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1) a) $\begin{cases} \frac{dx}{dt} = 2x + 2y \\ \frac{dy}{dt} = x + 3y \end{cases}$

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda) = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0 \iff \lambda = 1 \text{ ou } \lambda = 4$$

Calculando o autovetor de $\lambda = 1$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \Rightarrow x = -2y$$

Tomando $y = 1$, temos $V_{\lambda_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Logo $X_1 = e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Calculando o autovetor de $\lambda = 4$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow x = y$$

Tomando $y = 1$, temos $V_{\lambda_4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X_2 = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Logo a sol geral é:

$$X = c_1 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) $\begin{cases} \dfrac{dx}{dt} = -6x + 5y \\ \dfrac{dy}{dt} = -5x + 4y \end{cases}$ $\quad A = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1$$

Calculando o autovetor associado a $\lambda = -1$

$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ -5 & 5 & 0 \end{array}\right] \Rightarrow x = y \quad$ se $y = 1$

$$X_1 = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Como o autovalor tem multiplicidade geométrica 1 e multiplicidade algebrica maior que 1, temos que:

$$X_2 = t e^{-t} V + e^{\lambda t} w = \cancel{e^{-t}(tV + w)}$$

onde w é o auto vetor generalizado

$(A - \lambda I) w = V$

$$\begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} -5a + 5b = 1 \\ -5a + 5b = 1 \end{cases} \Leftrightarrow b = \frac{1}{5} - a$$

se $a = 0$

$$w = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$X_2 = e^{-t} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/5 \end{pmatrix} \right)$$

Logo a sol. geral é:

$$X = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} \right)$$

(c) $\begin{cases} \dfrac{dx}{dt} = 4x + 5y \\ \dfrac{dy}{dt} = -2x + 6y \end{cases}$ $\qquad A = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 34 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100 - 4.34}}{2} = \frac{10 \pm 6i}{2}$$

$\lambda_1 = 5 - 3i$

$\lambda_2 = 5 + 3i$

Calculando o auto vet. associado a λ_1

$$\left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & \frac{-1-3i}{2} & 0 \\ -2 & 1+3i & 0 \end{array}\right] \Rightarrow x + y\left(\frac{1-3i}{2}\right) = 0$$

se $y = 2$

$$V_{\lambda_1} = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix} \Rightarrow V_{\lambda_2} = \begin{bmatrix} -1-3i \\ 2 \end{bmatrix}$$

Logo a sol geral do sistema é:

$$X = c_1 e^{5t}\left(3\cos(3t)\begin{pmatrix}1\\2\end{pmatrix} - \operatorname{sen}(3t)\begin{pmatrix}-3\\0\end{pmatrix}\right) + c_2 e^{5t}\left(\cos(3t)\begin{pmatrix}-3\\0\end{pmatrix} + \operatorname{sen}(3t)\begin{pmatrix}1\\2\end{pmatrix}\right)$$

2) a)

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda = 0$$

$$\Rightarrow \lambda_1 = 0 \text{ e } \lambda_2 = 2$$

o auto vet. assoc. a $\lambda_1 = 0$ é:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \Rightarrow x = y \quad \text{tomando } y = 0$$

$$V_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

o auto vet. assoc. a $\lambda_2 = 2$ é:

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & \end{array}\right] \Rightarrow x = -y, \text{ tomando } y = +1$$

$$V_{\lambda_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b) Sim. Seja $b = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\phi = \begin{bmatrix} 1 & -e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

$$\phi^{-1} = \frac{1}{a_{11} a_{22} - a_{12} \cdot a_{21}} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2e^{2t}} & \frac{1}{2e^{2t}} \end{pmatrix}$$

$$\int \phi^{-1} \cdot b \, dt = \int \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2e^{2t}} & \frac{1}{2e^{2t}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} dt = \int \begin{pmatrix} \frac{a+b}{2} \\ \frac{-a+b}{2e^{2t}} \end{pmatrix} dt = \begin{pmatrix} \frac{t(a+b)}{2} \\ \frac{-a+b}{4e^{2t}} \end{pmatrix}$$

$$\phi \cdot \int \phi . b \, dt = \begin{pmatrix} 1 & -e^{2t} \\ 1 & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{t(a+b)}{2} \\ \frac{-a+b}{4e^{2t}} \end{pmatrix} = \begin{pmatrix} \frac{t(a+b)}{2} - \frac{\cancel{e^{2t}}(-a+b)}{4\cancel{e^{2t}}} \\ \frac{t(a+b)}{2} + \frac{\cancel{e^{2t}}(-a+b)}{4\cancel{e^{2t}}} \end{pmatrix}$$

agora basta tomar $a = -b$, por exemplo, $a = 1$ e $b = -1$

Logo $x_p = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}$

c) Usando o item anterior temos

$$x_p = \begin{pmatrix} -\frac{t \cdot 2}{2} + \frac{8}{4} \\ -t \frac{2}{2} \quad \frac{8}{4} \end{pmatrix} = \begin{pmatrix} -t+2 \\ -t-2 \end{pmatrix}$$

A sol. geral é:

$$X = \begin{pmatrix} 1 & -e^{2t} \\ 1 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -t+2 \\ -t-2 \end{pmatrix}$$

3) a) Diagonalizando $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ obtemos:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1/3 & \frac{2}{3} & -\frac{1}{3} \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{N} = X$$

b) $e^A = M e^X M^{-1}$

$$e^A = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^3 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$e^A = \begin{pmatrix} -1 & -1 & e^3 \\ 1 & 0 & e^3 \\ 0 & 1 & e^3 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} \frac{e^3+2}{3} & \frac{e^3-3}{3} & \frac{e^3-1}{3} \\ \frac{e^3-1}{3} & \frac{e^3+2}{3} & \frac{e^3-1}{3} \\ \frac{e^3-1}{3} & \frac{e^3-1}{3} & \frac{e^3+2}{3} \end{pmatrix}$$

c) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 = \lambda^2(-\lambda+3) = 0$

$\Leftrightarrow \lambda_1 = 0$ ou $\lambda_2 = 3$

Achando o autovetor assoc. a $\lambda_1 = 0$ temos:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow x = -y - z$$

• se $y=0$ e $z=1$

$$V_{\lambda_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad X_1 = C_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

• Se $y=1$ e $z=0$

$$V'_{\lambda_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad X_2 = C_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Vamos achar o auto vet. assoc. a $\lambda = 3$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} x+y-2z \\ y = z \qquad y=z \\ -2x+2z=0 \Leftrightarrow x = z \end{array}$$

• Se $y = 1$

$$V_{\lambda_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad X_3 = C_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Logo a sol. geral é:

$$X = C_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

d)

$$X = \begin{bmatrix} -1 & -1 & e^{3t} \\ 0 & 1 & e^{3t} \\ 1 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Esse sist. satisfaz $\frac{d\,X(t)}{dt} X_0 = A\,X(t) X_0$. Tomando $S(t) = X(t) X(0)^{-1}$

$$X(t).X(0)^{-1} = \begin{bmatrix} -1 & -1 & e^{3t} \\ 0 & 1 & e^{3t} \\ 1 & 0 & e^{3t} \end{bmatrix} \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}_{\begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}} = \underbrace{\begin{bmatrix} \frac{2+e^{3t}}{3} & \frac{e^{3t}-1}{3} & \frac{e^{3t}-1}{3} \\ \frac{-1+3e^{t}}{3} & \frac{e^{3t}+2}{3} & \frac{e^{3t}-1}{3} \\ \frac{-1+e^{3t}}{3} & \frac{e^{3t}-1}{3} & \frac{e^{3t}+2}{3} \end{bmatrix}}_{= e^{At}}$$