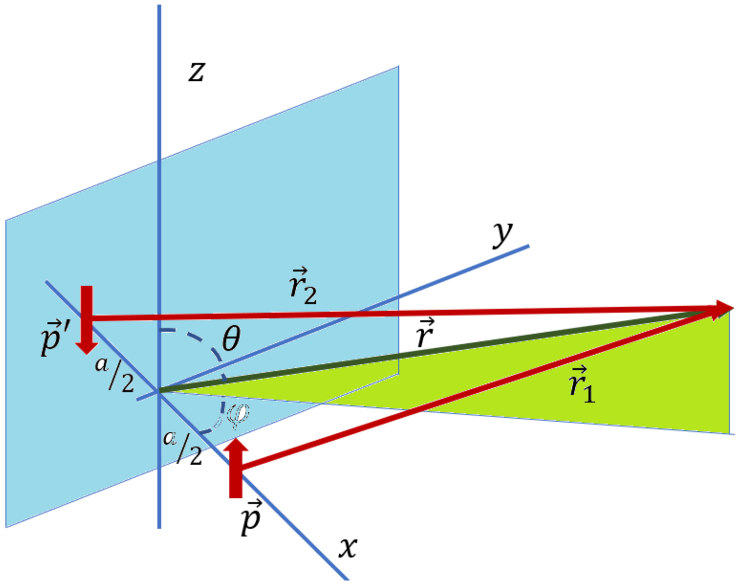
5. Um dipolo que oscila harmonicamente com frequência ω, $\vec{p}(t) = pe^{-i\omega t}\hat{e}_z$, é colocado a uma distância $a/2$ de um plano condutor infinito, produzindo um dipolo imagem, como esquematizado na figura.



a) Usando a expressão derivada em aula para o potencial vetor de um dipolo elétrico, mostre que, na região de radiação, isto é, $x > 0; r \gg a$, o potencial vetor é dado por

$$\vec{A}(\vec{r},t) = -i\frac{\mu_0}{4\pi}\omega p\left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right)e^{-i\omega t}\hat{e}_z,$$

onde $\vec{r}_1 = \vec{r} - \frac{a}{2}\hat{e}_x$; $\vec{r}_2 = \vec{r} + \frac{a}{2}\hat{e}_x$.

b) Usando a condição $r \gg a$, mostre que

$$r_1 \approx r - \frac{a}{2}\sin\theta\cos\varphi\,; \quad r_2 \approx r + \frac{a}{2}\sin\theta\cos\varphi.$$

c) Com esse resultado, mostre que a expressão para o potencial vetor fica

$$\vec{A}(\vec{r},t) = -\frac{\mu_0}{2\pi}\frac{\omega p}{r}e^{i(kr-\omega t)}\sin\left(\frac{ka}{2}\sin\theta\cos\varphi\right)\hat{e}_z.$$

d) Escrevendo $\hat{e}_z$ em coordenadas esféricas, mostre que na zona de radiação o campo magnético produzido pelo dipolo é dado por

$$\vec{B}(\vec{r},t) \cong i\frac{\omega^2 p}{2\pi\epsilon_0 c^3}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\sin\left(\frac{ka}{2}\sin\theta\cos\varphi\right)\hat{e}_\varphi.$$

e) Sabendo que, na região de radiação, $\vec{E}(\vec{r},t) = c\vec{B}(\vec{r},t) \times \hat{e}_r$, mostre que a distribuição angular da potência média radiada é dada por

$$\frac{d\langle P\rangle}{d\Omega} = \frac{\omega^4 p^2}{8\pi^2\epsilon_0 c^3}\left[\sin\theta\sin\left(\frac{ka}{2}\sin\theta\cos\varphi\right)\right]^2.$$

f) Considerando apenas a condição de longo comprimento de onda, $ka \ll 1$, mostre que a potência total radiada pelo dipolo é dada por

$$\langle P\rangle = \frac{\omega^6 p^2 a^2}{30\pi\epsilon_0 c^5}.$$

6. O cálculo do campo produzido por cargas com movimento arbitrário é bastante complexo, exigindo, via de regra, cálculo numérico, devido à relação implícita que define o tempo retardado. No entanto, num trabalho recente, Ruhlandt, Mühle e Enderlein apresentaram um método para obter as linhas de força do campo electromagnético, dadas por curvas tridimensionais parametrizadas, em que o tempo retardado é utilizado como parâmetro.

O artigo, em anexo, deve ser analisado detalhadamente neste problema. Todas as equações das seções 2, 3 e 4 que não forem triviais devem ser derivadas, com exceção das equações da subseção 4A, que, apesar de relevante, nesta prova não precisa ser estudada em detalhe para economia de tempo.

Electric field lines of relativistically moving point charges

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Electric field lines of relativistically moving point charges

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Generation of electromagnetic fields by moving charges is a fascinating topic where the tight connection between classical electrodynamics and special relativity becomes particularly apparent. One can gain direct insight into the fascinating structure of such fields by visualizing the electric field lines. However, the calculation of electric field lines for arbitrarily moving charges is far from trivial. Here, we derive an equation for the direction that points from the retarded position of a moving charge towards a specific field line position, which allows for a simple construction of these lines. We analytically solve this equation for several special but important cases: for an arbitrary rectilinear motion, for the motion within the wiggler magnetic field of a free electron laser, and for the motion in a synchrotron.
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I. INTRODUCTION

Electric and magnetic fields generated by arbitrarily moving point charges are a fascinating topic where relativistic physics meets classical electrodynamics. In particular, accelerated point charges are the generators for almost all electromagnetic radiation, such as that emitted by oscillating electric dipoles, synchrotrons, or free electron lasers. As is well known, the electric field $\mathbf{E}$ of an arbitrarily moving point charge q can be found with the help of Liénard–Wiechert potentials and has the explicit form[1,2]

$$\mathbf{E}(\mathbf{r},t) = q\left\{\frac{\mathbf{R}-R\boldsymbol{\beta}}{\gamma^2(R-\mathbf{R}\cdot\boldsymbol{\beta})^3} + \frac{\mathbf{R}\times\left[(\mathbf{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{c(R-\mathbf{R}\cdot\boldsymbol{\beta})^3}\right\}_{t'}, \tag{1}$$

where the three-dimensional vector $\mathbf{R}$ is the spatial part of the four-dimensional null-vector

$$\left\{c(t-t'), \mathbf{r}-\mathbf{r}_0(t')\right\}. \tag{2}$$

This null-vector defines the retarded time $t' < t$ via

$$t - t' = \frac{R}{c} = \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c} \tag{3}$$

at which the right hand side of Eq. (1) has to be evaluated. Here, $\mathbf{r}_0(t)$ is the particle's trajectory as a function of time t. This geometry of the situation is visualized in Fig. 1. The symbol $\boldsymbol{\beta}(t) = c^{-1}d\mathbf{r}_0(t)/dt$ is the particle's velocity divided by the speed of light c, γ is the usual Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$, and a dot denotes differentiation with respect to time. For finding the electric field at a given position $\mathbf{r}$ and time t, one has first to solve the retarded time equation (3), and then second to evaluate the right hand side of (1) at time t', which is typically a numerically demanding task.

Another way of visualizing an electric field is to use electric field lines–continuous lines tangential to the electric field vector. Visualization of field lines can lead to a better understanding of complex field configurations generated by non-trivial particle trajectories, and knowledge of field lines can also be used to estimate the electric field strength, due to the interconnection between local field line density and field strength as embodied in the zero divergence of the electric field in source-free space. Thus, the question how to efficiently calculate and draw field lines for arbitrarily moving point charges has been repeatedly considered in the literature.[3–10] Here, we present an efficient and relatively simple way to find and draw electric field lines of an arbitrarily moving charge by deriving a compact auxiliary equation for a unit vector pointing from the retarded position of the charge to a specific field line position. We then find analytic solutions of the problem for several important cases.

II. ARBITRARY MOTION

Let us describe a field line at time t by a parametric three-dimensional curve $\mathbf{p}(s)$ which is parametrized by the variable s. Along all its positions, it has to be parallel to the electric field vector, which means that it has to obey the differential equation

$$\frac{d\mathbf{p}(s)}{ds} \propto \mathbf{E}[\mathbf{p}(s)]. \tag{4}$$

Taking into account the non-trivial form of the electric field as given in Eq. (1), finding analytic solutions to this equation seems to be a formidable task. Note that any Cartesian position $\mathbf{r}$ can be geometrically referenced to the retarded position $\mathbf{r}_0(t')$ by $\mathbf{r} = \mathbf{r}_0(t') + \mathbf{R}(t')$, where t' is the retarded time of the particle's position when it contributes to the electric field at position $\mathbf{r}$, see also Fig. 1. In particular, this holds true also for positions $\mathbf{r} = \mathbf{p}(s)$ on a field line. Our core idea is to use the retarded time t' to parametrize a field line, by setting $s = t'$. Thus, the time t' has a double meaning: it denotes the retarded time t' and it parametrizes the field line, and we find for the field line positions the relation

$$\mathbf{p}(t') = \mathbf{r}_0(t') + \mathbf{R}(t') = \mathbf{r}_0(t') + c(t-t')\hat{\boldsymbol{\lambda}}(t'), \tag{5}$$

where we have used the fact that the length of the vector $\mathbf{R}(t')$ is $c(t-t')$, so that the vector $\hat{\boldsymbol{\lambda}}(t')$ on the right hand side of Eq. (5) is a unit vector pointing from the retarded position $\mathbf{r}_0(t')$ of the point charge to a position $\mathbf{p}(t')$. Now, let us consider Eq. (4). Because we require that the vector $d\mathbf{p}/ds \equiv d\mathbf{p}/dt'$ and the vector $\mathbf{E}$ have only to be parallel at all positions $\mathbf{p}(t')$, we can choose any proportionality factor in Eq. (4) between these two vectors. Let us thus set $d\mathbf{p}/dt'$

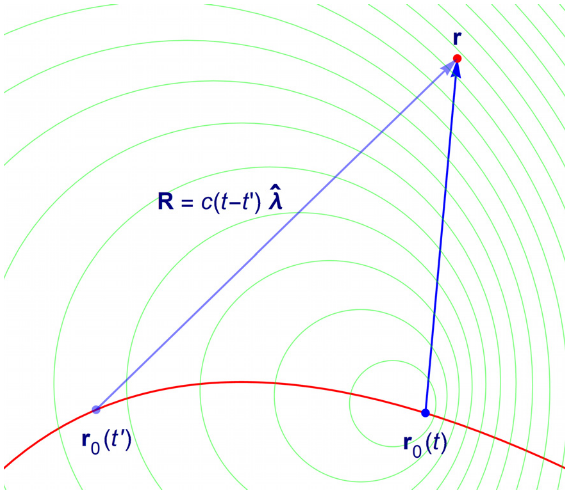

Fig. 1. A point charge moving along an arbitrary trajectory $\mathbf{r}_0(t)$ (curved solid line) generates an electromagnetic field throughout space. The field at any given position $\mathbf{r}$ at time t originates from the charge when it is at retarded position $\mathbf{r}_0(t')$. Circles are lines of constant retarded time t'. The vector $\mathbf{R} = c(t-t')\hat{\boldsymbol{\lambda}}$ is the spatial part of the null-vector of Eq. (2) which connects the space-time events $\{ct', \mathbf{r}_0(t')\}$ and $\{ct, \mathbf{r}\}$, so that $\hat{\boldsymbol{\lambda}}$ is a unit vector.

equal to $-c\gamma^2 R^2 (1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta})^3 \mathbf{E}/q$ so that we find the field-line determining equation

$$\frac{d\mathbf{p}(t')}{dt'} = -c\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right) - \gamma^2 \mathbf{R} \times \left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right], \tag{6}$$

where a hat over a vector symbolizes normalization (unit vector). Now, by inserting Eq. (5) into the last equation, we find the following auxiliary equation for the unit vector $\hat{\boldsymbol{\lambda}}(t') = \hat{\mathbf{R}}$:

$$\frac{d\hat{\boldsymbol{\lambda}}}{dt'} = \gamma^2\left[\left(\hat{\boldsymbol{\lambda}}-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right]\times\hat{\boldsymbol{\lambda}}. \tag{7}$$

This equation is the core result of our paper: When we can solve this equation and determine $\hat{\boldsymbol{\lambda}}(t')$ for all times $t' < t$, then we can use Eq. (5) to find the full field line. Thus, t' plays the role of a curve parameter and does not have to be found *a priori* from an implicit retarded time equation such as Eq. (3), as has to be done when calculating the electric field. The final condition of Eq. (7), i.e., the direction $\hat{\boldsymbol{\lambda}}(t'=t)$, defines into which direction a field line starts from a point charge at time t.

Although we cannot present a general solution of Eq. (7) for an arbitrary motion $\mathbf{r}_0(t')$, we consider in the next chapters several important and quite general cases for which analytical solutions can be found.

III. RECTILINEAR MOTION

Let us assume that the velocity and acceleration are always co-linear, i.e., $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$. In that case, our auxiliary equation for $\hat{\boldsymbol{\lambda}}$ simplifies to

$$\frac{d\hat{\boldsymbol{\lambda}}}{dt'} = \gamma^2\left(\hat{\boldsymbol{\lambda}}\times\dot{\boldsymbol{\beta}}\right)\times\hat{\boldsymbol{\lambda}} = \gamma^2\left[\dot{\boldsymbol{\beta}} - \hat{\boldsymbol{\lambda}}\left(\dot{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{\lambda}}\right)\right]. \tag{8}$$

Multiplying both sides with unit vector $\hat{\boldsymbol{\beta}}$ leads to an equation for the component $\lambda_\parallel = \hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{\lambda}}$ of $\hat{\boldsymbol{\lambda}}$ parallel to the constant direction of motion

$$d\lambda_\parallel = \frac{1-\lambda_\parallel^2}{1-\beta^2}d\beta. \tag{9}$$

This equation can be integrated and has the solution

$$\lambda_\parallel = \frac{n_\parallel + \beta}{1+n_\parallel\beta}, \tag{10}$$

where $n_\parallel$ is an integration constant. For a $\hat{\boldsymbol{\lambda}}$-component $\lambda_\perp$ that is orthogonal to the direction of motion, we can find a similar equation by multiplying both sides of Eq. (8) with a unit vector perpendicular to $\hat{\boldsymbol{\beta}}$. This results in

$$d\lambda_\perp = -\frac{\lambda_\perp\lambda_\parallel}{1-\beta^2}d\beta = -\frac{\lambda_\perp\left(n_\parallel+\beta\right)}{\left(1+n_\parallel\beta\right)\left(1-\beta^2\right)}d\beta, \tag{11}$$

which can also be explicitly integrated and has the solution

$$\lambda_\perp = \frac{n_\perp}{\gamma\left(1+n_\parallel\beta\right)} \tag{12}$$

with a second integration constant $n_\perp$. By adding $\lambda_\parallel^2$ and $\lambda_\perp^2$ together, one can check that $n_\parallel^2 + n_\perp^2 = 1$ so that the integration constants are the components of a unit vector $\hat{\mathbf{n}}$. Putting this all together leads to the compact result

$$\hat{\boldsymbol{\lambda}} = \frac{\hat{\mathbf{n}} + (\gamma-1)(\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}} + \gamma\boldsymbol{\beta}}{\gamma(1+\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}. \tag{13}$$

Inserting this expression into Eq. (5) and after some algebraic transformations, one finds the result for the electric field line itself to be

$$\begin{aligned}\mathbf{p}(t') &= \mathbf{r}_0(t') + c(t-t')\boldsymbol{\beta}(t') \\ &\quad + c(t-t')\left[\frac{(\gamma^{-1}-1)(\hat{\mathbf{n}}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}+\hat{\mathbf{n}}}{\gamma(1+\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}\right]_{t'}.\end{aligned} \tag{14}$$

Here, $\boldsymbol{\beta}$ and γ in the square bracket are evaluated at time t'. Please note that the expression $\mathbf{r}_0(t') + c(t-t')\boldsymbol{\beta}(t')$ in the above equation would represent the position of the moving charge *if* it would continue to move uniformly with its instantaneous velocity $\dot{\mathbf{r}}_0(t') = c\boldsymbol{\beta}(t')$ from its position $\mathbf{r}_0(t')$ at time t'. Thus, the vector in the second line of Eq. (14) points from this virtual position to the field line position corresponding to t'.

The expression found for $\mathbf{p}(t')$ gives an explicit parametric representation of a field line at time t, where the parametric variable is the retarded time t'. For finding a particular field line, one first defines $\hat{\mathbf{n}}$ and then traces the line for decreasing values of t' starting from $t' = t$.

To better understand the physical meaning of the unit vector $\hat{\mathbf{n}}$, let us check Eq. (14) against the well-known case of a point charge moving uniformly with velocity $\dot{\mathbf{r}}_0 = c\boldsymbol{\beta}$. For this case, the electric field reads

$$\mathbf{E}(\mathbf{r},t) = \frac{q\gamma\Delta\mathbf{r}}{\left\{\gamma^2(\Delta\mathbf{r}\cdot\hat{\boldsymbol{\beta}})^2 + \left[\Delta\mathbf{r} - (\Delta\mathbf{r}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}}\right]^2\right\}^{3/2}}, \tag{15}$$

where we have used the abbreviation $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0(t') - c(t-t')\boldsymbol{\beta}$. This expression describes an isotropic electric field which is "squeezed" by a factor γ^{-1} along the direction of motion. Thus, if a field line is directed along unit vector $\hat{\mathbf{n}}$ in the particle's rest frame, it will point along direction

$$\hat{\mathbf{n}}' = \frac{(\gamma^{-1} - 1)(\hat{\mathbf{n}} \cdot \hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}} + \hat{\mathbf{n}}}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \boldsymbol{\beta})^2}} \tag{16}$$

in the observer's lab frame. Comparing Eq. (16) with Eq. (14) shows that Eq. (14) indeed describes straight field lines along directions $\hat{\mathbf{n}}'$ starting from the instantaneous position $\mathbf{r}_0(t') + c(t - t')\boldsymbol{\beta}$ of the uniformly moving charge at time t, and that $\hat{\mathbf{n}}$ in Eq. (7) is the starting direction of the field line within the rest frame of the moving charge. To summarize, Eq. (14) describes the field line position as pointing from the virtual position $\mathbf{r}_0(t') + c(t - t')\boldsymbol{\beta}$ into the direction of the squeezed unit vector in the charge's rest frame. Thus, if $\beta = \text{const.}$, this direction is also constant and the field lines are straight lines originating from the virtual position at time t of the charge.

In what follows, we consider several special case of rectilinear motion and calculate images of the corresponding field lines. All numerical calculations for the figures in this paper have been done with MATHEMATICA, and the code can be found at Ref. 11. Animated GIFs for all the discussed examples below can be found in Ref. 12.

A. Uniformly accelerated motion

As a first application of Eq. (14), we consider the well-known classical example of a uniformly accelerated charge[13] which is at rest at time zero, then (relativistically) accelerates along the (horizontal) x-direction with constant acceleration to the speed $c/\sqrt{2}$ within one unit of time, and then continues to move uniformly with that constant velocity. For such a motion, the particle's x-position as a function of time is given by

$$x_0(t) = \begin{cases} 0, & \text{if } t \le 0 \\ c\left(\sqrt{1+t^2} - 1\right), & \text{if } 0 < t \le 1 \\ c\left(\sqrt{2} - 1 + (t-1)/\sqrt{2}\right), & \text{if } t > 1. \end{cases} \tag{17}$$

Figure 2 shows the electric field lines overlaid with a density plot of the decadic logarithm of the electric field amplitude for the time $t = 4$. As can be seen, Eq. (14) nicely reproduces the field lines of the static charge at large distances and those of the uniformly moving charge at small distances from the particle, with the acceleration-related transition zone in between. Due to the zero-divergence of the electric field, the field lines outside and inside the transition zone have to be continuously connected which leads to their strong inclination towards radial lines starting from the point charge. This transition zone of strongly inclined electric field lines forms a shell that travels with the speed of light. For a detailed discussion, see Ref. 13.

B. Uniformly decelerated motion

The second example considers the opposite situation: A uniformly moving charge (uniform speed $c/\sqrt{2}$) starts to decelerate at time zero with constant deceleration so that it stops moving at time one. Now, its position is given by

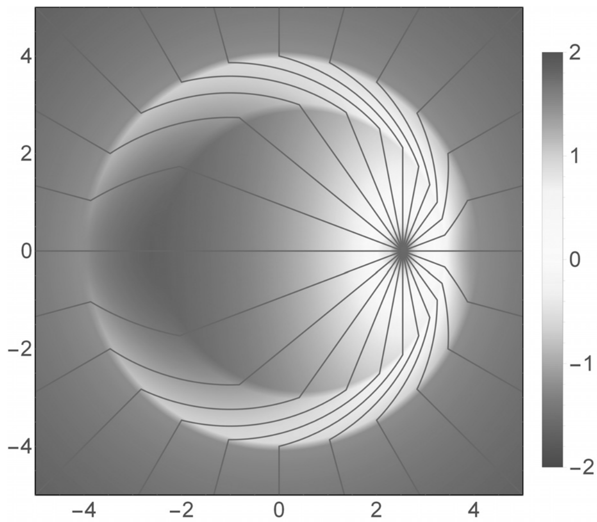

Fig. 2. Electric field (density plot) and electric field lines for an accelerated point charge. Its position along the horizontal axis (x-axis) is given in Eq. (17). The picture shown refers to time $t = 4$, and the unit of length is chosen in such a way that the numerical value of the speed of light is unity. Here, we show field lines that start, in the particle's rest frame, from its position at angles $\phi = 15^\circ$ to $\phi = 360^\circ$ with respect to the horizontal axis in steps of 15°.

$$x_0(t) = \begin{cases} ct/\sqrt{2}, & \text{if } t \le 0 \\ c\left(1 + t/\sqrt{2} - \sqrt{1+t^2}\right), & \text{if } 0 < t \le 1 \\ c\left(1 + 1/\sqrt{2} - \sqrt{2}\right), & \text{if } t > 1. \end{cases} \tag{18}$$

The resulting field lines and electric field for $t = 4$ are presented in Fig. 3. Although the motion of the charge is a simple time-reversal of the first example, the field lines and electric field look significantly different, which is, of course, a direct consequence of the retarded time effect. However, one sees again a transition zone of highly inclined field lines that moves away with the speed of light. This is the shock wave of light ("Bremsstrahlung") which is generated when charged particles (electrons) jam into a solid, the classical process of x-ray generation in x-ray vacuum tubes.[14]

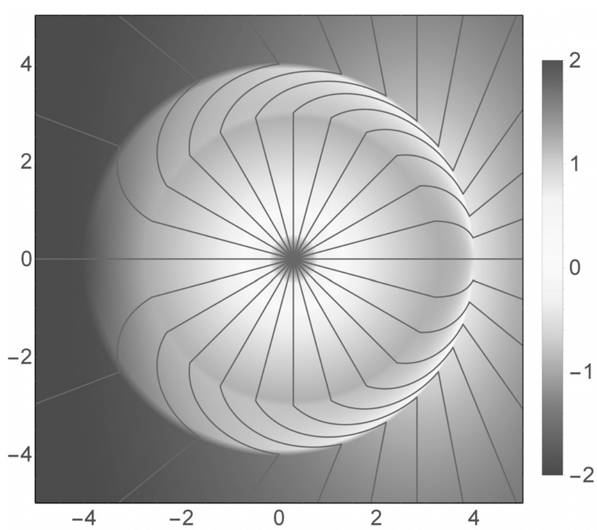

Fig. 3. Same as Fig. 2 but for a decelerating charge, the horizontal position of which is described in Eq. (18). Shown is a snapshot for time $t = 4$.

C. Oscillating charge

As a third and last example for rectilinear motion we consider a charge which oscillates with angular frequency ω and amplitude a along the vertical direction. Thus, its vertical position is given by

$$y_0(t) = a \sin \omega t, \tag{19}$$

so that $\hat{\boldsymbol{\beta}} = \hat{\mathbf{y}}$ and $\beta = a\omega \cos \omega t / c$. The resulting field lines and electric field for the numerical values $\omega = \pi$ and $\omega a/c = 0.8$ are presented in Fig. 4 for $t = 0$.

IV. PLANAR MOTION

Remarkably, the expression of Eq. (14) was already presented by Arutyunyan in the 1980s, see Refs. 15–17. There, it was claimed that it is valid also for non-rectilinear motions such as that of a charge in a synchrotron. Unfortunately, this is not true because its derivation was based on the assumption that $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$ so that the term with $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}$ in Eq. (7) drops out. In Secs. IV A and IV B, we consider two cases of planar motion (motion confined to a plane) where this is no longer true, but where we can still find analytical solutions to Eq. (7).

A. Wiggler/undulator radiation

Let us consider the motion of a point charge in a magnetic wiggler/undulator:[18–20] A point charge moves with constant speed $c\beta_0$ along the x-direction and wiggles along the orthogonal y-direction with arbitrarily time-dependent velocity $c\beta_\perp(t)$. Thus, we now have

$$\boldsymbol{\beta}(t) = \beta_0 \hat{\mathbf{x}} + \beta_\perp(t) \hat{\mathbf{y}} \tag{20}$$

and

$$\dot{\boldsymbol{\beta}}(t) = \dot{\beta}_\perp(t) \hat{\mathbf{y}}. \tag{21}$$

In what follows, we always consider field lines in the plane of motion so that $\hat{\boldsymbol{\lambda}}$ lies in the xy-plane. Let us denote the angle between $\hat{\boldsymbol{\lambda}}$ and the horizontal x-axis by $\psi(t)$ so that $\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\beta}} = \dot{\beta}_\perp \cos \psi(t) \hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ is a unit vector pointing out of the xy-plane. Also, we have $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = \dot{\beta}_\perp(t) \beta_0 \hat{\mathbf{z}}$. Inserting these expressions into Eq. (7), and taking into account that $d\psi/dt'$ equals the modulus of $d\hat{\boldsymbol{\lambda}}/dt'$, we find

$$\frac{d\psi}{dt'} = \frac{\dot{\beta}_\perp(t')}{1 - \beta_0^2 - \beta_\perp^2(t')} (\cos \psi - \beta_0). \tag{22}$$

This equation can be solved analytically and has the explicit solution

$$\psi(t') = 2\operatorname{arctan}\left\{ \frac{1}{\zeta} \tanh\left[\frac{1}{2} \operatorname{artanh}\left[\gamma_0 \beta_\perp(t')\right] - \frac{1}{2} \operatorname{artanh}[\gamma_0 \beta_\perp(t)] + \operatorname{artanh}\left(\zeta \tan \frac{\phi}{2} \right) \right] \right\}, \tag{23}$$

where ϕ is the final angle of $\psi(t')$ at time $t' = t$, and where we have introduced the abbreviations

$$\zeta = \sqrt{\frac{1 + \beta_0}{1 - \beta_0}} \quad \text{and} \quad \gamma_0 = \sqrt{1 - \beta_0^2}. \tag{24}$$

Knowing the solution for ψ, the unit vector $\hat{\boldsymbol{\lambda}}$ is given by

$$\hat{\boldsymbol{\lambda}}(t') = \{ \cos \psi(t'), \sin \psi(t') \}, \tag{25}$$

which, when inserted into Eq. (5), solves the problem of finding a parametric description for the electric field lines. The expressions found are valid for an arbitrary transverse motion described by $\beta_\perp(t)$. Let us consider the special case of a harmonic transverse oscillation with frequency ω. Then, the particle's trajectory is described by

$$\mathbf{r}_0(t) = c\beta_0 t \hat{\mathbf{x}} + \frac{c\beta_\perp}{\omega} \sin(\omega t) \hat{\mathbf{y}}. \tag{26}$$

Let us consider the following numerical values: $\omega = \pi$, $\beta_0 = 1/\sqrt{2}$ and $\beta_\perp = 0.1$. Thus, the particle moves uniformly with $1/\sqrt{2}$ light speed along the horizontal axis while oscillating with maximum 0.1 light speed vertically. The resulting field lines and electric field are shown in Fig. 5. It is nice that one sees that regions of strong transverse field-line orientation (with respect to the line of sight from the particle) and thus field line density correspond to regions of large electric field strength.

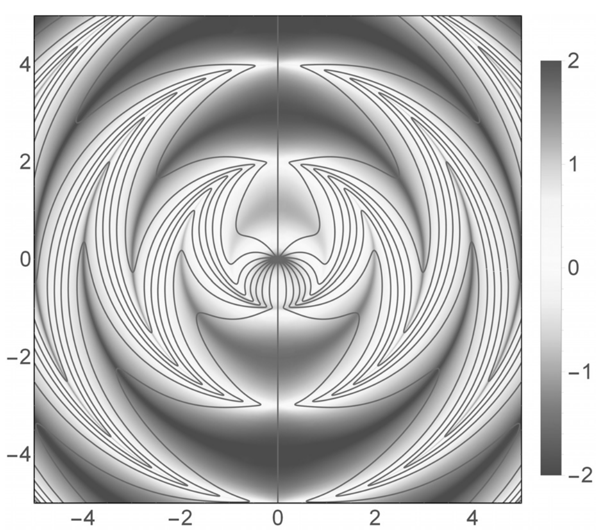

Fig. 4. Same as Fig. 2 but for an vertically oscillating motion, see Eq. (19), with oscillation angular frequency $\omega = \pi$, and oscillation amplitude a chosen in such a way that the maximum value of $\beta = \omega a/c$ is equal to 0.8. Shown is a snapshot at time $t = 0$. Shown are field lines that start, in the particle's rest frame, from its position at angles $\phi = 10°$ to $\phi = 350°$ with respect to the horizontal axis in steps of 20°.

B. Synchrotron

The last example considered refers to the motion of a point charge in a synchrotron:[21–25] a motion with uniform speed around a circle with radius a and angular frequency ω. Thus, the time-dependent coordinate of the particle is described by

$$\mathbf{r}_0(t) = a(\cos \omega t \, \hat{\mathbf{x}} + \sin \omega t \, \hat{\mathbf{y}}), \tag{27}$$

so that the constant modulus of $\boldsymbol{\beta}$ is $\beta = a\omega/c$ and the constant modulus of $\dot{\boldsymbol{\beta}}$ is $\dot{\beta} = \omega\beta = a\omega^2/c$. Let us denote the angle between $\hat{\boldsymbol{\lambda}}$ and $\mathbf{r}_0$ by ψ. Then we find the determining equation for $\psi(t')$ from Eq. (7) to be

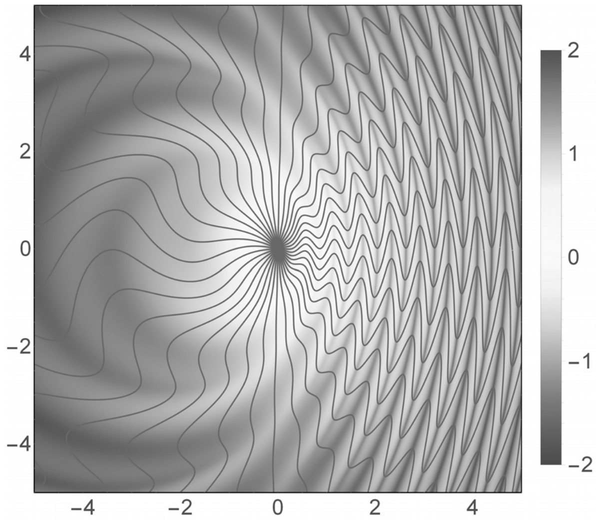

Fig. 5. Same as Fig. 2 but for a wiggling motion as described in Eq. (26). Shown is a snapshot at time $t = 0$ in the xy-plane.

$$\frac{d\psi}{dt'} + \omega = \gamma^2 \left(\dot{\beta} \sin\psi - \beta\dot{\beta}\right), \tag{28}$$

where the ω on the left side comes from the uniform rotational motion of $\mathbf{r}_0$. After replacing $\dot{\beta}$ by $\omega\beta$ and subtracting ω on both sides, this can be rewritten into

$$\frac{d\psi}{dt'} = \gamma^2 \omega (\beta \sin\psi - 1). \tag{29}$$

Again, this equation admits an analytical solution which reads

$$\psi(t') = 2\arctan\left\{ \beta - \frac{1}{\gamma}\tan\left[\frac{\gamma\omega(t'-t)}{2} + \arctan\left[\gamma\left(\beta - \tan\frac{\phi}{2}\right)\right]\right]\right\}, \tag{30}$$

where ϕ now is the final value of $\psi(t')$ for $t' = t$. Now, with the solution for $\psi(t')$ at hand, the unit vector $\hat{\boldsymbol{\lambda}}$ is given in Cartesian (x, y)-coordinates by

$$\hat{\boldsymbol{\lambda}}(t') = \left\{ \cos\left[\omega t' + \psi(t')\right], \sin\left[\omega t' + \psi(t')\right] \right\}, \tag{31}$$

which again solves the full problem. As a numerical example, let us first consider the case of charge moving with 0.6 light speed around a circle of $a = 100$ m. Thus, we find for the angular frequency a value of $\omega = 0.6\,c/a \sim 1.8 \times 10^6$ s^{-1}, which corresponds to an oscillation period of $\sim 3.5\,\mu$s. The resulting field line structure and electric field are presented in Fig. 6. For comparison, we consider also a travel speed of $0.9\,c$, which corresponds to an angular frequency of $\omega = 0.9\,c/a \sim 1.8 \times 10^6$ s^{-1}, or an oscillation period of $\sim 2.3\,\mu$s. The field line structure and electric field for this case are shown in Fig. 7. Although both Figs. 6 and 7 look qualitatively similar, one can see how the field line structures develop a more pronounced shock-wave structure for velocities closer to the speed of light. It should be noted that the polarization of this shock wave is everywhere parallel to the field lines, thus becoming close to perpendicular to its propagation direction for large velocities. Also, a comparison of

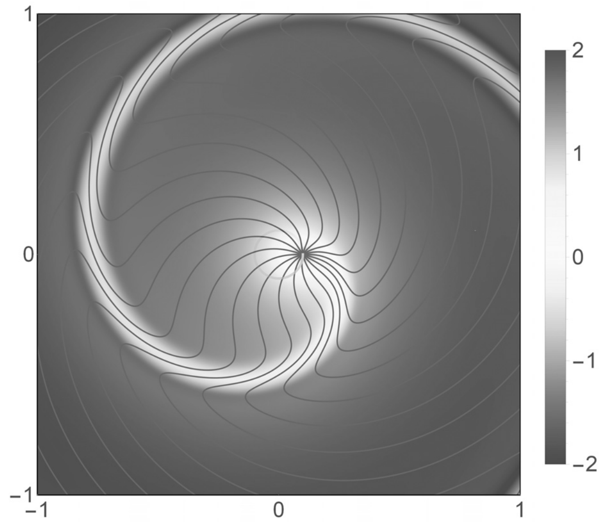

Fig. 6. Electric field lines and electric field amplitude for a circularly moving point charge (indicated by the circular line in the middle) at 0.6 light speed. Here, the unit of length is 1 km, and the radius of the circular motion is 0.1 km. High field intensities coincide with strong bunching of electric field lines, demonstrating nicely the tight connection between field intensity and field line density.

Figs. 6 and 7 shows nicely how the pulse width of the emitted light, i.e., the width of the shock wave, decreases with increasing speed.

V. CONCLUSION

We have presented an elementary derivation of a differential equation, Eq. (7), the solution of which leads to a simple description of electric field lines for an arbitrarily moving charge. We have presented several analytical solutions of this equation for a quite broad class of cases. Even if one cannot find analytical solutions to Eq. (7), its simplicity should make numerical integration straightforward. Thus, it provides a powerful tool for visualizing the electric field structure generated by a point charge moving along arbitrarily complex and relativistic trajectories.

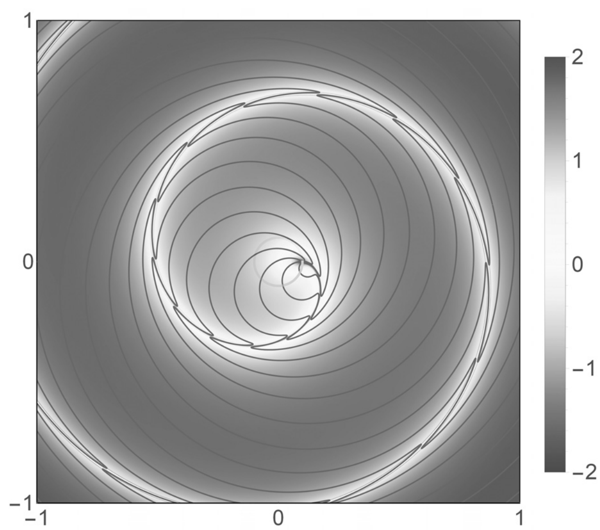

Fig. 7. Same as Fig. 6 but for a circularly moving point charge at 0.9 light speed.

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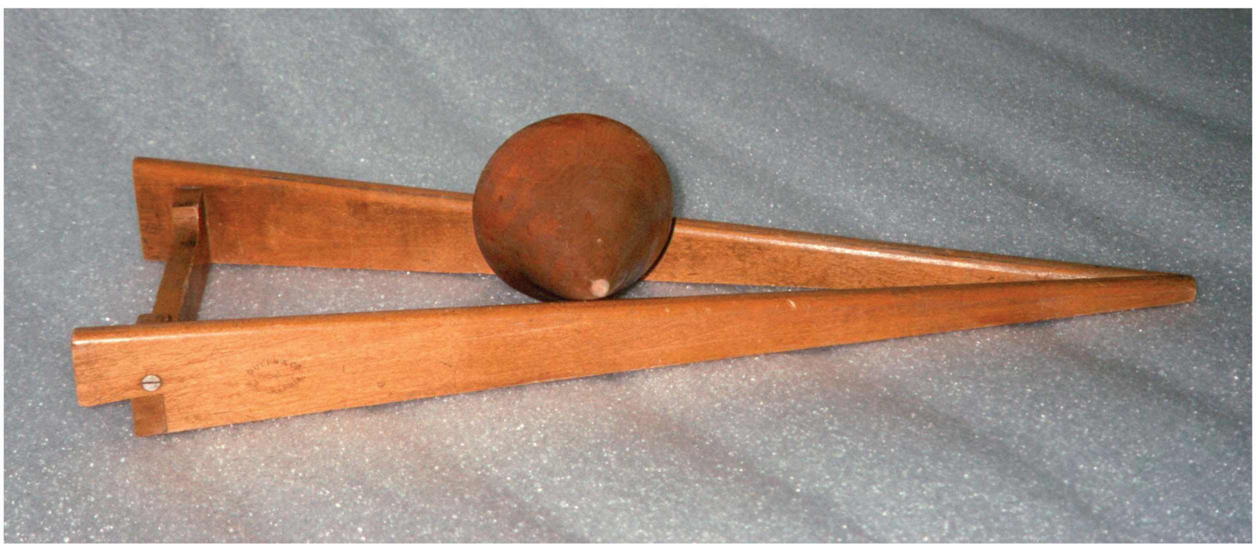

Cone Rolling up a Hill Illusion

This is a very old demonstration and is to be found in the 1713 lecture demonstration book by Hauksbee and Whiston. When the system is started at the left-hand side, it moves steadily in what appears to be the uphill direction. In reality, because of the differing slopes of the two guide rails and the cones, the center of mass of the system is actually getting lower as the system moves. I photographed it during the summer 2005 AAPT meeting at the University of Utah where Vacek Miglus and I examined all of the early apparatus in the physics demonstration room. (Picture and text by Thomas B. Greenslade, Jr., Kenyon College)