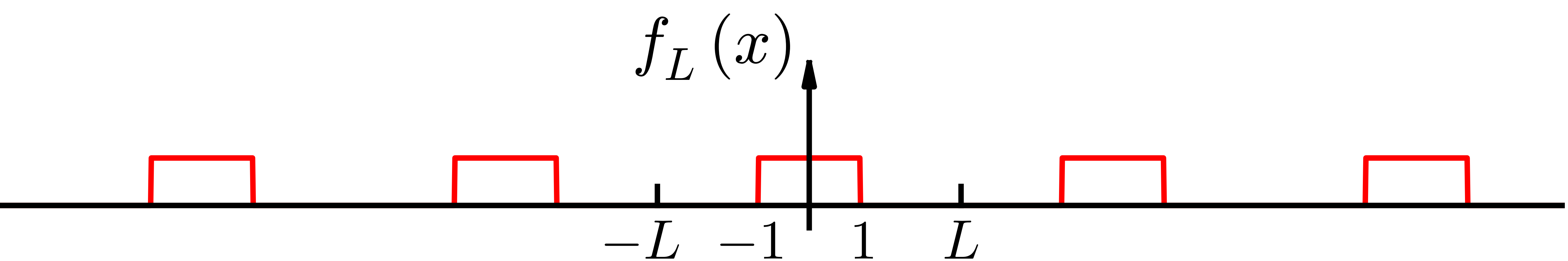
Victor Sacek IAG/USP

Integral de Fourier

Onda retangular

$$f_L(x) = \begin{cases} 0 & \text{se} \quad -L < x < -1 \\ 1 & \text{se} \quad -1 < x < 1 \\ 0 & \text{se} \quad 1 < x < L \end{cases} \qquad p = 2L$$



Onda retangular

$$f_L(x) = \begin{cases} 0 & \text{se} & -L < x < -1 \\ 1 & \text{se} & -1 < x < 1 \\ 0 & \text{se} & 1 < x < L \end{cases} \qquad p = 2L$$

Série de Fourier

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nux + b_n \sin nux \qquad u = \frac{2\pi}{p} = \frac{\pi}{L}$$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x) \qquad w_n = \frac{n\pi}{L}$$

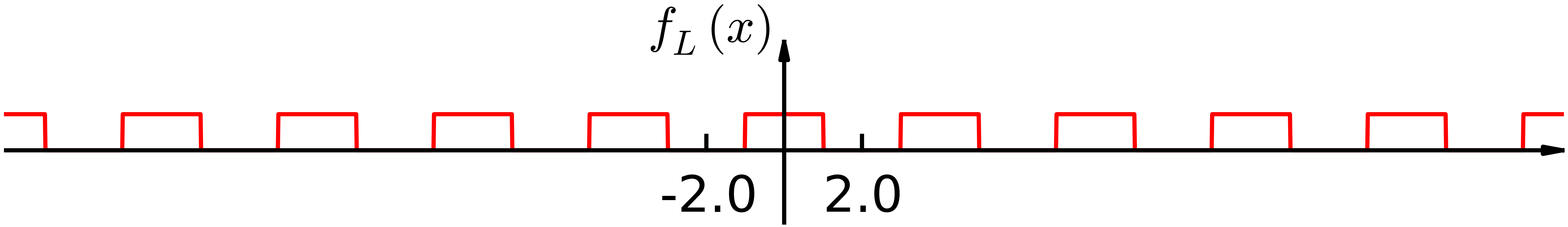
Onda retangular

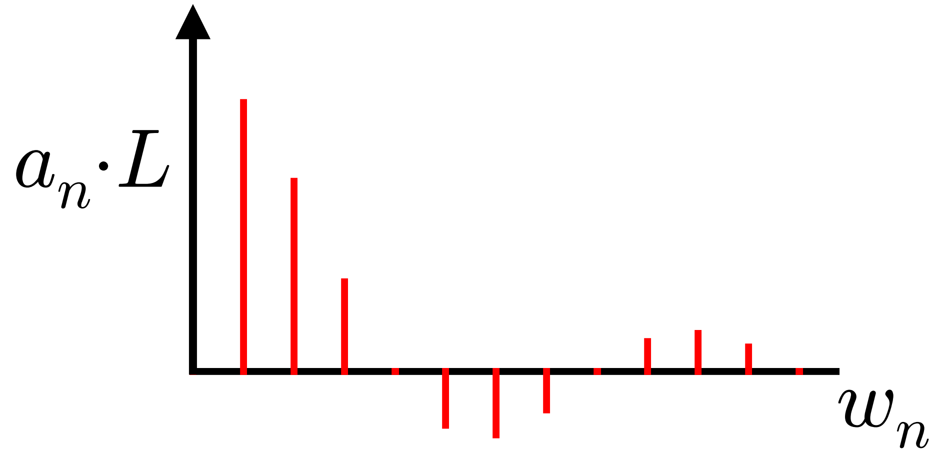
$$f_L(x) = \begin{cases} 0 & \text{se} \quad -L < x < -1 \\ 1 & \text{se} \quad -1 < x < 1 \\ 0 & \text{se} \quad 1 < x < L \end{cases} \qquad p = 2L$$

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

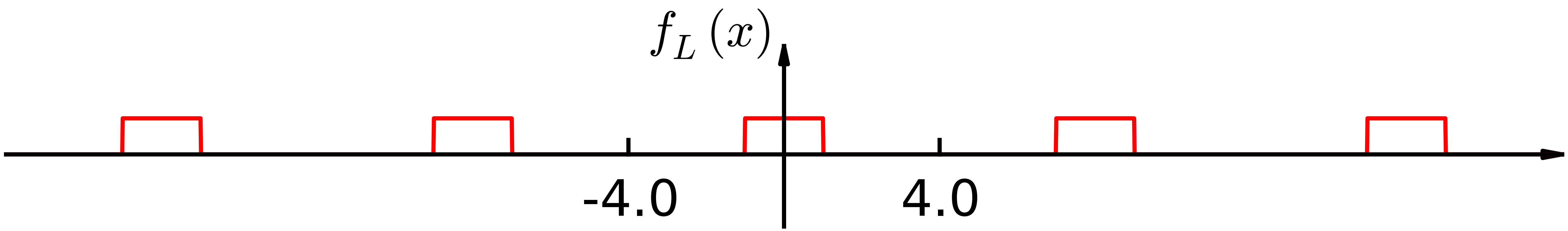
$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L} dx \quad = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L}$$

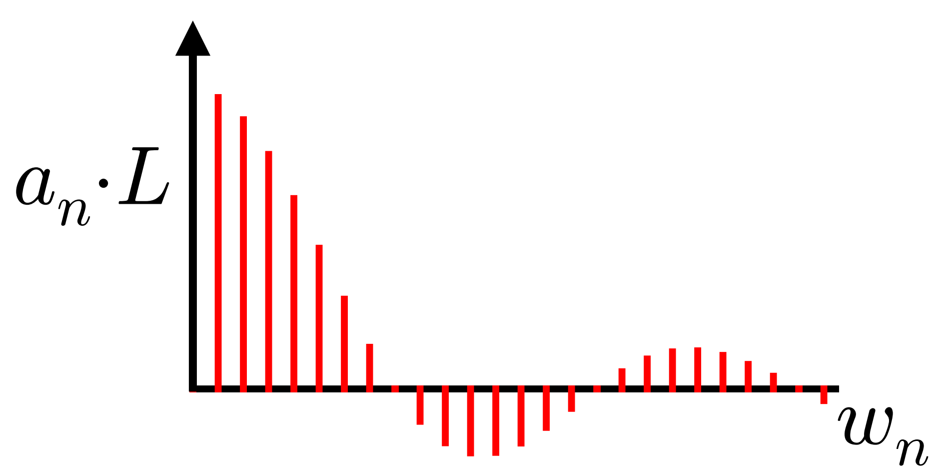
$$L = 2 \qquad w_n = \frac{n\pi}{L} = \frac{n\pi}{2} \qquad \Delta w = w_{n+1} - w_n = \frac{\pi}{2}$$



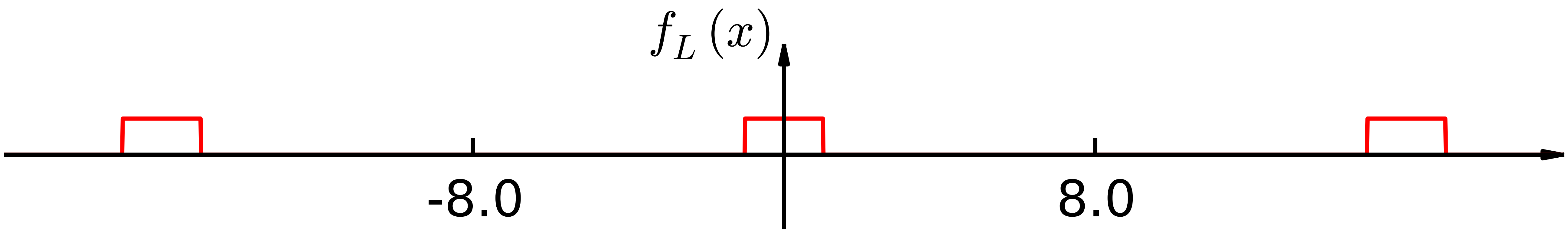


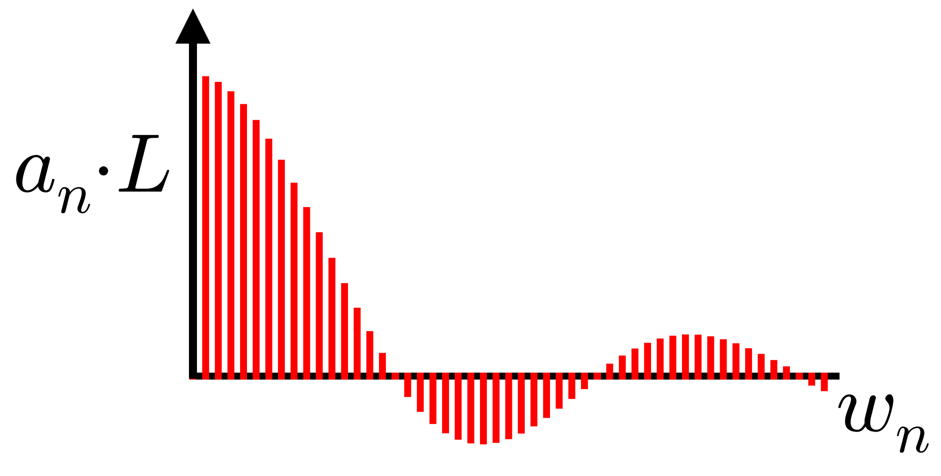
$$L = 4 \qquad\qquad w_n = \frac{n\pi}{L} = \frac{n\pi}{4} \qquad\qquad \Delta w = w_{n+1} - w_n = \frac{\pi}{4}$$





$$L = 8 \qquad w_n = \frac{n\pi}{L} = \frac{n\pi}{8} \qquad \Delta w = w_{n+1} - w_n = \frac{\pi}{8}$$





Onda retangular

$$f_L(x) = \begin{cases} 0 & \text{se} \quad -L < x < -1 \\ 1 & \text{se} \quad -1 < x < 1 \\ 0 & \text{se} \quad 1 < x < L \end{cases}$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{caso contrário} \end{cases}$$

Função periódica qualquer de período *2L*

Série de Fourier: $$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x)$$

$$w_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv \right.$$

$$\left. + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \qquad \frac{1}{L} = \frac{\Delta w}{\pi}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv \right.$$

$$\left. + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv \right.$$

$$\left. + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg[(\cos w_n x)\Delta w\int_{-L}^{L} f_L(v)\cos w_n v dv$$

$$+ (\sin w_n x)\Delta w\int_{-L}^{L} f_L(v)\sin w_n v dv\Bigg]$$

$$L \to \infty \qquad 1/L \to 0 \qquad \Delta w = \pi/L \to 0$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\Bigg[\cos wx\int_{-\infty}^{\infty} f(v)\cos wv dv$$

$$+ \sin wx\int_{-\infty}^{\infty} f(v)\sin wv dv\Bigg]dw$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv dv\right] dw$$

$$\boxed{A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv}$$

$$\boxed{B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv}$$

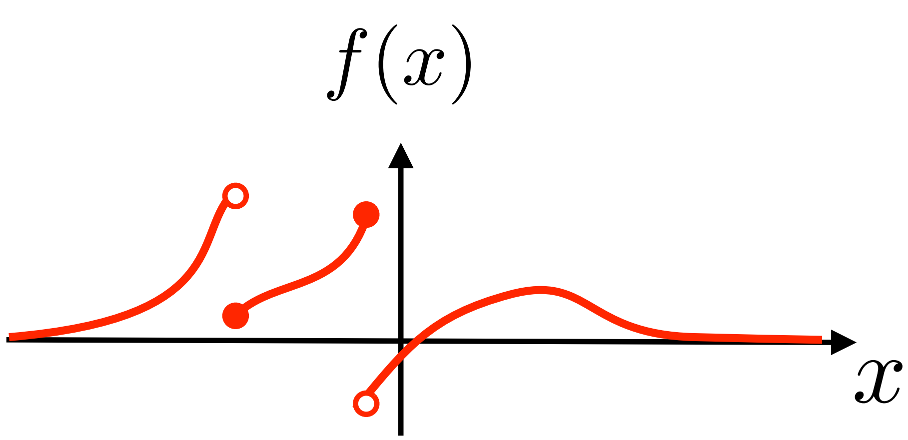
$$\boxed{f(x) = \int_0^{\infty}[A(w)\cos wx + B(w)\sin wx]\, dw}$$

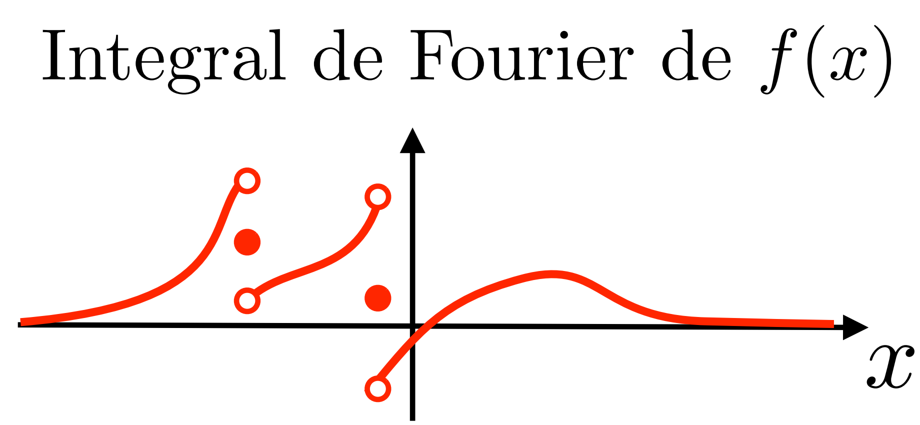
Integral de Fourier

Teorema

Se $f(x)$ é contínua por partes em um número finito de intervalos, tem derivada "pela direita" e "pela esquerda" em cada ponto e $\int_{-\infty}^{\infty} |f(x)| dx$ existe, então $f(x)$ pode ser representada pela integral de Fourier.

Em pontos onde $f(x)$ é descontínua, o valor da integral de Fourier é igual ao valor médio dos limites esquerdos e direitos de $f(x)$ nesses pontos.





Série de Fourier

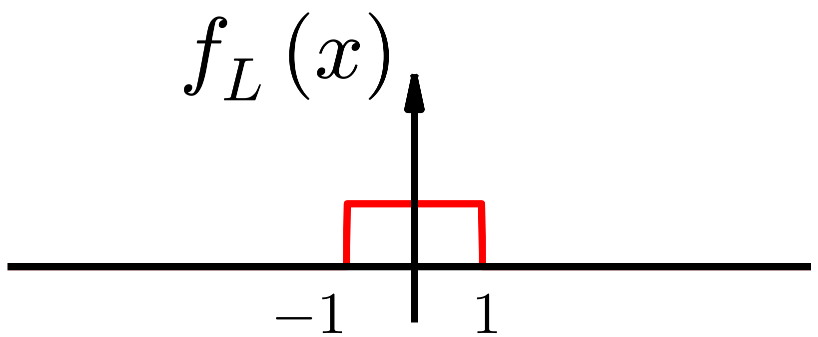
$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x)$$

Integral de Fourier

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] \, dw$$

Exemplo

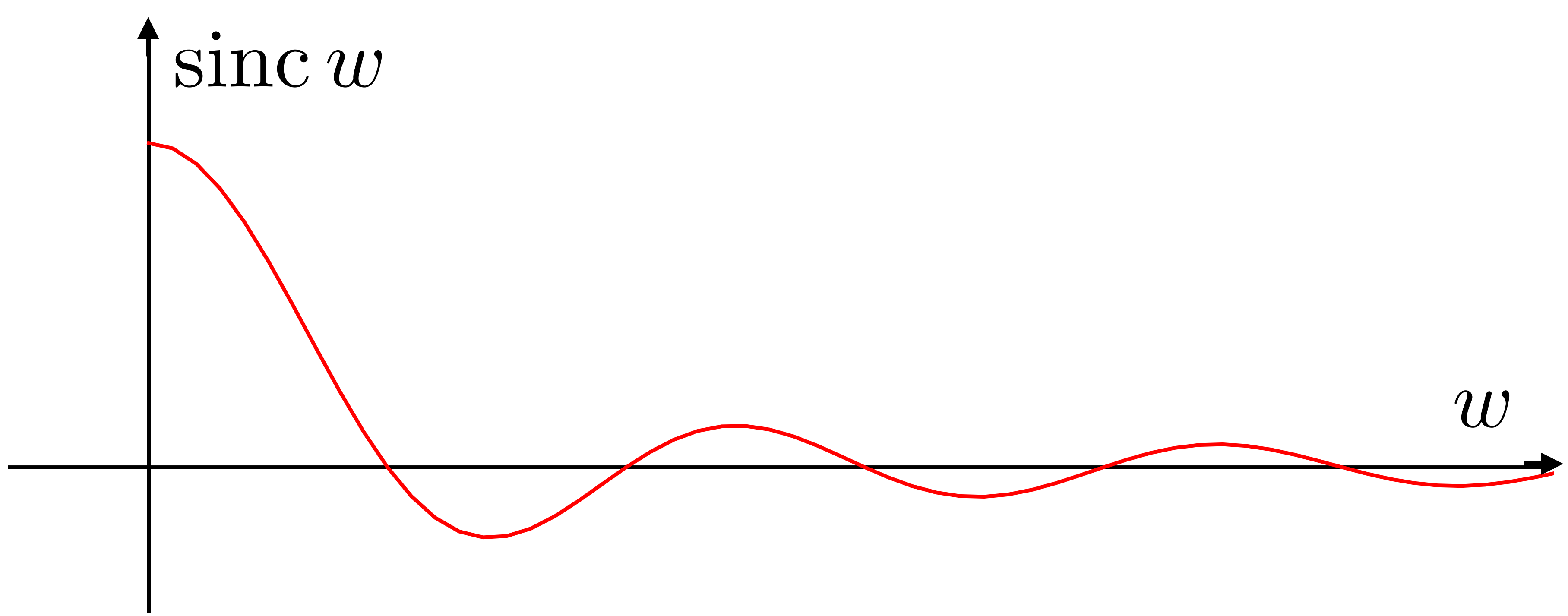
$$f(x) = \begin{cases} 1 & \text{se } |x| < 1 \\ 0 & \text{se } |x| > 1 \end{cases}$$



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv = \frac{1}{\pi}\int_{-1}^{1} \cos wv\,dv = \left.\frac{\sin wv}{\pi w}\right|_{-1}^{1}$$

$$= \frac{2\sin w}{\pi w}$$

$$= \frac{2}{\pi}\ \mathrm{sinc}\, w$$



Exemplo

$$f(x) = \begin{cases} 1 & \text{se } |x| < 1 \\ 0 & \text{se } |x| > 1 \end{cases}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv = \frac{1}{\pi}\int_{-1}^{1} \sin wv dv = 0$$

Exemplo

$$A(w) = \frac{2\sin w}{\pi w} \qquad\qquad B(w) = 0$$

$$f(x) = \int_0^\infty \left[A(w)\cos wx + B(w)\sin wx\right] dw$$

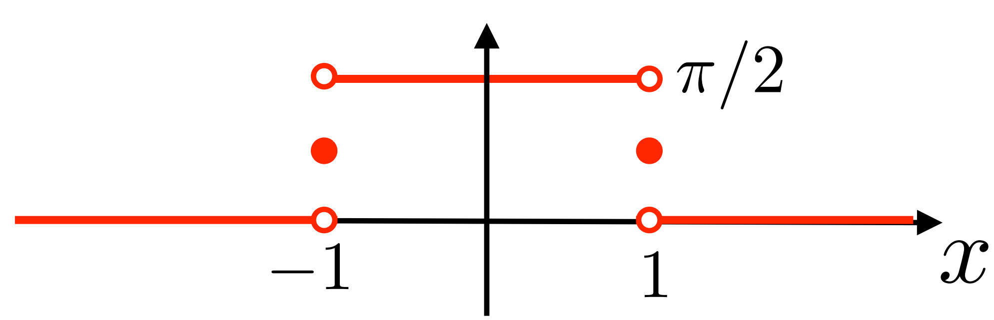
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw$$

Exemplo

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w} dw$$

Mas $$f(x) = \begin{cases} 1 & \text{se } |x| < 1 \\ 0 & \text{se } |x| > 1 \end{cases}$$

Então $$\int_0^\infty \frac{\cos wx \sin w}{w} dw = \begin{cases} \pi/2 & \text{se} & |x| < 1 \\ \pi/4 & \text{se} & |x| = 1 \\ 0 & \text{se} & |x| > 1 \end{cases}$$



Integral de Fourier

$$f(x) = \int_0^\infty \left[A(w)\cos wx + B(w)\sin wx\right] dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

Substituindo os coeficientes na integral de Fourier

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\left[\cos wv\cos wx + \sin wv\sin wx\right] dvdw$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wv - wx)dv\right] dw$$

Integral de Fourier

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} \underbrace{f(v)}_{\substack{\text{não depende}\\ \text{de } w}} \underbrace{\cos(wv - wx)}_{\substack{\text{função par}\\ \text{de } w}} dv\right]}_{\substack{\text{função par}\\ \text{de } w}} dw$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\cos(wv - wx)dv\right] dw$$

$$\cos(wv - wx) = \cos(wx - wv)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right] dw$$

Integral de Fourier

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right] dw$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} i\,f(v)\underbrace{\sin(wx - wv)}_{\substack{\text{função ímpar}\\ \text{de } w}}dv\right] dw = 0$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\left[\cos(wx - wv) + i\sin(wx - wv)\right]dv\right] dw$$

$$\boxed{e^{it} = \cos t + i\sin t}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{i(wx - wv)}dv\right] dw$$

Integral de Fourier

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{i(wx-wv)}dv\right]dw$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}dv\right]}_{\hat{f}(w)}e^{iwx}dw$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}dw$$

Integral de Fourier

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

Transformada de Fourier de f

$$\hat{f} = \mathcal{F}(f)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

Transformada de Fourier Inversa de $\hat{f}$

$$f = \mathcal{F}^{-1}(\hat{f})$$

Exemplo 1

Qual é transformada de Fourier de $f(x) = \begin{cases} 1 & \text{se } |x| < 1 \\ 0 & \text{se } |x| > 1 \end{cases}$?

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left. \frac{e^{-iwx}}{-iw} \right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} \overbrace{\left(e^{-iw} - e^{iw}\right)}^{-2i \sin w}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

Exemplo 11

Qual é transformada de Fourier de $f(x) = \begin{cases} e^{-ax} & \text{se} \quad x, a > 0 \\ 0 & \text{se} \quad x < 0 \end{cases}$?

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+iw)x}}{-(a+iw)} \right|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a+iw)}$$

Interpretação física da transformada de Fourier: espectro

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

$\hat{f}(w)$ mede a intensidade de $f(x)$ no intervalo de frequência entre w e $w + \Delta w$

Linearidade da Transformada de Fourier

$$\boxed{\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)}$$

$$\mathcal{F}\{af(x) + bg(x)\} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{-iwx} dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-iwx} dx$$

$$= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

Transformada de Fourier da derivada de *f(x)*

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancelto{0}{f(x)e^{-iwx}\Big|_{-\infty}^{\infty}} - (-iw) \underbrace{\int_{-\infty}^{\infty} f(x) e^{-iwx} dx}_{\mathcal{F}\{f(x)\}} \right]$$

$$\boxed{\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}}$$

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Transformada de Fourier da derivada de *f(x)*

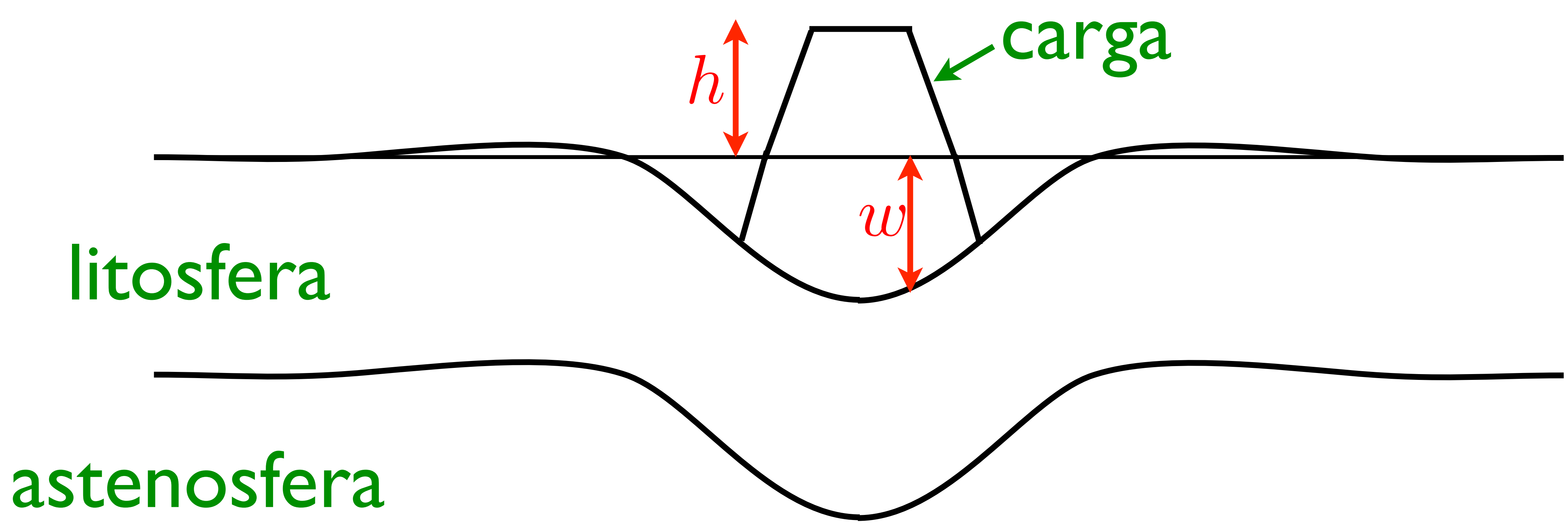
$$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f''(x)\} = (iw)^2\mathcal{F}\{f(x)\} = -w^2\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f'''(x)\} = (iw)^3\mathcal{F}\{f(x)\} = -iw^3\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f''''(x)\} = (iw)^4\mathcal{F}\{f(x)\} = w^4\mathcal{F}\{f(x)\}$$

Aplicação:
Flexura da litosfera



Deslocamento Vertical

Gravidade

Topografia

$$D\frac{d^4w}{dx^4} + (\rho_m - \rho_s)gw = \rho_c g h(x)$$

Rigidez flexural

Dens. do Manto

Dens. do Sedimento

Dens. do Crosta

Aplicação: Flexura da litosfera

$$D\frac{d^4w}{dx^4} + (\rho_m - \rho_s)gw = \rho_c gh(x)$$

$$\mathcal{F}\{D\frac{d^4w}{dx^4} + (\rho_m - \rho_s)gw\} = \mathcal{F}\{\rho_c gh(x)\}$$

$$D\mathcal{F}\{\frac{d^4w}{dx^4}\} + (\rho_m - \rho_s)g\mathcal{F}\{w\} = \rho_c g\mathcal{F}\{h(x)\}$$

Aplicação: Flexura da litosfera

$$D\mathcal{F}\{\frac{d^4w(x)}{dx^4}\} + (\rho_m - \rho_s)g\mathcal{F}\{w(x)\} = \rho_c g\mathcal{F}\{h(x)\}$$

$$D(-ik)^4W(k) + (\rho_m - \rho_s)gW(k) = \rho_c gH(k)$$

$$\boxed{W(k) = \frac{\rho_c gH(k)}{Dk^4 + (\rho_m - \rho_s)g}}$$

Transformada de Fourier de funções reais

$f(x)$ é função real de variável real

$$\Big\downarrow \mathcal{F}$$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$F(w)$ é função complexa de variável real

$$\Big\downarrow \mathcal{F}^{-1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{iwx} dw$$

Se x é o tempo,
w é a frequência angular

Se x é uma coordenada espacial,
w é o número de onda

Transformada de Fourier de funções reais

Se $f(x)$ é real, então sua transformada $F(w)$ tem a seguinte propriedade:

$$F(-w) = \bar{F}(w)$$

$$X(-w) + iY(-w) = X(w) - iY(w)$$

$X(-w) = X(w)$ é par

$Y(-w) = -Y(w)$ é ímpar

$$\boxed{\begin{aligned} z &= a + bi \\ \bar{z} &= a - bi \end{aligned}}$$

Transformada de Fourier de funções reais

Se $f(x)$ é real e par, então sua transformada $F(w)$ é real e par também

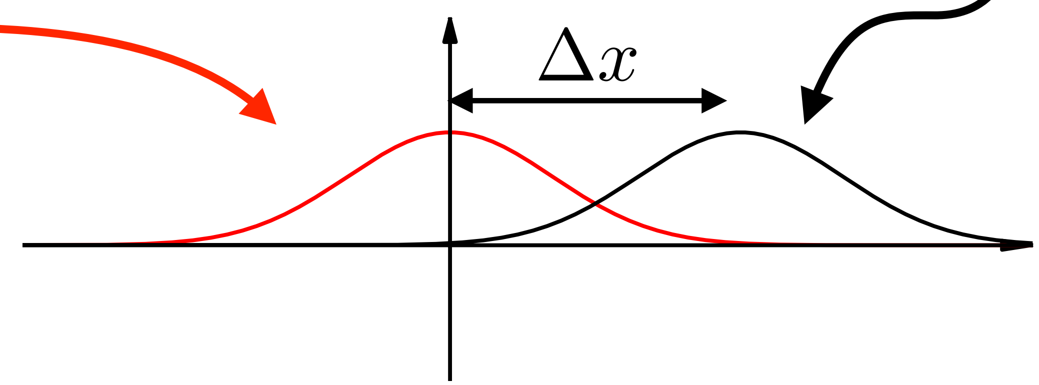
$$F(w) = X(w) + iY(w) = X(w)$$

Se $f(x)$ é real e ímpar, então sua transformada $F(w)$ é imaginária e ímpar

$$F(w) = X(w) + iY(w) = iY(w)$$

Atraso em x

$$f(x) \xrightarrow{\mathcal{F}} F(w) \qquad\qquad f(x-\Delta x) \xrightarrow{\mathcal{F}} ?$$



Mudança de variável

$$x - \Delta x \to \lambda$$
$$x \to \lambda + \Delta x$$
$$dx \to d\lambda$$

$$\mathcal{F}\{f(x-\Delta x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\Delta x)e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)e^{-iw(\lambda+\Delta x)}d\lambda$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)e^{-iw\lambda}\underbrace{e^{-iw\Delta x}}_{cte.}d\lambda \;=\; \boxed{e^{-iw\Delta x}F(w)}$$

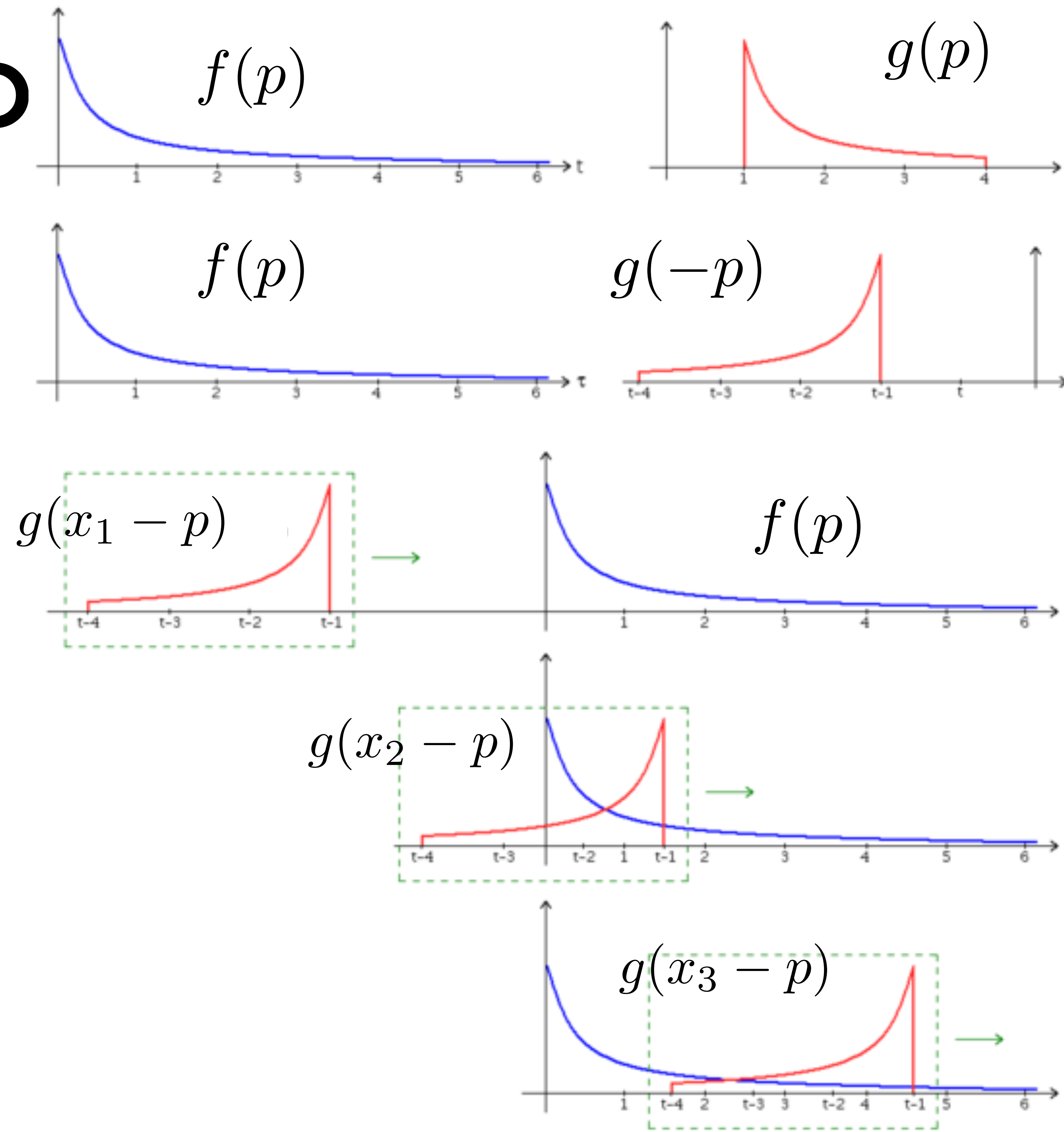
Convolução

A convolução $f * g$ de funções f e g é definida como:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

Convolução

$$\int_{-\infty}^{\infty} f(p)g(x-p)dp$$



Transformada de Fourier da convolução

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(x-p) dp \; e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(x-p) \; e^{-iwx} dx dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) g(q) e^{-iw(p+q)} dq \; dp$$

Transformada de Fourier da convolução

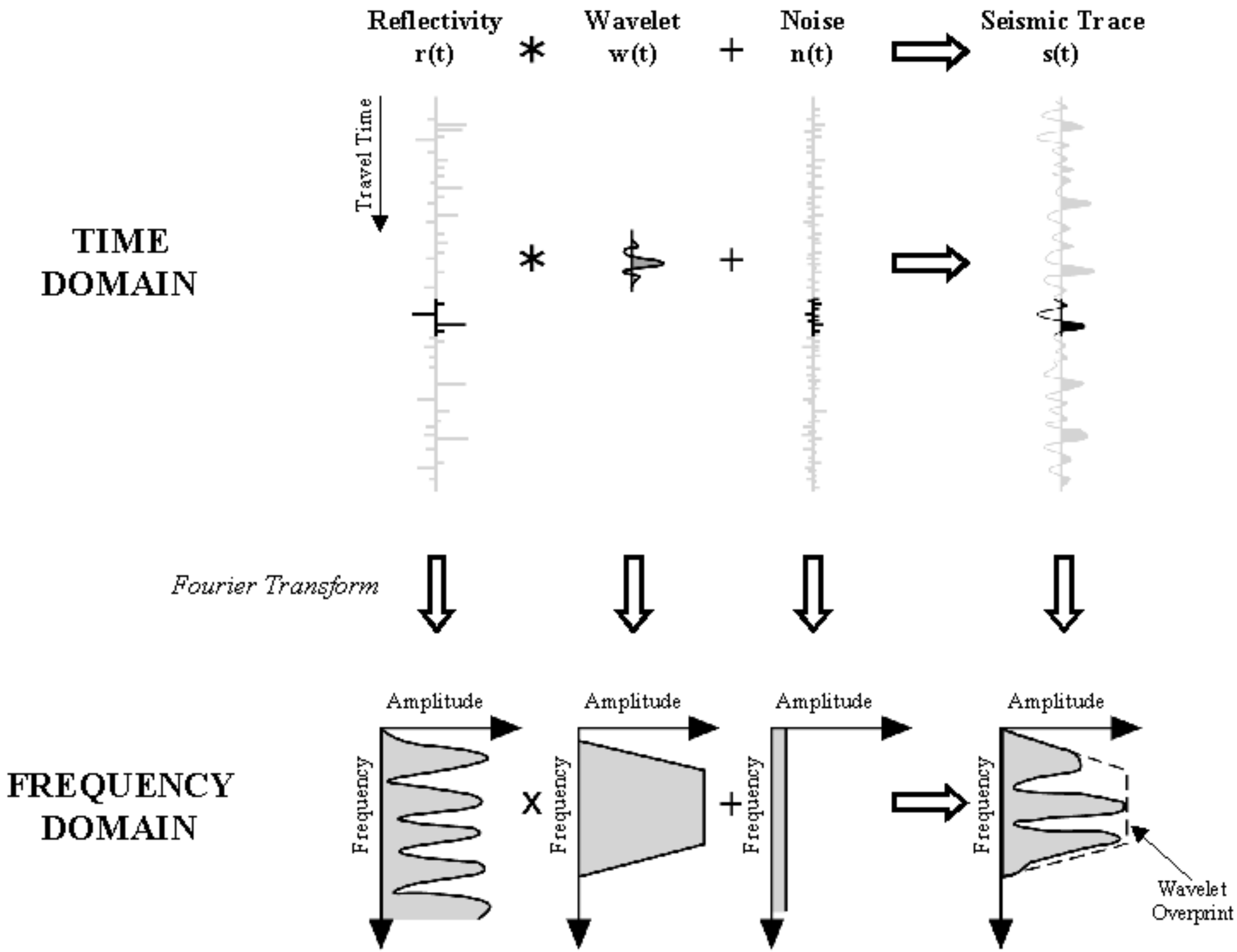
$$\mathcal{F}(f*g) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(q)e^{-iw(p+q)}dq\,dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(q)e^{-iwp}e^{-iwq}dq\,dp$$

$$= \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} f(p)e^{-iwp}dp}_{\sqrt{2\pi}\mathcal{F}(f)}\underbrace{\int_{-\infty}^{\infty} g(q)e^{-iwq}dq}_{\sqrt{2\pi}\mathcal{F}(g)}$$

$$\boxed{\mathcal{F}(f*g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)}$$

Aplicação em sísmica



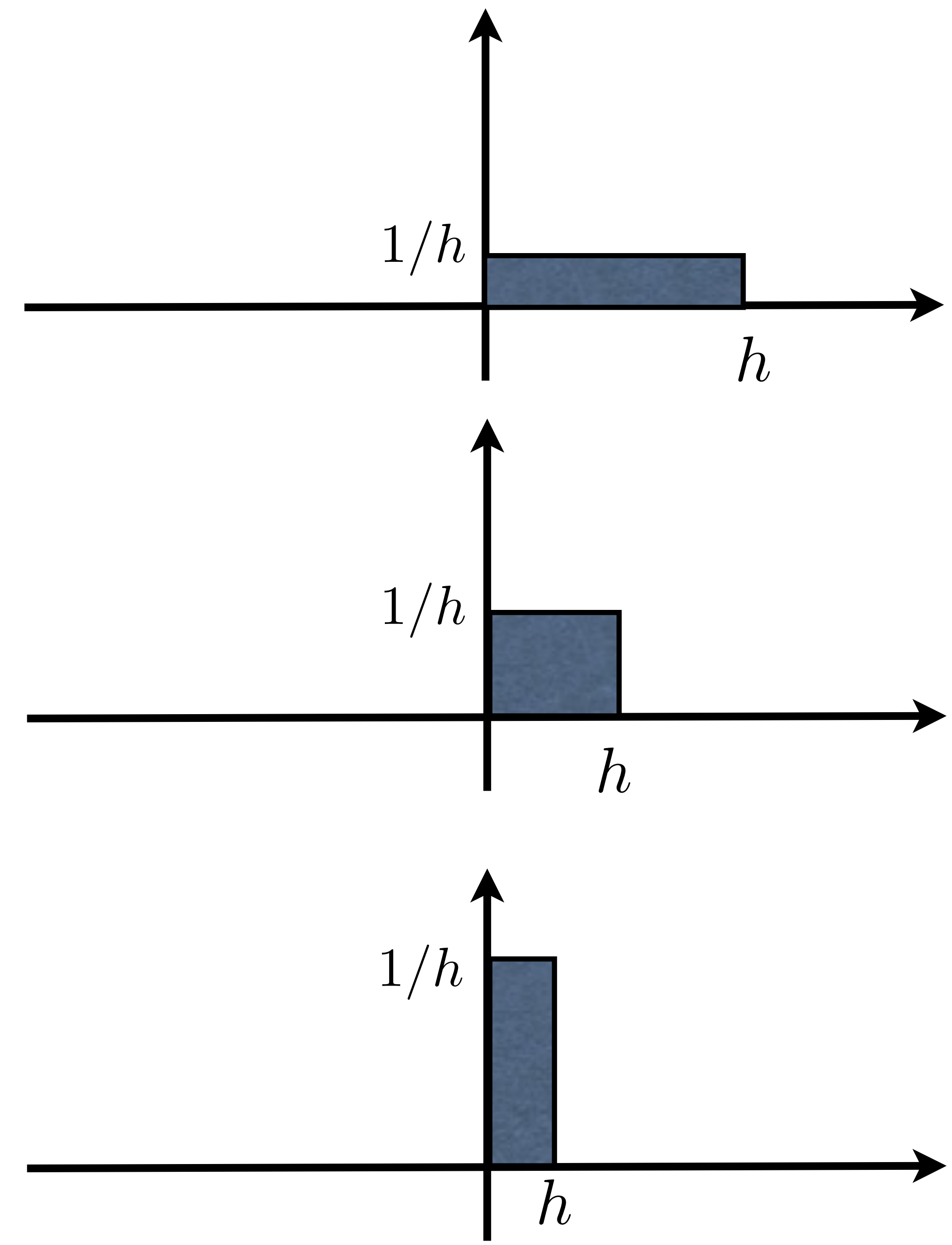
Impulso de Dirac

Seja $\Pi(x,h)$ um pulso retangular de largura h e área unitária

$$\Pi(x,h) = \begin{cases} 0, & -\infty < x < 0 \\ 1/h, & 0 \le x < h \\ 0, & h \le x < \infty \end{cases}$$

$$\int_{-\infty}^{\infty} \Pi(x,h)dx = \int_0^h \frac{1}{h}\, dx = 1$$

E se $h \to 0$?



Impulso de Dirac

$h \to 0$

$1/h \to \infty$

Mas $\text{Área} = 1$

$$\lim_{h \to 0} \Pi(x,h) = \delta(x)$$

Delta de Dirac

Impulso de Dirac

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

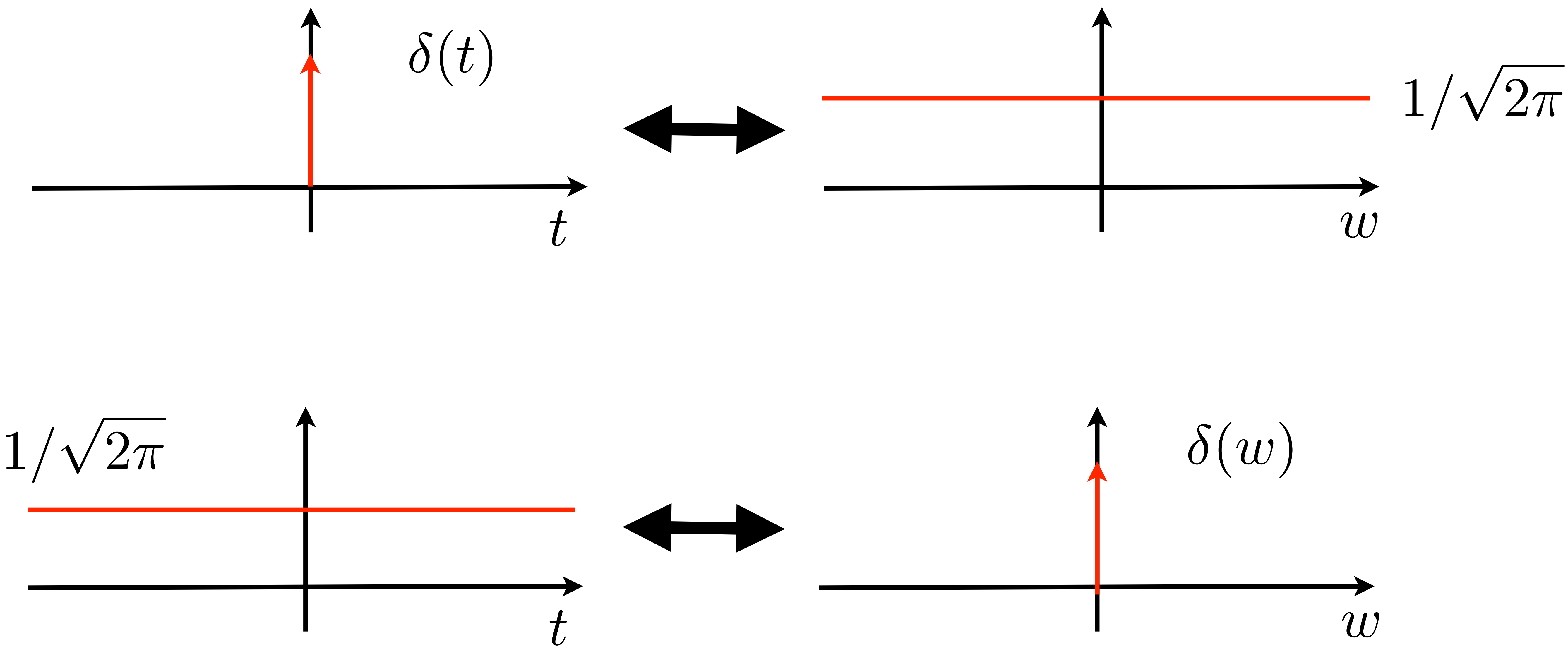
Impulso de Dirac

$$\mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}e^{-iw0} = \frac{1}{\sqrt{2\pi}}$$

$$\mathcal{F}^{-1}(\delta(w)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(w)e^{iwx}dw = \frac{1}{\sqrt{2\pi}}e^{i0x} = \frac{1}{\sqrt{2\pi}}$$

Impulso de Dirac

Tempo | Frequência



Impulso de Dirac

$$\mathcal{F}(\delta(x-x_0)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x_0)e^{-iwx}dx \quad = \frac{1}{\sqrt{2\pi}}e^{-iwx_0}$$

$$\overset{-1}{\mathcal{F}}(\delta(w-w_0)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(w-w_0)e^{iwx}dw \quad = \frac{1}{\sqrt{2\pi}}e^{iw_0x}$$

Transformada de Fourier de Funções Periódicas

$f(t)$ periódica $\qquad f(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikw_0 t} \qquad w_0 = 2\pi/p$

$$\mathcal{F}\left[f(t)\right] = \mathcal{F}\left(\sum_{k=-\infty}^{+\infty} c_k e^{ikw_0 t}\right)$$

$$F(w) = \sum_{k=-\infty}^{+\infty} c_k \mathcal{F}\left(e^{ikw_0 t}\right) = \sqrt{2\pi} \sum_{k=-\infty}^{+\infty} c_k \delta(w - kw_0)$$