Prof. Luciano Nakabashi

More on Optimal Control

Autonomous Problems

- As a special case of problem

$$\text{Maximize} \qquad \mathrm{V} = \int_0^T G(t, y, u)e^{-\rho t} dt$$

$$\text{Subject to} \qquad \dot{y} = f(t, y, u)$$

and boundary conditions

- Both the G and f functions may contain no t argument. Therefore, the problem above may take the form

$$\text{Maximize} \qquad \mathrm{V} = \int_0^T G(y, u)e^{-\rho t} dt$$

$$\text{(24)} \quad \text{Subject to} \qquad \dot{y} = f(y, u)$$

and boundary conditions

Autonomous Problems

- Since the integrand $G(y, u)e^{-\rho t}$ still explicitly contains t, the problem is, strictly speaking, nonautonomous.
- However, **by using the current-value Hamiltonian**, we can in effect take the discount factor $\boldsymbol{e^{-\rho t}}$ **out of consideration**.
- **All the revised maximum-principle conditions** of the general current-value Hamiltonian **still hold**.
- But the current-value Hamiltonian of the autonomous problem (24) has an additional property not available in problem (13).
- Since H_c now specializes to the form

(25) $H_c = G(y, u) + m(t)f(y, u)$

Autonomous Problems

- which is free of the t argument, **its value evaluated along the optimal paths of all variables must be constant over time**. That is,

(26) $\frac{dH_c^*}{dt} = 0$ or $H_c^* = constant$ [autonomous problem]

- This result is, of course, nothing but a replay of the previous autonomous problem result.

Sufficient Conditions

- A basic **sufficiency theorem** due to 0. L. Mangasarian states that for the optimal control problem

	Maximize	$V = \int_0^T F(t, y, u)dt$	
(27)	Subject to	$\dot{y} = f(t, y, u)$	
	and	$y(0) = y_0$	$(y_0, T\ given)$

- The necessary conditions of the maximum principle are also sufficient for the global maximization of V if

1. the $\boldsymbol{F}$ **and** $\boldsymbol{f}$ **functions are differentiable and concave in** the variables $(\boldsymbol{y}, \boldsymbol{u})$ jointly, and
2. in the optimal solution it is true that

Sufficient Conditions

(28) $\lambda(t) \geq 0$ for all $t \in [0, T]$ if f is no linear in y or in u

- **If f is linear in y and in u, then $\lambda(t)$ needs no sign restriction**.
- With the Hamiltonian

  (29) $H = F(t, y, u) + \lambda(t) f(t, y, u)$
- the optimal control path $\boldsymbol{u}^*(\boldsymbol{t})$ - **along with** the associated $\boldsymbol{y}^*(\boldsymbol{t})$ and $\boldsymbol{\lambda}^*(\boldsymbol{t})$ **paths** - **must satisfy** the maximum principle, so that

  (30) $\left.\frac{\partial H}{\partial u}\right|_{u^*} = F_u(t, y^*, u^*) + \lambda^* f_u(t, y^*, u^*) = 0$
- This implies that

Sufficient Conditions

(31) $F_u(t, y^*, u^*) = -\lambda^* f_u(t, y^*, u^*)$

- Moreover, from the costate equation of motion, $\dot{\boldsymbol{\lambda}} = -\boldsymbol{\partial H / \partial y}$, we should have

$\dot{\lambda}^* = -F_y(t, y^*, u^*) - \lambda^* f_y(t, y^*, u^*)$

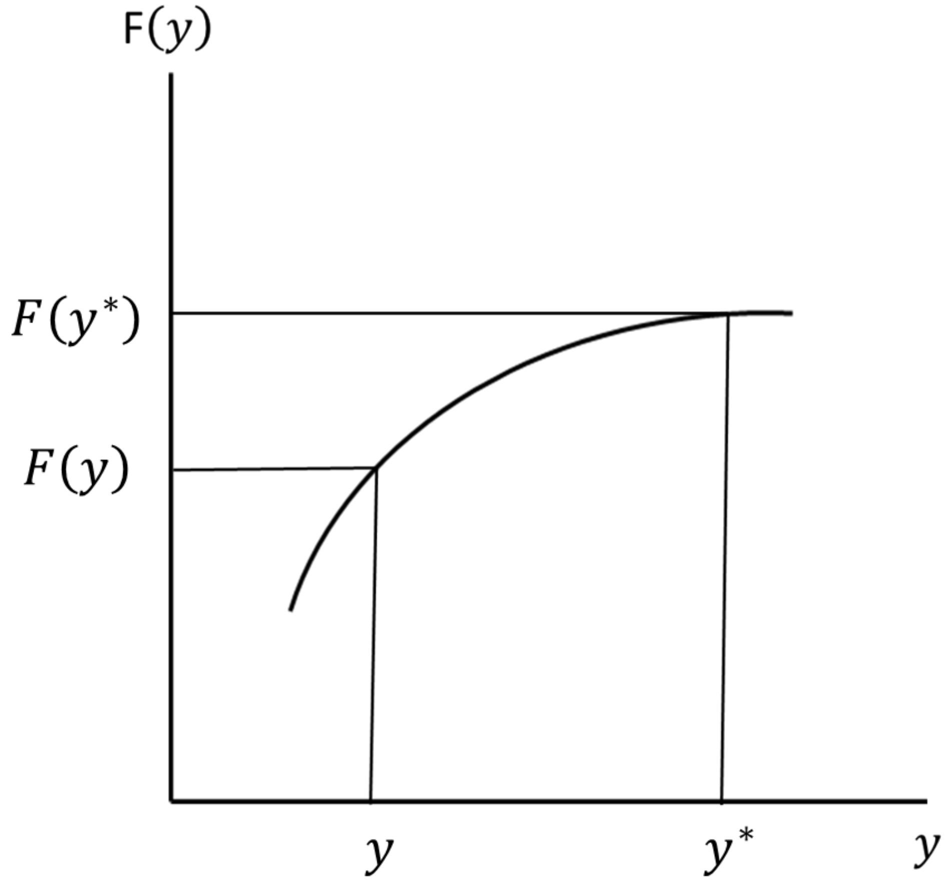
- which implies that

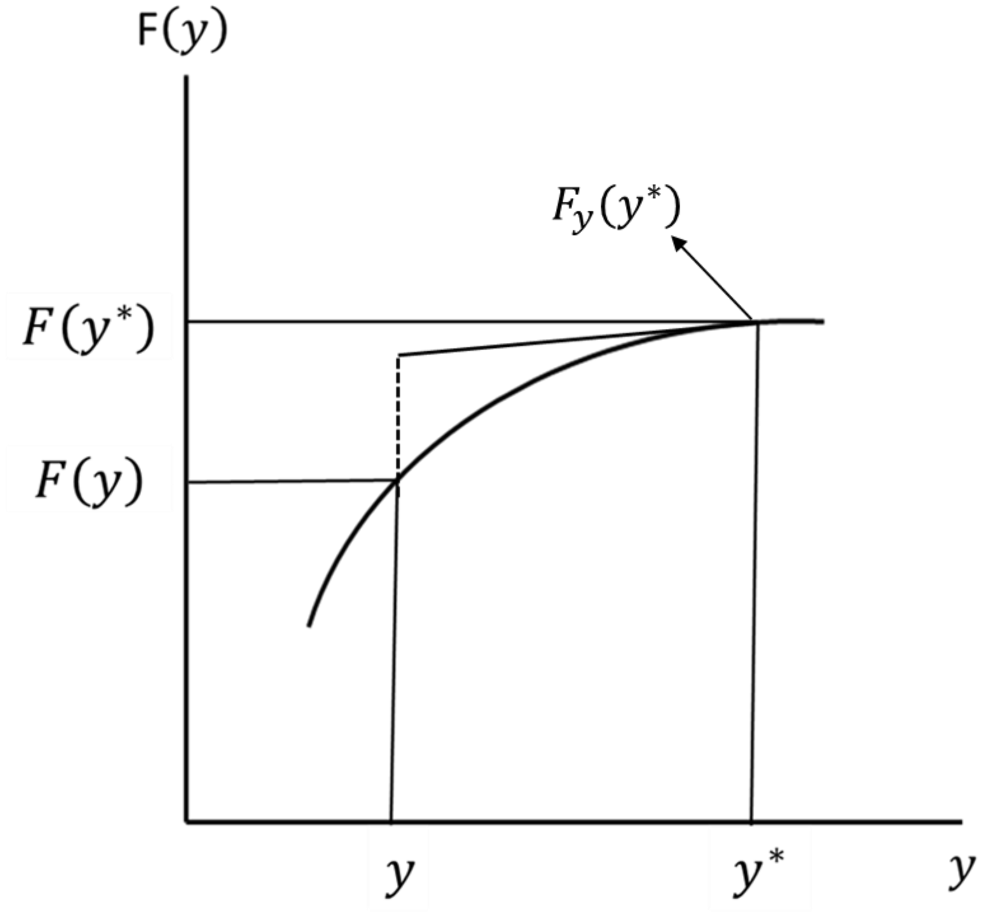
(32) $F_y(t, y^*, u^*) = -\dot{\lambda}^* - \lambda^* f_y(t, y^*, u^*)$

- Finally, **assuming** for that the problem has **a vertical terminal line**, the initial condition and the transversality condition should give us

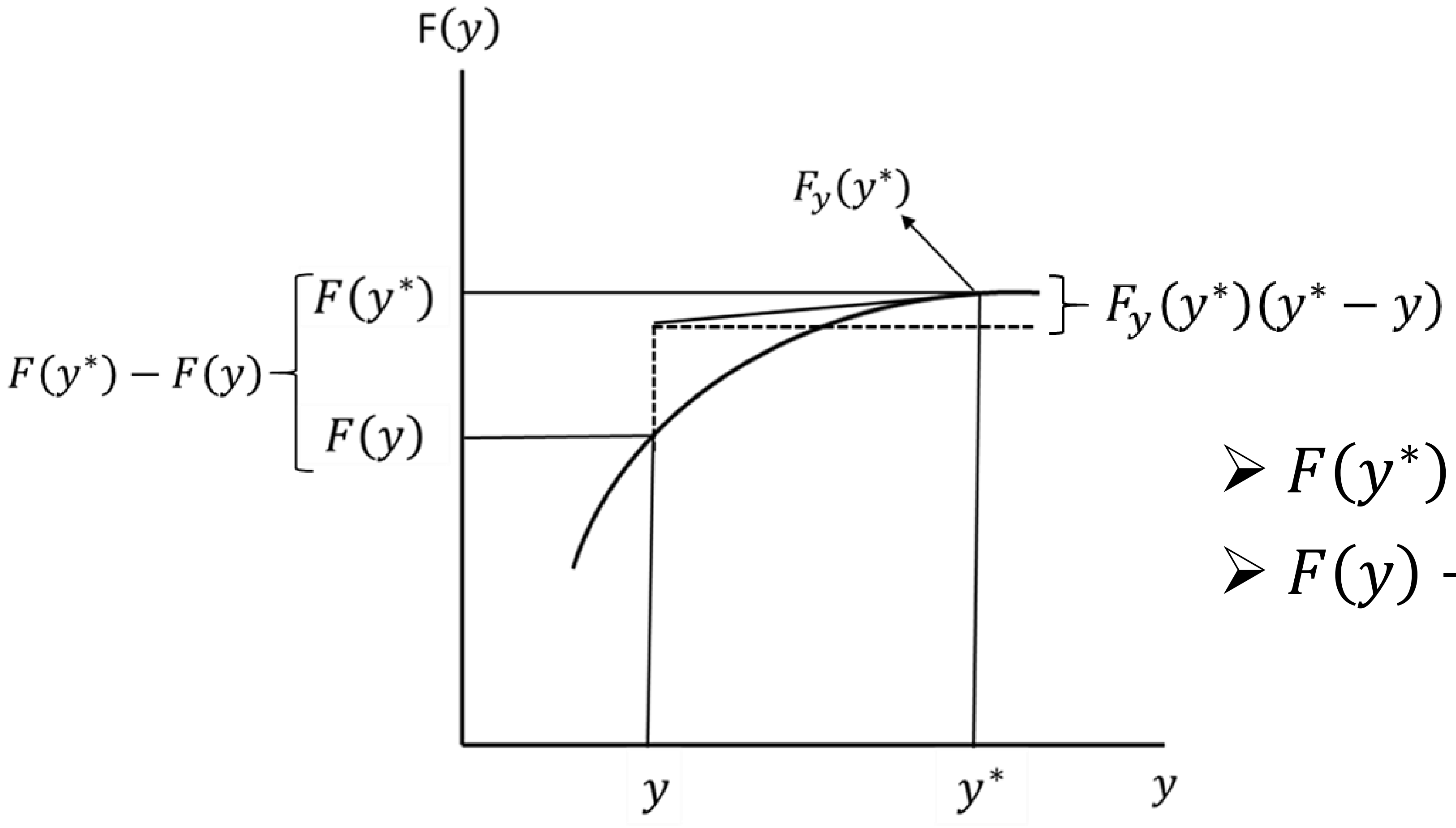
(33) $y_0^* = y_0 \; (given)$ and $\lambda^*(T) = 0$

Concavity





Concavity



- $F(y^*) - F(y) \geq F_y(y^*)[y^* - y]$
- $F(y) - F(y^*) \leq F_y(y^*)[y - y^*]$

Sufficient Conditions

- Now let **both the $\boldsymbol{F}$ and $\boldsymbol{f}$ functions be concave in $(\boldsymbol{y}, \boldsymbol{u})$**. Then, for two distinct points (t, y^*, u^*) and (t, y, u) in the domain, we have:

(34) $F(t, y, u) - F(t, y^*, u^*) \leq F_y(t, y^*, u^*)(y - y^*) + F_u(t, y^*, u^*)(u - u^*)$

(34') $f(t, y, u) - f(t, y^*, u^*) \leq f_y(t, y^*, u^*)(y - y^*) + f_u(t, y^*, u^*)(u - u^*)$

- Upon integrating both sides of (34) over $[0, T]$, that inequality becomes

(35)

$$\int_0^T F(t, y, u)dt - \int_0^T F(t, y^*, u^*)dt \leq \int_0^T F_y(t, y^*, u^*)(y - y^*) + F_u(t, y^*, u^*)(u - u^*)dt$$

- By **(31)** $F_u(t, y^*, u^*) = -\lambda^* f_u(t, y^*, u^*)$ **and (32)** $F_y(t, y^*, u^*) = -\dot{\lambda}^* - \lambda^* f_y(t, y^*, u^*)$:

(36)

$$V - V^* \leq \int_0^T \left[-\dot{\lambda}^*(y - y^*) - \lambda^* f_y(t, y^*, u^*)(y - y^*) - \lambda^* f_u(t, y^*, u^*)(u - u^*)\right]dt$$

Sufficient Conditions

- Let $\boldsymbol{w} = -\boldsymbol{\lambda}^*$ and $\boldsymbol{v} = \boldsymbol{y} - \boldsymbol{y}^*$. Then $\boldsymbol{dw} = -\dot{\boldsymbol{\lambda}}^* \boldsymbol{dt}$ and $\boldsymbol{dv} = (\dot{\boldsymbol{y}} - \dot{\boldsymbol{y}}^*)\boldsymbol{dt}$. So,

  $\int_0^T -\dot{\lambda}^*(y - y^*)dt \quad \left(= \int_0^T v dw\right)$

  $= [-\lambda^*(y - y^*)]_0^T - \int_0^T -\lambda^* (\dot{y} - \dot{y}^*)dt \quad \left(= [wv]_0^T - \int_0^T w dv\right)$

  $= -\lambda^*(T)(y_T - y_T^*) + \lambda^*(0)(y_0 - y_0^*) + \int_0^T \lambda^* (\dot{y} - \dot{y}^*)dt$

  $= \int_0^T \lambda^* (\dot{y} - \dot{y}^*)dt$ **since** $\boldsymbol{y_0^*} = \boldsymbol{y_0}$ $(given)$ and $\boldsymbol{\lambda}^*(\boldsymbol{T}) = \mathbf{0}$

  (37) $\int_0^T -\dot{\lambda}^*(y - y^*)dt = \int_0^T \lambda^*[f(t, y, u) - f(t, y^*, u^*)]\, dt$

- by the equation of motion $\dot{y} = f(t, y, u)$

Sufficient Conditions

- Using (37) $\int_0^T -\dot{\lambda}^*(y - y^*)dt = \int_0^T \lambda^*[f(t, y, u) - f(t, y^*, u^*)]\, dt$ into (36) yield:

(36) $V - V^* \leq \int_0^T \left[-\dot{\lambda}^*(y - y^*) - \lambda^* f_y(t, y^*, u^*)(y - y^*) - \lambda^* f_u(t, y^*, u^*)(u - u^*)\right] dt$

(38)

$$V - V^* \leq \int_0^T \lambda^* \{f(t, y, u) - f(t, y^*, u^*) - \left[f_y(t, y^*, u^*)(y - y^*) + f_u(t, y^*, u^*)(u - u^*)\right]\} dt \leq 0$$

- The last inequality follows from the assumption of $\lambda^*(t) \geq 0$ in (28), and the fact that the bracketed expression in the integrand is ≤ 0 by (34').

(34') $f(t, y, u) - f(t, y^*, u^*) \leq f_y(t, y^*, u^*)(y - y^*) + f_u(t, y^*, u^*)(u - u^*)$

- Consequently, the final result is

(39) $V \leq V^*$

Sufficient Conditions

- Which establishes $\boldsymbol{V^*}$ **to be a (global) maximum**, as claimed in the theorem.
- The above theorem is based on the F and f functions being concave.
- **If those functions are strictly concave**, **the weak inequalities** in (34) and (34') **will become strict inequalities**, as will the inequalities in (36), (38), and (39).
- The maximum principle will then be sufficient for a unique global maximum of V.
- Although the proof of the theorem has proceeded on the assumption of a vertical terminal line, **the theorem is also valid for other problems with a fixed $\boldsymbol{T}$** (fixed terminal point or truncated vertical terminal line).

The Arrow Sufficiency Theorem

- **Another sufficiency theorem**, due to **Kenneth J. Arrow**, uses a weaker condition than Mangasarian's theorem, and can be considered as a generalization of the latter.
- Here, we shall **describe its essence** without reproducing the proof.
- **At any point of time**, **given** the values of the state and costate variables $\boldsymbol{y}$ **and** $\boldsymbol{\lambda}$, **the Hamiltonian** function **is maximized by** a particular u , $\boldsymbol{u}^*$, which depends on t, y, and λ.

  (40) $u^* = u^*(t, y, \lambda)$
- **When (40) is substituted into the Hamiltonian**, **we obtain** what is referred to as **the maximized Hamiltonian function**

The Arrow Sufficiency Theorem

(41) $H^0(t, y, \lambda) = F(t, y, u^*) + \lambda f(t, y, u^*)$

- Note that **the concept of $\boldsymbol{H^0}$ is different from that of the optimal Hamiltonian $\boldsymbol{H^*}$**. Since $\boldsymbol{H^*}$ **denotes the Hamiltonian evaluated along** all the optimal paths, that is, evaluated at $\boldsymbol{y^*(t)}$, $\boldsymbol{u^*(t)}$, **and** $\boldsymbol{\lambda^*(t)}$ for every point of time, the y, u, and λ arguments can all be substituted out, leaving H^* as a function of t alone: $\boldsymbol{H^* = H^*(t)}$.
- In contrast, $\boldsymbol{H^0}$ **is evaluated along** $\boldsymbol{u^*(t)}$ **only**; thus, **while the $\boldsymbol{u}$ argument is substituted out**, **the other arguments remain**, so that $\boldsymbol{H^0(t, y, \lambda)}$ is still a function with three arguments.

The Arrow Sufficiency Theorem

- **The Arrow theorem states that**, in the optimal control problem (27), the conditions of the maximum principle **are sufficient for the global maximization of $\boldsymbol{V}$, if** the maximized Hamiltonian function $\boldsymbol{H^0}$ defined in (41) **is concave i**n the variable $\boldsymbol{y}$ **for all $\boldsymbol{t}$** in the time interval $[0, T]$, **for given $\boldsymbol{\lambda}$**.
- **If** both the $\boldsymbol{F}$ **and** $\boldsymbol{f}$ functions **are concave in $(\boldsymbol{y}, \boldsymbol{u})$ and $\boldsymbol{\lambda} \geq \mathbf{0}$**, as stipulated by Mangasarian, **then $\boldsymbol{H} \equiv \boldsymbol{F} + \boldsymbol{\lambda f}$ is also concave in** $(\boldsymbol{y}, \boldsymbol{u})$, and from this it follows that $\boldsymbol{H^0}$ **is concave in $\boldsymbol{y}$**, as stipulated by Arrow.
- **But $\boldsymbol{H^0}$ can be concave in $\boldsymbol{y}$ even if $\boldsymbol{F}$ and $\boldsymbol{f}$ are not concave in** $(\boldsymbol{y}, \boldsymbol{u})$, which makes the **Arrow condition a weaker requirement**.

Example

Maximize	$V = \int_0^T -(1+u^2)^{1/2} dt$
Subject to	$\dot{y} = u$
and	$y(0) = A; \quad y_T = free; \; (A, T \; given)$

- For the Mangasarian theorem, we note that neither the F function nor the f function depends on y, so **the concavity condition relates to $\boldsymbol{u}$ alone**.
- From the $F = -(1+u^2)^{1/2}$ function, we obtain

$$F_u = -\frac{1}{2}(1+u^2)^{-1/2} 2u = -u(1+u^2)^{-1/2}$$

$$F_{uu} = -(1+u^2)^{-1/2} + \frac{1}{2} u(1+u^2)^{-3/2} 2u$$

Example

$$F_{uu} = -(1 + u^2)^{-1/2} + u^2(1 + u^2)^{-1}(1 + u^2)^{-1/2}$$
$$F_{uu} = (1 + u^2)^{-1/2}[u^2(1 + u^2)^{-1} - 1] =$$
$$F_{uu} = (1 + u^2)^{-1/2}\left[\frac{u^2 - (1 + u^2)}{(1 + u^2)}\right] = -(1 + u^2)^{-3/2} < 0$$

- **Thus $\boldsymbol{F}$ is concave in $\boldsymbol{u}$**. As to the f function, $\boldsymbol{f = u}$, **since it is linear in $\boldsymbol{u}$, it is automatically concave in $\boldsymbol{u}$**.
- Besides, the fact that f is linear makes condition (28) $\lambda(t) \geq 0$ irrelevant. Consequently, **the conditions of Mangasarian are satisfied, and the optimal solution found earlier does maximize $\boldsymbol{V}$** (and minimize the distance) globally.

Example

- In the present example, the Hamiltonian is

$$H = -(1+u^2)^{1/2} + \lambda u$$

$$\frac{\partial H}{\partial u} = -\frac{1}{2}(1+u^2)^{-1/2}2u + \lambda = 0$$

$$\Rightarrow\ u(1+u^2)^{-1/2} = \lambda \qquad \Rightarrow\ u^2 = \lambda^2(1+u^2) \qquad \Rightarrow\ u^2 = \lambda^2 + \lambda^2 u^2$$

$$\Rightarrow\ u^2(1-\lambda^2) = \lambda^2 \qquad \Rightarrow\ u = \frac{\lambda}{(1-\lambda^2)^{1/2}} \qquad \Rightarrow \boldsymbol{u = \lambda\left(1-\lambda^2\right)^{-1/2}}$$

- **is substituted into $\boldsymbol{H}$ to eliminate $\boldsymbol{u}$, the resulting $\boldsymbol{H^0}$ expression contains $\boldsymbol{\lambda}$ alone, with no y. Thus $\boldsymbol{H^0}$ is linear and hence concave in $\boldsymbol{y}$** for given λ, and it **satisfies the Arrow sufficient condition**.