Optimal Control with Constraints

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The political-business-cycle model

- Suppose that **a party has just won the election at $\boldsymbol{t = 0}$**, and the **next election is to be held $\boldsymbol{T}$** years later at $\boldsymbol{t = T}$.
- **The incumbent party** then has a total of $\boldsymbol{T}$ **years** in which to impress the voters with its accomplishments in order to win their votes.
- At any time in the period $t \in [0, T]$, the pair of realized values **of $\boldsymbol{U}$ and $\boldsymbol{p}$ will determine** a specific **value of $\boldsymbol{v}$.**
- Such values of $\boldsymbol{v}$ for different points of time **must** all **enter into** the **objective functional** of the **incumbent** party.
- If the **voters** have a short collective memory and **are influenced more by the events occurring near election time**, then the v values of the later part of the period $[0, T]$ should be assigned heavier weights.

The political-business-cycle model

- **The political-business-cycle model**

Maximize $\quad V = \int_0^T v(U, p) e^{rt} dt$

$p = \phi(U) + a\pi$

(31) Subject to $\quad \dot{\pi} = b(p - \pi); \qquad (\boldsymbol{b > 0})$

$\pi(0) = \pi_0; \quad \pi(T) = free; \qquad (\pi_0, T\ given)$

- contains an equality constraint

(32) $p = \phi(U) + a\pi$ [**Augmented Phillips Curve**]

$\boldsymbol{\phi' < 0; \qquad 0 < a \leq 1}$

- $\boldsymbol{U}$ **is the unemployment** rate; $\boldsymbol{p}$ **is the inflation** rate; $\boldsymbol{\pi}$ **is expected inflation** rate.

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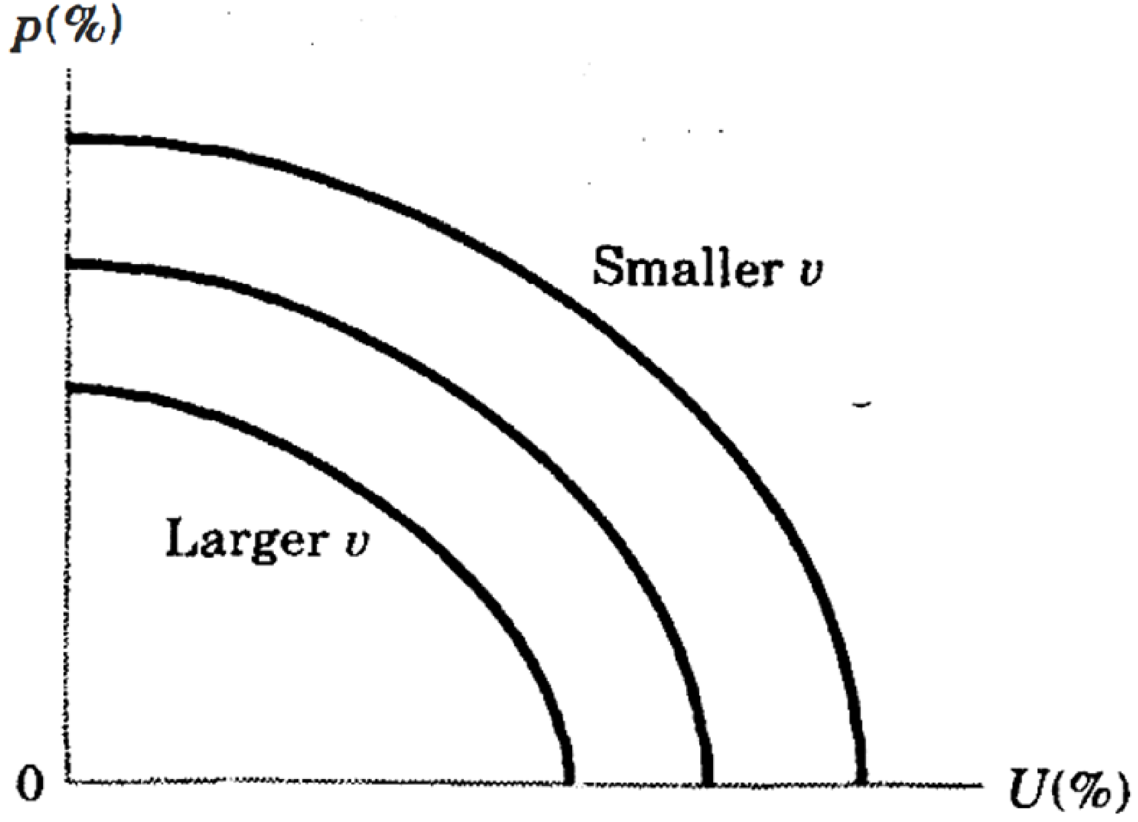
(33) $\boldsymbol{v(U,p);\ \ v_U < 0;\ v_p < 0}$

- v **is the aggregate vote function**; a measure of the vote-getting power **of the incumbent party**.
- $r > 0$ **denotes the rate of decay of memory**. It shows that the v **values at later points of time** are **weighted more heavily**.
- **Figure 1 captures the tradeoff between U and p.**
- U and p are both conducive to vote loss

Figure 1 – Isovote Curves



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- **Expectations are assumed to be formed adaptively**, according to the differential equation

  (34) $\dot{\pi} = b(p - \pi);$ $\qquad (\boldsymbol{b > 0})$

- For a variable to qualify as a **state variable**, it **must come with** a given **equation of motion** in the problem.
- Since (34) constitutes an equation of motion for π, we can take $\boldsymbol{\pi}$ **as a state variable**.
- The variable U, on the other hand, does not come with an equation of motion. But since $\boldsymbol{U}$ **can affect** $\boldsymbol{p}$ via (32) **and then** dynamically drive $\boldsymbol{\pi}$ via (34), we can use it as a **control variable**.

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- To use $\boldsymbol{U}$ **as a control variable**, however, requires the implicit assumption that the **government** in power does **have the ability to implement any target rate of unemployment** it chooses at any point of time.
- As to the remaining variable, p, (32) $\boldsymbol{p = \phi(U) + a\pi}$ shows **that its value** at any time t will **become determinate once** the values of **the state and control variables are known**.
- Now that $\boldsymbol{p}$ is retained in the model, it **ought to be taken as another control variable**. Thus **the constraint equation**

  (35) $p - \phi(U) - a\pi = 0$

- **is in line with the general format of** $\boldsymbol{g(t, y, u_1, u_2) = c}$, although there is no explicit t argument in it.

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- We can write **the Lagrangian**

(36) $\mathcal{L} = v(U,p)e^{rt} + \lambda b(p-\pi) + \theta[\phi(U) + a\pi - p]$

- If the following specific functions are adopted:

(37) $v(U,p) = -U^2 - hp;$ $\qquad (\boldsymbol{h > 0})$

(38) $\phi(U) = j - kU$ $\qquad (\boldsymbol{j, k > 0})$

- **Using these specific functions, the Lagrangian becomes**:

(39) $\mathcal{L} = (-U^2 - hp)e^{rt} + \lambda b(p-\pi) + \theta[j - kU + a\pi - p]$

- Accordingly, the maximum principle calls for the conditions

The political-business-cycle model

$$\boldsymbol{\mathcal{L} = \left(-U^2 - hp\right)e^{rt} + \lambda b(p - \pi) + \theta[j - kU + a\pi - p]}$$

(40) $\dfrac{\partial \mathcal{L}}{\partial U} = -2Ue^{rt} - \theta k = 0 \quad \Rightarrow \boldsymbol{U = -\frac{1}{2}\theta k e^{-rt}}$

(41) $\dfrac{\partial \mathcal{L}}{\partial p} = -he^{rt} + \lambda b - \theta = 0 \quad \Rightarrow \boldsymbol{\theta = \lambda b - he^{rt}}$

(42) $\dfrac{\partial \mathcal{L}}{\partial \theta} = j - kU + a\pi - p = 0$

(43) $\dot{\pi} = \dfrac{\partial \mathcal{L}}{\partial \lambda} = b(p - \pi)$

(44) $\dot{\lambda} = -\dfrac{\partial \mathcal{L}}{\partial \pi} = \lambda b - \theta a$

- By using (41) into (40):

(45) $U = -\frac{1}{2}(\lambda b - he^{rt})ke^{-rt} \quad \Rightarrow \boldsymbol{U = \frac{1}{2}k(h - \lambda b e^{-rt})}$

The optimal costate path.

- From (44) $\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial \pi} = \lambda b - \theta a$

  $\dot{\lambda} = \lambda b - \theta a = \lambda b - (\lambda b - he^{rt})a = \lambda b(1-a) + ahe^{rt}$

  (46) $\dot{\lambda} - b(1-a)\lambda = ahe^{rt}$
- **Equation (46)** is readily recognized as a **first-order linear differential equation** with a constant coefficient but a variable term.
- The general solution of (46) is:

  (47) $\lambda(t) = Ae^{b(1-a)t} + \frac{ah}{B}e^{rt}$
- Where $\boldsymbol{B = r - b(1-a)}$ and A is an arbitrary constant.

The optimal costate path.

- Note that the two constants A and B are fundamentally different in nature; $\boldsymbol{B}$ **is merely a shorthand symbol we have chosen in order to simplify the notation**, but $\boldsymbol{A}$ **is an arbitrary constant** to be definitized.
- To definitize A, we can make use of the transversality condition for the vertical-terminal-line problem, $\boldsymbol{\lambda(T) = 0}$.
- Letting $t = T$ in (47) $\lambda(T) = Ae^{b(1-a)T} + \frac{ah}{B}e^{rT}$, applying the transversality condition, and solving for A, we find that $\boldsymbol{A = -\frac{ah}{B}e^{BT}}$.
- It follows that the definite solution - **the optimal costate path - is**

(48) $\boldsymbol{\lambda^*(t)} = -\frac{ah}{B}e^{BT}e^{b(1-a)t} + \frac{ah}{B}e^{rt} = \boldsymbol{\frac{ah}{B}\left[e^{rt} - e^{BT+b(1-a)t}\right]}$

The optimal control path

- Now that we have found $\lambda^*(t)$, all it takes is to substitute (48) $\lambda^*(t) = \frac{ah}{B}\left[e^{rt} - e^{BT+b(1-a)t}\right]$ into (45) $U = \frac{1}{2}k(h - \lambda b e^{-rt})$ to derive the optimal control path.
- The result is

$$U^*(t) = \frac{1}{2}k\left\{h - \frac{ah}{B}\left[e^{rt} - e^{BT+b(1-a)t}\right]be^{-rt}\right\}$$

$$U^*(t) = \frac{1}{2}k\left\{h - \frac{ah}{B}\left[e^{rt}e^{-rt} - e^{BT+[b(1-a)-r]t}\right]b\right\}$$

$$U^*(t) = \frac{1}{2}\frac{hk}{B}\left\{B - ab\left[1 - e^{B(T-t)}\right]\right\}; \qquad \boldsymbol{B = r - b(1-a)}$$

(49) $$U^*(t) = \frac{hk}{2B}\left[(r-b) + abe^{B(T-t)}\right]$$

The optimal control path

- **Equation (49) is this control path that the incumbent party should follow in the interest of its reelection in year $\boldsymbol{T}$**.
- What are the economic implications of this path?
- First, we note that $\boldsymbol{U^*(t)}$ **is a decreasing function of $\boldsymbol{t}$**. Specifically, we have $\left\{U^*(t) = \frac{hk}{2B}\left[(r-b) + abe^{B(T-t)}\right]\right\}$

(50) $\frac{dU^*}{dt} = -\frac{1}{2}abhke^{B(T-t)} < 0$

- because $\boldsymbol{k, h, b, a}$ **and exponential expression are all positive**.
- The vote-maximizing economic policy is to set **a high unemployment level at $\boldsymbol{t = 0}$,** and then **let the rate of unemployment fall steadily throughout the electoral period $[\boldsymbol{0, T}]$**.

The optimal control path

- In fact, **the optimal levels of unemployment at time $\mathbf{0}$ and time $\boldsymbol{T}$ can be exactly determined**. They are $\left\{U^*(t) = \frac{hk}{2B}\left[(r-b) + abe^{B(T-t)}\right]\right\}$

$U^*(0) = \frac{hk}{2B}\left[(r-b) + abe^{BT}\right]$; and

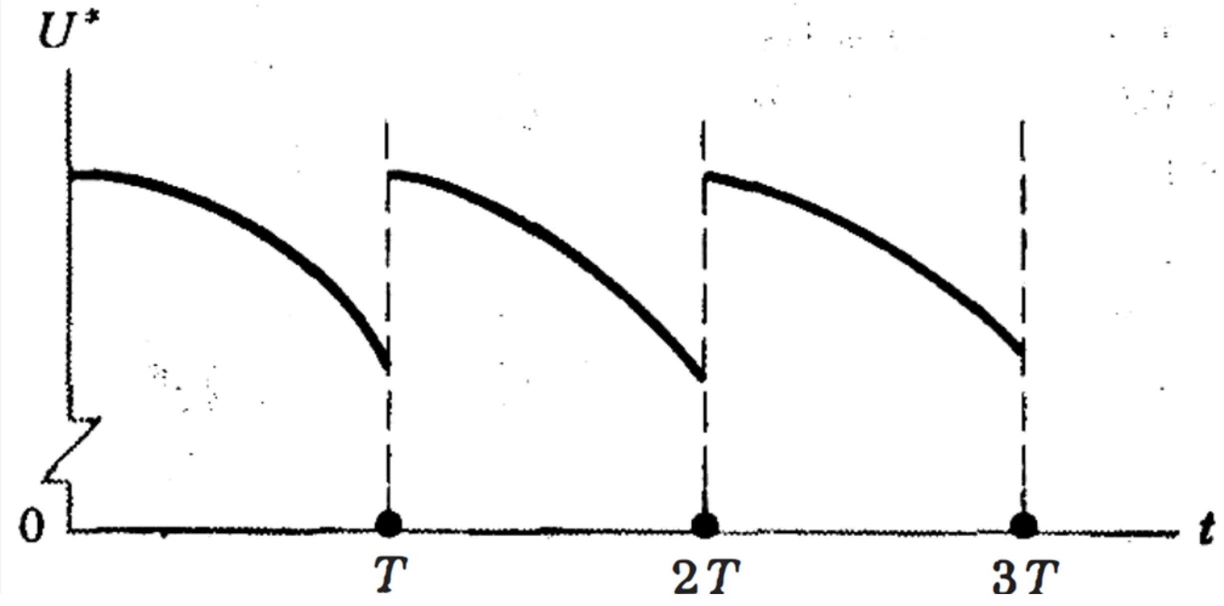
$U^*(T) = \frac{hk}{2B}\left[(r-b) + ab\right] \quad \Rightarrow U^*(T) = \frac{hk}{2}$

- Note that the terminal unemployment level, $\boldsymbol{hk/2}$**, is a positive quantity.**
- Since $\boldsymbol{U^*(T)}$ **represents the lowest point** on the $U^*(T)$ path, the $U^*(T)$ values at all values of t in $[0, T]$ **must** uniformly **be positive**.
- This means that **not imposing any restriction on $\boldsymbol{U}$ does not cause any trouble regarding the sign of $\boldsymbol{U}$** in the present case.

The optimal control path

- However, to be economically meaningful, $\boldsymbol{U^*(0)}$ **must be less than unity or**, more realistically, **less than some maximum** tolerable unemployment rate $\boldsymbol{U_{max} < 1}$.
- The typical optimal unemployment path, $\boldsymbol{U^*(t)}$**, is illustrated in Fig. 2**, where we also show the repetition of similar $U^*(t)$ patterns over successive electoral periods generates political business cycles.

Figure 2 – The Political Business Cycles



The optimal state path

- **The politically inspired cyclical tendency** in the control variable U **must also spill over to the state variable $\boldsymbol{\pi}$**, and hence also **to the actual rate of inflation $\boldsymbol{p}$**.
- The general pattern would be for the optimal rate of inflation to be relatively low at the beginning of each electoral period, **but undergo a steady climb**.
- In other words, **the time profile of $\boldsymbol{p}^*$ tends to be the opposite of that of $\boldsymbol{U}^*$**, since the Phillips Curve **depicts a trade-off between the two: $\boldsymbol{p = \phi(U) + a\pi; \phi' < 0; 0 < a \leq 1}$.**

Current-Value Hamiltonian and Lagrangian

- When **the constrained problem involves a discount factor**, **it is possible to use the current-value Hamiltonian $\boldsymbol{H_c}$ in lieu of $\boldsymbol{H}$**.
- In that case, the Lagrangian $\mathcal{L}$ **should be replaced by** the current-value Lagrangian $\boldsymbol{\mathcal{L}_c}$.
- Consider the inequality-constraint problem

	Maximize	$\mathrm{V} = \int_0^T \Phi(t, y, u) e^{-\rho t} dt$
(51)	Subject to	$\dot{y} = f(t, y, u)$
		$g(t, y, u) \leq c$
	and	boundary conditions

Current-Value Hamiltonian and Lagrangian

- **The regular Hamiltonian and Lagrangian are**

(52) $H = \Phi(t, y, u)e^{-\rho t} + \lambda(t) f(t, y, u)$

(52') $\mathcal{L} = \Phi(t, y, u)e^{-\rho t} + \lambda(t) f(t, y, u) + \theta(t)[c - g(t, y, u\,)]$

- And **the maximum principle calls for** (assuming interior solution):

(53) $\frac{\partial \mathcal{L}}{\partial u} = 0;$ for all $t \in [0, T]$

(54) $\frac{\partial \mathcal{L}}{\partial \theta} = c - g(t, y, u\,) \geq 0;$ $\theta \geq 0;$ $\theta \frac{\partial \mathcal{L}}{\partial \theta} = 0$

(55) $\dot{y} = \frac{\partial \mathcal{L}}{\partial \lambda}$ [equation of motion for y]

(56) $\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial y}$ [equation of motion for λ]

- Plus an appropriate transversality condition.

Current-Value Hamiltonian and Lagrangian

- By introducing new multipliers

(57) $m(t) = \lambda(t)e^{\rho t}$ [implying $\boldsymbol{\lambda(t) = m(t)e^{-\rho t}}$]

(57') $n(t) = \theta(t)e^{\rho t}$ [implying $\boldsymbol{\theta(t) = n(t)e^{-\rho t}}$]

- we can introduce the **current-value versions of $\boldsymbol{H}$ and $\boldsymbol{\mathcal{L}}$ as follows**:

(58) $H_c = He^{\rho t} = \Phi(t, y, u) + m(t)f(t, y, u)$

(58') $\mathcal{L}_c = \mathcal{L}e^{\rho t} = \Phi(t, y, u) + m(t)f(t, y, u) + n(t)[c - g(t, y, u\,)]$

- It can readily be verified that

(59) $\frac{\partial \mathcal{L}_c}{\partial u} = \frac{\partial \mathcal{L}}{\partial u}e^{\rho t}$; $\frac{\partial \mathcal{L}_c}{\partial n} = \frac{\partial \mathcal{L}}{\partial \theta}$; and $\frac{\partial \mathcal{L}_c}{\partial m} = \frac{\partial \mathcal{L}}{\partial \lambda}$

Current-Value Hamiltonian and Lagrangian

- Therefore, **conditions (53), (54), and (55) can be equivalently expressed with $\boldsymbol{\mathcal{L}_c}$**, and the new multipliers m and n as follows:

(60) $\frac{\partial \mathcal{L}_c}{\partial u} = 0;$ for all $t \in [0, T]$

(61) $\frac{\partial \mathcal{L}_c}{\partial n} \geq 0;$ $n \geq 0;$ $n \frac{\partial \mathcal{L}}{\partial n} = 0$

(62) $\dot{y} = \frac{\partial \mathcal{L}_c}{\partial m}$ [equation of motion for y]

- The only **major modification required when we use $\boldsymbol{\mathcal{L}_c}$ in the equation of motion for the costate variable, (56)**.
- To revise the equation of motion for the costate variable, **(56) $\boldsymbol{\dot{\lambda} = -\partial \mathcal{L} / \partial y}$**, we shall transform each side of this equation into an expression involving the new variable m.

Current-Value Hamiltonian and Lagrangian

- For the left-hand side, **by differentiating (57)** $\boldsymbol{m(t) = \lambda(t) e^{\rho t}}$:

(63) $\dot{m} = \dot{\lambda} e^{\rho t} + \rho \lambda e^{\rho t} = \dot{\lambda} e^{\rho t} + \rho m \quad \Rightarrow \boldsymbol{\dot{\lambda} = \dot{m} e^{-\rho t} - \rho m e^{-\rho t}}$

- Using the definition of $\mathcal{L}$ in (**58')** $\boldsymbol{\mathcal{L}_c = \mathcal{L} e^{\rho t}}$, we can **rewrite (56)** $\boldsymbol{\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial y}}$ **as**

(64) $\boldsymbol{\dot{\lambda}} = -\frac{\partial \mathcal{L}}{\partial y} = -\boldsymbol{\frac{\partial \mathcal{L}_c}{\partial y} e^{-\rho t}}$

- Equating (63) and (64):

(65) $-\frac{\partial \mathcal{L}_c}{\partial y} e^{-\rho t} = \dot{m} e^{-\rho t} - \rho m e^{-\rho t}$

(66) $\dot{m} = -\frac{\partial \mathcal{L}_c}{\partial y} + \rho m$ [**equation of motion for** $\boldsymbol{m}$]

Sufficient Conditions

- **The Mangasarian and Arrow sufficient conditions**, previously discussed in the context of unconstrained problems, turn out to be **valid also for constrained problems** when the terminal time T is fixed.
- Let us use the symbol $\boldsymbol{u}$ to represent the **vector of control variables**. As before, let $\boldsymbol{H}^{\mathbf{0}}$ denote the **maximized Hamiltonian**, the Hamiltonian evaluated **along the $\boldsymbol{u}^*(\boldsymbol{t})$ path**.
- The Hamiltonian is understood to **be maximized subject to** all the constraints of the $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{y}, \boldsymbol{u}) = \boldsymbol{c}$ form **or** the $\boldsymbol{g}(\boldsymbol{t}, \boldsymbol{y}, \boldsymbol{u}) \leq c$ form present in the problem.
- Besides, **since every integral constraint is reflected in $\boldsymbol{H}$ via the new costate variable $\boldsymbol{\mu}$**, it must also be similarly reflected in $\boldsymbol{H}^{\mathbf{0}}$.

Sufficient Conditions

- For simplicity, **we can consolidate the Mangasarian and Arrow sufficient conditions into a single statement**.
- The maximum-principle conditions are sufficient for the global maximization of the objective functional if:

(67) **Either the concavity of $\mathcal{L}$ is in $(\boldsymbol{y}, \boldsymbol{u})$, jointly, for all $\boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}]$; or Arrow's condition that $\boldsymbol{H}^{\mathbf{0}}$ is concave in the $\boldsymbol{y}$ variable alone for all $\boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}]$, for a given $\boldsymbol{\lambda}$.**

- These conditions **are also applicable to infinite-horizon problems**, but in this case, the $\boldsymbol{\lambda}$ **must satisfy**

(68) $\lim_{t \to \infty} \lambda(t)[y(t) - y^*(t)] \geq 0$

Sufficient Conditions

- A few comments about (67) may be added here.
- First, the concavity of $\mathcal{L}$ is in (y, u) means concavity in the variables y and u jointly.
- Second, **since $\boldsymbol{H}$ and $\boldsymbol{\mathcal{L}}$ are composed of the $\boldsymbol{F}$, $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{G}$ functions as follows**:

(69) $H = F + \lambda f - \mu G$

(70) $\mathcal{L} = H + \theta[c - g]$

- it is clear that **(67) will be satisfied if the following are simultaneously true**:

Sufficient Conditions

1) F is concave in (y, u)
2) λf is concave in (y, u)
3) μG is convex in (y, u)
4) and θg is convex in (y, u) for all $t \in [0, T]$

- **In the case of an inequality integral constraint**, however, **where μ is a nonnegative constant**, **the convexity of μG is ensured by the convexity of G** itself**.**
- Similarly, in the case of an inequality constraint, **where $\theta \geq 0$**, **the convexity of θg is ensured by the convexity of g** itself.
- Finally, if the current-value Hamiltonian and Lagrangian are used, (67) can be easily adapted by replacing $\mathcal{L}$ by $\mathcal{L}_c$ and H^0 by H_c^0.