Optimal Control with Constraints

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Optimal Control with Constraints

- In the present chapter, we turn **to constraints** that apply **throughout** the planning period $[\mathbf{0}, \boldsymbol{T}]$.
- **The treatment of constraints** in optimal control theory **relies** heavily **on the Lagrange-multiplier technique**.
- But since optimal control problems contain not only state variables, but also control variables, **it is necessary to distinguish between two major categories of constraints**.
- In **the first** category**, control variables are present in the constraints**, either with or without the state variables alongside.
- In **the second** category, **control variables are absent**, so that the **constraints only affect the state variables**.

Constraints Involving Control Variables

- Four basic types of constraints can be considered:

1. **Equality constraints.**
2. **Inequality constraints.**
3. **Equality integral constraints.**
4. **Inequality integral constraints.**

- We shall in general include the state variables alongside the control variables in the constraints.

Equality Constraints

- Let there be **two control variables** in a problem**,** $\boldsymbol{u_1}$ **and** $\boldsymbol{u_2}$, that are required to satisfy the condition

(1) $g(t, y, u_1, u_2) = c$

- We shall refer to the $\boldsymbol{g}$ function as **the constraint function**, and the constant $\boldsymbol{c}$ as **the constraint constant**.
- The control problem may then be stated as

	Maximize	$\mathrm{V} = \int_0^T F(t, y, u_1, u_2)dt$
(2)	Subject to	$\dot{y} = f(t, y, u_1, u_2)$
		$g(t, y, u_1, u_2) = c$
	and	boundary conditions

Equality Constraints

- This is a simple version of the problem with $\boldsymbol{m}$ **control variables and** $\boldsymbol{q}$ **equality constraints**, where **it is required** that $\boldsymbol{q} < \boldsymbol{m}$.
- **The maximum principle calls for the maximization of the Hamiltonian**

  (3) $H = F(t, y, u_1, u_2) + \lambda(t) f(t, y, u_1, u_2)$
- for all $t \in [0, T]$.
- But this time **the maximization of** H is subject to the constraint $g(t, y, u_1, u_2) = c$.
- Accordingly, **we form the Lagrangian expression**

  (4) $\mathcal{L} = H + \theta(t)[c - g(t, y, u_1, u_2)]$

  (4’) $\mathcal{L} = F(t, y, u_1, u_2) + \lambda(t) f(t, y, u_1, u_2) + \theta(t)[c - g(t, y, u_1, u_2)]$

Equality Constraints

- Where **the Lagrange multiplier $\boldsymbol{\theta}$ is made dynamic**, **as a function of $\boldsymbol{t}$**.
- This is necessary by the fact that the $\boldsymbol{g}$ constraint **must be satisfied at every $\boldsymbol{t}$** in the planning period.
- Assuming an interior solution for each u_j, we require that:

(5) $\dfrac{\partial \mathcal{L}}{\partial u_j} = \dfrac{\partial F}{\partial u_j} + \lambda \dfrac{\partial f}{\partial u_j} - \theta \dfrac{\partial g}{\partial u_j} = 0$ for all $t \in [0, T]$; $(j = 1,2)$

- Simultaneously, we must also set

(6) $\dfrac{\partial \mathcal{L}}{\partial \theta} = c - g(t, y, u_1, u_2) = 0$ for all $t \in [0, T]$

- to ensure that the **constraint will always be in force**.

Equality Constraints

- Together, (5) and (6) constitute the first-order condition for the constrained maximization of H.
- **The rest of the maximum-principle conditions includes**:

(7) $\dot{y} = \frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial H}{\partial \lambda}$ [equation of motion for y]

- And

(8) $\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial H}{\partial y} + \theta \frac{\partial g}{\partial y}$ [equation of motion for λ]

- plus an appropriate transversality condition.
- **Equation of motion for y is the same** whether we differentiate the Lagrangian or the original Hamiltonian function with respect to λ.

Equality Constraints

- On the other hand, **it would make a difference in the equation of motion for $\boldsymbol{\lambda}$,** (8), whether **we differentiate $\boldsymbol{\mathcal{L}}$ or $\boldsymbol{H}$ with respect to $\boldsymbol{y}$**.
- This is because, as the constraint in problem (2) specifically prescribes, the $\boldsymbol{y}$ **variable impinges upon the range of choice of the control variables**, and such effects must be taken into account in determining the path for the costate variable λ.
- While it is feasible to solve a problem with equality constraints in the manner outlined above, **it is usually simpler to use substitution to reduce the number of variables we have to deal with**.

Inequality Constraints

- We first remark that when the g **constraints are in the inequality form**, there is **no need to insist that the number of control variables exceed the number of constraints**.
- For simplicity, we shall illustrate this type of problem with **two control variables and two inequality constraints**:

	Maximize	$V = \int_0^T F(t, y, u_1, u_2)dt$
(9)	Subject to	$\dot{y} = f(t, y, u_1, u_2)$
		$g^1(t, y, u_1, u_2) \leq c_1$
		$g^2(t, y, u_1, u_2) \leq c_2$
	and	boundary conditions

Inequality Constraints

- **The Hamiltonian defined in (3) is still valid for the present problem**.
- But since the Hamiltonian is now to be maximized with respect to u_1 and u_2 subject to the two inequality constraints, **we need to invoke the Kuhn-Tucker conditions**.
- Besides, for these conditions to be necessary, **a constraint qualification** must be satisfied.
- According to a theorem of **Arrow, Hurwicz, and Uzawa**, **any of the following conditions will satisfy the constraint qualification**:

1) **All** the constraint functions $\boldsymbol{g^i}$ **are concave in the control variables** $\boldsymbol{u_j}$ [here, concave in (u_1, u_2)].

Inequality Constraints

2) **All** the constraint functions $\boldsymbol{g^i}$ **are linear in the control variables** $\boldsymbol{u_j}$ [here, concave in (u_1, u_2)] - a special case of (1).
3) **All** the constraint functions $\boldsymbol{g^i}$ **are convex in the control variables** $\boldsymbol{u_j}$. **In addition**, there **exists a point in** the control region $\boldsymbol{u_0} \in \boldsymbol{U}$ [here, u_0 is a point (u_{10}, u_{20})] such that, when evaluated at u_0, **all constraints** $\boldsymbol{g^i} < \boldsymbol{c}$ (That is, the constraint set has a nonempty interior.)
4) **The** $\boldsymbol{g^i}$ **functions satisfy the rank condition**: Taking only **those constraints** that turn out to be effective or **binding** (satisfied as strict equalities), **form the matrix of partial derivative** $\left[\partial \boldsymbol{g^i} / \partial \boldsymbol{u_j}\right]_{\boldsymbol{e}}$(where e indicates "effective constraints only"), and evaluate the partial derivatives at the optimal values of the y and u variables.

Inequality Constraints

4) The rank condition is that the **rank of this matrix be equal to the number of effective constraints**.

- We now augment the Hamiltonian into a Lagrangian function:

(10) $$\mathcal{L} = F(t, y, u_1, u_2) + \lambda(t) f(t, y, u_1, u_2) + \theta_1(t)[c_1 - g^1(t, y, u_1, u_2)] + \theta_2(t)[c_2 - g^2(t, y, u_1, u_2)]$$

- The essence of $\mathcal{L}$ may become more transparent if we suppress all the arguments and simply write

(10') $$\mathcal{L} = F + \lambda f + \theta_1[c_1 - g^1] + \theta_2[c_2 - g^2]$$

- The first-order condition for maximizing $\mathcal{L}$, assuming interior solutions

(11) $$\frac{\partial \mathcal{L}}{\partial u_j} = 0$$

Inequality Constraints

- **as well as**

(12) $\frac{\partial \mathcal{L}}{\partial \theta_i} = c_i - g^i \geq 0; \qquad \theta_i \geq 0; \qquad \theta_i \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$

- for all $t \in [0, T]$; $(i = 1, 2 \ and \ j = 1, 2)$.
- Condition (12) differs from (6) because the constraints in the present problem are inequalities.
- **The $\boldsymbol{\partial \mathcal{L}/\partial \theta_i \geq 0}$ condition merely restates the $\boldsymbol{ith}$ constraint.**
- The complementary-slackness condition $\boldsymbol{\theta_i(\partial \mathcal{L}/\partial \theta_i) = 0}$ ensures that those terms **in (10) $\boldsymbol{\theta_i[c_i - g^i]}$ will vanish in the solution**, so that the value of $\boldsymbol{\mathcal{L}}$ **will be identical with that of $\boldsymbol{H = F + \lambda f}$ after maximization.**

Inequality Constraints

- **If the latter problem contains additional nonnegativity restrictions**

$$u_j \geq 0$$

- then, **by the Kuhn-Tucker conditions, we should replace the $\boldsymbol{\partial \mathcal{L} / \partial u_j = 0}$ conditions in (11) with**

(13) $\dfrac{\partial \mathcal{L}}{\partial u_j} \leq 0; \qquad u_j \geq 0; \qquad u_j \dfrac{\partial \mathcal{L}}{\partial u_j} = 0$

- It should be pointed out that the symbol $\mathcal{L}$ in (13) denotes the same Lagrangian as defined in (10), without separate $\theta(t)$ type of multiplier terms appended on account of the additional constraints $u_j(t) \geq 0$.
- **Other maximum-principle conditions include the equations of motion for $\boldsymbol{y}$ and $\boldsymbol{\lambda}$.** These are the same as in (7) and (8):

Inequality Constraints

(14) $\dot{y} = \frac{\partial \mathcal{L}}{\partial \lambda}$; and $\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial y}$ [equations of motion for y and λ]

- Wherever appropriate, of course, transversality conditions must be added, too.

Isoperimetric Problem

- **When an equality integral constraint is present**, **the control problem is known as an isoperimetric problem**.
- **Two features** of such a problem **are worth noting:**

1. **The costate variable associated with the integral constraint is**, as in the calculus of variations, **constant over time.**
2. **Although the constraint is in the nature of a strict equality**, the integral aspect of it obviates the **need to restrict the number of constraints relative to the number of control variables**.

- We shall illustrate the solution method with a problem that contains one state variable, one control variable, and one integral constraint:

Isoperimetric Problem

	Maximize	$V = \int_0^T F(t, y, u)dt$	
(15)	Subject to	$\dot{y} = f(t, y, u)$	
		$\int_0^T G(t, y, u)dt = k;$	$(k\ given)$
	and	$y(0) = y_0;\ \ y(T) = free;$	$(y_0, T\ given).$

- **The approach** to be used here **is to introduce a new state variable $\boldsymbol{\Gamma(t)}$** into the problem such that the integral constraint can be replaced by a condition in terms of $\Gamma(t)$. To this end, **let us define**

(16) $\Gamma(t) = -\int_0^t G(t, y, u)dt$

- Where **the upper limit of integration is $\boldsymbol{t}$** not the terminal time T.

Isoperimetric Problem

- **The derivative** of this variable $\left[\Gamma(t) = -\int_0^t G(t,y,u)dt\right]$ is

(17) $\dot{\Gamma}(t) = -G(t,y,u)dt$ [equation of motion for $\Gamma(t)$]

- and the initial and terminal values of $\Gamma(t)$ in the planning period are

(18) $\Gamma(0) = -\int_0^0 G(t,y,u)dt = 0$

- And

(19) $\Gamma(T) = -\int_0^T G(t,y,u)dt = -k$ from (15)

- From (19), it is clear that **we can replace the given integral constraint by a terminal condition on the $\boldsymbol{\Gamma(t)}$ variable**.

Isoperimetric Problem

- **By incorporating $\boldsymbol{\Gamma(t)}$ into the problem** as a new state variable, we can restate (15) as

$$
\begin{array}{lll}
 & \text{Maximize} & V = \int_0^T F(t, y, u)dt \\
(20) & \text{Subject to} & \dot{y} = f(t, y, u) \\
 & & \dot{\Gamma}(t) = -G(t, y, u)dt \\
 & \text{and} & y(0) = y_0; \ \ y(T) = free; \qquad (y_0, T \ given) \\
 & & \Gamma(0) = 0; \quad \Gamma(T) = -k; \qquad (k \ given)
\end{array}
$$

- **This new problem is an unconstrained problem with two state variables, $\boldsymbol{y}$ and $\boldsymbol{\Gamma}$.**

Isoperimetric Problem

- While the y variable has a vertical terminal line, **the new Γ variable has a fixed terminal point**.
- Inasmuch **as this problem is now an unconstrained problem**, **we can work with the Hamiltonian** without first expanding it into a Lagrangian function.
- Defining the Hamiltonian as

(21) $H = F(t, y, u) + \lambda(t) f(t, y, u) - \mu G(t, y, u)$

- we have the **following conditions from the maximum principle**:

$$\underset{u}{\text{Max}}\, H(t, y, u, \lambda, \mu) \qquad \text{for all } t \in [0, T]$$

Isoperimetric Problem

(22) Subject to

$$\dot{y} = \frac{\partial H}{\partial \lambda}$$ [equation of motion for y]

$$\dot{\lambda} = -\frac{\partial H}{\partial y}$$ [equation of motion for λ]

$$\dot{\Gamma} = \frac{\partial H}{\partial \mu}$$ [equation of motion for Γ]

$$\dot{\mu} = -\frac{\partial H}{\partial \Gamma}$$ [equation of motion for μ]

$$\lambda(T) = 0$$ [transversality condition]

- **What distinguishes (22)** from the conditions for the usual unconstrained problem **is the presence of the pair of equations of motion for $\boldsymbol{\Gamma}$ and $\boldsymbol{\mu}$**.

Isoperimetric Problem

- **Since the $\boldsymbol{\Gamma}$ variable** is an artifact whose mission is **only** to **guide** us **to add the $\boldsymbol{\mu G(t, y, u)}$ term to the Hamiltonian**, we can **safely omit its equation of motion** from (22) at no loss.
- On the other hand, **the equation of motion for $\boldsymbol{\mu}$** does impart **a significant piece of information**.
- Since the $\boldsymbol{\Gamma}$ **does not appear in the Hamiltonian**, it follows that

(23) $\dot{\mu} = -\frac{\partial H}{\partial \Gamma} = 0 \qquad \Rightarrow \qquad u(t) = constant$

- **The costate variable associated with the integral constraint is constant over time**.

Inequality Integral Constraint

- Consider the case where **the integral constraint enters** the problem **as an inequality**

	Maximize	$V = \int_0^T F(t, y, u)dt$	
(24)	Subject to	$\dot{y} = f(t, y, u)$	
		$\int_0^T G(t, y, u)dt \leq k;$	$(k\ given)$
	and	$y(0) = y_0;\ \ y(T) = free;$	$(y_0, T\ given)$

- Define a new state variable $\boldsymbol{\Gamma(t)}$ **the same as in (16)**:

$$\Gamma(t) = -\int_0^t G(t, y, u)dt$$

Inequality Integral Constraint

- The **derivative of $\boldsymbol{\Gamma(t) = -\int_0^t G(t, y, u)dt}$ is simpl**y

  (25) $\dot{\Gamma}(t) = -\,G(t, y, u)dt$ [equation of motion for $\Gamma(t)$]

- and its initial and terminal values are [**(24)** $\int_0^T G(t, y, u)dt \leq k$]

  (26) $\boldsymbol{\Gamma(0)} = -\int_0^0 G(t, y, u)dt = \mathbf{0}$ and

  $\boldsymbol{\Gamma(T)} = -\int_0^T G(t, y, u)dt \geq -\boldsymbol{k}$ [**by (24)**]

- **Using (25) and (26), we can restate problem (24) as**

Inequality Integral Constraint

	Maximize	$\mathrm{V} = \int_0^T F(t, y, u)dt$	
(27)	Subject to	$\dot{y} = f(t, y, u)$	
		$\dot{\Gamma}(t) = -\, G(t, y, u)dt$	
	and	$y(0) = y_0;\;\; y(T) = free;$	$(y_0, T\ given)$
		$\Gamma(0) = 0;\quad \boldsymbol{\Gamma(T) \geq -k};$	$(k\ given)$

- Like the problem in (20), **this is an unconstrained problem with two state variables**.
- But, unlike (20), the new variable Γ in (27) has a truncated vertical terminal line.

Inequality Integral Constraint

- The Hamiltonian of problem (27) is simply

(28) $H = F(t, y, u) + \lambda(t) f(t, y, u) - \mu G(t, y, u)$ [**same as (21)**]

- If the constraint qualification is satisfied, then **the maximum principle requires**

$$\underset{u}{\text{Max}}\, H(t, y, u, \lambda, \mu) \qquad \text{for all } t \in [0, T]$$

(29) Subject to

$$\dot{y} = \frac{\partial H}{\partial \lambda} \qquad \text{[equation of motion for } y\text{]}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \qquad \text{[equation of motion for } \lambda\text{]}$$

$$\dot{\Gamma} = \frac{\partial H}{\partial \mu} \qquad \text{[equation of motion for } \Gamma\text{]}$$

$$\dot{\mu} = -\frac{\partial H}{\partial \Gamma} \qquad \text{[equation of motion for } \mu\text{]}$$

Inequality Integral Constraint

and to $\lambda(T) = 0$ [transversality condition for y]

$$\mu(T) \geq 0; \quad [\Gamma(T) + k] \geq 0; \quad \mu(T)[\Gamma(T) + k] = 0$$

[**transversality condition for $\boldsymbol{\Gamma}$**]

- Note, again, that **because the Hamiltonian is independent of $\boldsymbol{\Gamma(T)}$**, we have

(30) $\dot{\mu} = -\frac{\partial H}{\partial \Gamma} = 0 \qquad \Rightarrow \qquad \mu(t) = constant$

- Therefore, **the multiplier associated with any integral constraint**, whether equality or inequality, **is constant over time**.
- As in (22), **we can omit from (29) the equations of motion for $\boldsymbol{\Gamma}$ and $\boldsymbol{\mu}$**, as long as we bear in mind that μ is a nonnegative constant.

Inequality Integral Constraint

- In sum, **the conditions in (29) can be restated without reference to $\boldsymbol{\Gamma}$ as** follows:

$$\underset{u}{\text{Max}}\, H(t, y, u, \lambda, \mu) \qquad \text{for all } t \in [0, T]$$

(29') Subject to

$$\dot{y} = \frac{\partial H}{\partial \lambda} \qquad \text{[equation of motion for } y\text{]}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \qquad \text{[equation of motion for } \lambda\text{]}$$

$$\dot{\mu} = -\frac{\partial H}{\partial \Gamma} \qquad \text{[equation of motion for } \mu\text{]}$$

$$\mu = constant \geq 0; k - \int_0^T G(t, y, u)dt \geq 0$$

and

$$\mu\left[k - \int_0^T G(t, y, u)dt\right] = 0 \qquad [\textbf{by 26}]$$