P3 MAT 130

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1. a)

A partir do sistema temos que: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 - 2 = 0 \iff (\lambda - 1)(\lambda - 4) = 0$

- $\lambda = 1$

$$A - I = \left[\begin{array}{cc|c} 2-1 & 2 & 0 \\ 1 & 3-1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ se $y = 1$, $x = -2 \Rightarrow v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\quad \therefore x_1 = e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

- $\lambda = 4$

$$A - 4I = \left[\begin{array}{cc|c} 2-4 & 2 & 0 \\ 1 & 3-4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ se $y = 1$, $x = 1 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \therefore x_2 = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Logo $x = c_1 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0$

$\iff \lambda = -1$

$$A + I = \left[\begin{array}{cc|c} -6+1 & 5 & 0 \\ -5 & 4+1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ se $y = 1$, $x = 1 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore x_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A partir de v_1 vamos calcular o vetor generalizado v_2

$(A - \lambda I) v_2 = v_1 \Rightarrow \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \cdot v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \iff \begin{cases} -5a_1 + 5a_2 = 1 \\ -5a_1 + 5a_2 = 1 \end{cases}$, $a_2 = 0 \Rightarrow a_1 = -\frac{1}{5}$

$\therefore x_2 = e^{-t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/5 \\ 0 \end{bmatrix} \right)$

Logo $x = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/5 \\ 0 \end{bmatrix} \right)$

I.c $\begin{cases} \dot{x} = 4x + 5y \\ \dot{y} = -2x + 6y \end{cases}$ $A = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 34 = 0$$

$$\Leftrightarrow \lambda = \frac{10 \pm \sqrt{-36}}{2} = 5 \pm 3i$$

- $\lambda = 5 - 3i$

$$A - \lambda I = \left[\begin{array}{cc|c} -1+3i & 5 & 0 \\ -2 & 1+3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{-1-3i}{2} & 0 \\ -2 & 1+3i & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{-1-3i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \text{se } y = 2,\ x = 1+3i$$

- $\lambda = 5 + 3i$

$$A - \lambda I = \left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{-1+3i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \text{se } y = 2,\ x = 1-3i$$

$\therefore$ os autovetores serão da forma: $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 3 \\ 0 \end{pmatrix} i$

Solução geral será

$$x(t) = C_1 e^{5x}\left(\cos 3t \begin{pmatrix} \frac{1}{2} \end{pmatrix} - \operatorname{sen} 3t \begin{pmatrix} -3 \\ 0 \end{pmatrix}\right) + C_2 e^{5x}\left(\cos 3t \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \operatorname{sen} 3t \begin{pmatrix} \frac{1}{2} \end{pmatrix}\right)$$

2 a.

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & -1\\ -1 & 1-\lambda\end{vmatrix}=0 \iff \lambda^2-2\lambda=0 \iff \lambda=0,\ \lambda=2$$

- $\lambda=0$

$$\left(\begin{array}{cc|c}1 & -1 & 0\\ -1 & 1 & 0\end{array}\right)\Rightarrow\left(\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right) \Rightarrow x=y \Rightarrow v_1=\begin{pmatrix}1\\1\end{pmatrix}$$

- $\lambda=2$

$$\left(\begin{array}{cc|c}-1 & -1 & 0\\ -1 & -1 & 0\end{array}\right)\Rightarrow\left(\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right) \Rightarrow x=-y \Rightarrow v_2=\begin{pmatrix}-1\\+1\end{pmatrix}$$

b)

$$\Phi(t)=\begin{pmatrix}1 & -e^{2t}\\ 1 & e^{2t}\end{pmatrix}, \quad \Phi(t)^{-1}=\begin{pmatrix}1/2 & 1/2\\ \frac{-e^{-2t}}{2} & \frac{e^{-2t}}{2}\end{pmatrix}$$

$$\int \Phi^{-1}(t)\cdot\begin{pmatrix}a\\b\end{pmatrix}dt=\int\begin{pmatrix}\frac{a+b}{2}\\ \frac{-ae^{-2t}+be^{-2t}}{2}\end{pmatrix}dt=\begin{pmatrix}\frac{(a+b)\cdot t}{2}\\ (-a+b)\frac{e^{-2t}}{4}\end{pmatrix}$$

$$\begin{pmatrix}1 & -e^{2t}\\ 1 & e^{2t}\end{pmatrix}\begin{pmatrix}(a+b)\,t/2\\ (-a+b)\,\frac{e^{-2t}}{4}\end{pmatrix}=\begin{pmatrix}(a+b)\,t/2-(-a+b)\,1/4\\ (a+b)\,t/2+(-a+b)\,1/4\end{pmatrix}=x_p$$

Então se $a=-b$ é possível.

c)

$$x_p=\begin{pmatrix}-2\,t/2 & +8/4\\ -2\,t/2 & -8/4\end{pmatrix}=\begin{pmatrix}-t+2\\ -t-2\end{pmatrix}$$

$$x=\begin{pmatrix}1 & -e^{2t}\\ 1 & e^{2t}\end{pmatrix}\cdot C+\begin{pmatrix}-t+2\\ -t-2\end{pmatrix}$$

3.a.)

$$M^{-1}AM = D + N \Rightarrow A = M(D+N)M^{-1}$$

Adotando $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $M^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ $\Rightarrow M^{-1}AM = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Tenho que $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ e $N = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b. para resolver e^A irei diagonalizar a matriz A

com isso tenho que $A = C D C^{-1}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \iff -\lambda^3 + 3\lambda^2 = 0$$

$$\iff \lambda^2(-\lambda + 3) = 0$$

- $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ adotando $y = 0$ tenho que $x = -y \Rightarrow v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

adotando $y = 0$ $\quad x = -z \Rightarrow v_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

- $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ tenho $\begin{matrix} x = z \\ y = z \end{matrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Logo $A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$

$$\Rightarrow e^A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -1 & -1 & e^3 \\ 1 & 0 & e^3 \\ 0 & 1 & e^3 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} \frac{e^3+2}{3} & \frac{e^3-1}{3} & \frac{e^3-1}{3} \\ \frac{e^3-1}{3} & \frac{e^3+2}{3} & \frac{e^3-1}{3} \\ \frac{e^3-1}{3} & \frac{e^3-1}{3} & \frac{e^3+2}{3} \end{pmatrix}$$

c.

$$\frac{d}{dt} U(t) = A\,U(t)$$

Com os resultados obtidos no exercício anterior temos que:

$$u(t) = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 & -1 & e^{3t} \\ 1 & 0 & e^{3t} \\ 0 & 1 & e^{3t} \end{bmatrix} = S(t)$$

d. se $e^{tA} = S(t) \cdot S(0)^{-1} = S(t) \cdot \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$

$$= S(t) \cdot \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$e^{tA} = M e^{tD} M^{-1}$, se $e^{A} = M e^{D} M^{-1}$