The Rationale of the Maximum Principle

Prof. Luciano Nakabashi

The Calculus of Variations and Optimal Control Theory Compared

- Before, we state a special problem of optimal control as:

Maximize $\quad \mathrm{V} = \int_0^T F(t, y, u)dt$

Subject to $\quad \dot{y} = u$

$y(0) = A, \quad \mathrm{y}(T) = free \quad (A, T\ given)$

- By substituting the equation of motion into the integrand function:

Maximize $\quad \mathrm{V} = \int_0^T F(t, y, \dot{y})dt$

Subject to $\quad y(0) = A, \quad \mathrm{y}(T) = free \quad (A, T\ given)$

- For this problem, the Hamiltonian function is

The Calculus of Variations and Optimal Control Theory Compared

(44) $H = F(t, y, u) + \lambda u$

- Assuming this function to be differentiable with respect to u, we may list the following conditions by the maximum principle:

$$\frac{\partial H}{\partial u} = F_u + \lambda = 0 \quad \Longrightarrow \quad \lambda = -F_u$$

(45) $\dot{y} = \frac{\partial H}{\partial \lambda} = u \quad \Longrightarrow \quad F_{\dot{y}} = F_u$ [since $\dot{y} = u$]

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = -F_y \qquad \text{and } \lambda(t) = 0$$

- From (45), we have

(46) $\lambda = -F_u \quad \Longrightarrow \quad \lambda = -F_{\dot{y}}$

The Calculus of Variations and Optimal Control Theory Compared

- Differentiation of (46) with respect to t yields

(47) $\dot{\lambda} = -\frac{d}{dt}F_{\dot{y}}$

- In addition, since $\dot{\lambda} = -F_y$, (47) may be expressed as

(48) $F_y - \frac{d}{dt}F_{\dot{y}} = 0$

- which is identical with the **Euler equation**.
- Further differentiation of the $\partial H/\partial u$ expression in (45) yields

(49) $\frac{\partial^2 H}{\partial u^2} = F_{uu} = F_{\dot{y}\dot{y}} \leq 0$

- Equation (49) is the Legendre necessary condition for maximum in the Calculus of Variations.

The Calculus of Variations and Optimal Control Theory Compared

- Thus the maximum principle is perfectly consistent with the conditions of the calculus of variations.
- For a control problem with a vertical terminal line, the transversality condition is $\lambda(T) = 0$.
- By (46) $\left(\lambda = -F_{\dot{y}}\right)$, this may be written as $\left[-F_{\dot{y}}\right]_{t=T} = 0$, or, equivalently,

  (50) $\left[F_{\dot{y}}\right]_{t=T} = 0$
- This is precisely the transversality condition in the **calculus of variations in the vertical terminal line** problem.

More on Optimal Control

Prof. Luciano Nakabashi

An Economic Interpretation of the Maximum Principle

- Consider **a firm** that seeks to **maximize its profits** over the time interval $[\mathbf{0}, \boldsymbol{T}]$.
- There is a single **state variable**, capital stock $\boldsymbol{K}$.
- And there is a single **control variable** $\boldsymbol{u}$, representing some **business decision the firm has to make at each moment of time** (such as its advertising budget or inventory policy).
- The **firm starts out** at time 0 **with capital** $\boldsymbol{K_0}$, but the **terminal capital stock is left open**.
- At any moment of time, **the profit of the firm depends on the amount of capital** it currently holds as well as **on the policy** $\boldsymbol{u}$ it currently selects.

An Economic Interpretation of the Maximum Principle

- It follows that the optimal control problem is to

Maximize $\Pi = \int_0^T \pi(t, K, u)dt$

(1) Subject to $\dot{K} = f(t, K, u)$

$\mathrm{K}(0) = K_0,$ $\mathrm{K}(T) = free$ $(K_0, T\ given)$

- The maximum principle places conditions on three types of variables:

1. Control.
2. State.
3. Costate.

The control variable u and the state variable K have already been assigned their economic meanings. **What about the costate variable λ**?

An Economic Interpretation of the Maximum Principle

- Remember that:

$$\mathcal{V} = \int_0^T \underbrace{\left[H(t, y, u, \lambda) + y(t)\dot{\lambda}(t)\right]}_{\Omega_1} dt - \underbrace{\lambda(T) y_T}_{\Omega_2} + \underbrace{\lambda(0) y_0}_{\Omega_3}$$

- Thefore, in the present problem, we have the functional

(2) $\Pi^* = \int_0^T \left[H(t, K^*, u^*, \lambda^*) + K^*(t)\, \dot{\lambda}^*(t)\right] dt - \lambda^*(T) K^*(T) + \lambda^*(0) K_0$

- Partial differentiation of Π^* with respect to the (given) initial capital and the (optimal) terminal capital yields

(3) $\frac{\partial \Pi^*}{\partial K_0} = \lambda^*(0)$ and $\frac{\partial \Pi^*}{\partial K^*(T)} = -\lambda^*(T)$

An Economic Interpretation of the Maximum Principle

- Thus, $\boldsymbol{\lambda}^*(\mathbf{0})$, the optimally determined initial costate value, **is a measure of the sensitivity of the optimal total profit $\boldsymbol{\Pi}^*$ to the given initial capital**.
- If we had one more (infinitesimal) unit of capital initially, Π^* would be larger by the amount $\lambda^*(0)$.
- Therefore, the latter expression can be taken as the imputed value or **shadow price of a unit of initial capital**.
- In the other partial derivative in (3), the terminal value of the optimal costate path, $\boldsymbol{\lambda}^*(\boldsymbol{T})$ **is seen to be the negative of the rate of change of $\boldsymbol{\Pi}^*$ with respect to the optimal terminal capital stock**.
- If we wished to **preserve one more unit** (use up one less unit) of capital stock at the end of the planning period, then we would have to **sacrifice our total profit by the amount $\boldsymbol{\lambda}^*(\boldsymbol{T})$**.

The Hamiltonian and the Profit Prospect

- The Hamiltonian of problem (1) is

(4) $H = \pi(t, K, u)dt + \lambda(t)\, f(t, K, u)$

- The **first component on the right is simply the profit function at time t,** based on the current capital and the current policy decision taken at that time.
- In the **second component** of (4), the $f(t, K, u)$ function **indicates the rate of change of (physical) capital, K, corresponding to policy u.**
- When the f **function is multiplied by** the shadow price, $\lambda(t)$, it is converted to a **monetary value**.
- Hence, the second component of the Hamiltonian represents the "**rate of change of capital value corresponding to policy u**."

The Hamiltonian and the Profit Prospect

- Unlike the first term, which relates to the current-profit effect of u, the **second term** can be viewed as **the future-profit effect of $\boldsymbol{u}$**, since the objective of capital accumulation is to pave the way for the production of profits for the firm in the future.
- In sum, then, **the Hamiltonian** represents the overall **profit prospect of the various policy decisions**, with both **the immediate** and **the future effects** taken into account.
- The maximum principle requires the maximization of the Hamiltonian with respect to u. What this means is that **the firm must try at each point of time to maximize the overall profit prospect by the proper choice of $\boldsymbol{u}$**.

The Hamiltonian and the Profit Prospect

- To see this more clearly, examine the weak version of the "$Max\ H$" condition:

(5) $\frac{\partial H}{\partial u} = \frac{\partial \pi}{\partial u} + \lambda(t)\frac{\partial f}{\partial u} = 0$

- it is rewritten into the form

(6) $\frac{\partial \pi}{\partial u} = -\lambda(t)\frac{\partial f}{\partial u}$

- This condition shows that **the optimal choice $\boldsymbol{u}^*$ must balance** any **marginal increase in the current profit** made possible by the policy [the left-hand-side expression in (6)] **against the marginal decrease in the future profit** that the policy will induce via the change in the capital stock [the right-hand-side expression in (6)].

The Equations of Motion

- The maximum principle involves two equations of motion.
- The one for the state variable K. It specifies the way the policy decision of the firm will affect the rate of change of capital.
- The equation of motion for the costate variable is

(7) $\dot{\lambda} = -\frac{\partial H}{\partial K} = -\frac{\partial \pi}{\partial K} - \lambda(t)\frac{\partial f}{\partial K}$

- or, after multiplying through by - 1,

(8) $-\dot{\lambda} = \frac{\partial \pi}{\partial K} + \lambda(t)\frac{\partial f}{\partial K}$

- The left-hand-side of (8) denotes the rate of decrease of the shadow price over time.

The Equations of Motion

- The first term in (8) $\boldsymbol{\partial \pi / \partial K}$, represents the marginal **contribution of capital to current profit**.
- The second, $\boldsymbol{\lambda(t)\, \partial f / \partial K}$, represents the marginal **contribution of capital to the enhancement of capital valu**e.
- What the **maximum principle** requires is that the **shadow price of capital depreciate at the rate at which capital is contributing to the current and future profits of the firm**.

Transversality Conditions

- With **a free terminal state $\boldsymbol{K(t)}$** at a **fixed terminal time $\boldsymbol{T}$** (vertical terminal line), that condition is

  (9) $\lambda(T) = 0$

- This means that the **shadow price of capital should be driven to zero at the terminal time**.
- The reason for this is that the **valuableness of capital to the firm emanates solely from its potential for producing profits**.
- **Given the planning horizon $\boldsymbol{T}$, the tacit understanding is that only the profits made within the period $\boldsymbol{[0, T]}$ would matter**, and that whatever capital stock that still exists at time T, being too late to be put to use, would have no economic value to the firm.

Transversality Conditions

- For a **firm** that intends **to continue its existence beyond the planning period** $[0, T]$, it may be reasonable to **stipulate some minimum** acceptable level for the terminal capital, say $\boldsymbol{K_{min}}$.
- In that case, the transversality condition stipulates that

(10) $\lambda(T) \geq 0;$ $K_T \geq K_{min};$ $(K_T^* - K_{min})\lambda(T) = 0$

- **If $\boldsymbol{K_T^*}$ exceed $\boldsymbol{K_{min}}$, the restriction** placed upon the terminal capital stock is **nonbinding**. The outcome is the same as if there is no restriction, and the old condition $\lambda(T) = 0$ still apply.
- But **if the terminal shadow price $\boldsymbol{\lambda(T)}$ is optimally nonzero (positive), then the restriction $\boldsymbol{K_{min}}$ is binding**,
- In the sense that it is **preventing the firm from using up as much of its capital** toward the **end of the period** as it would otherwise do.

Transversality Conditions

- Finally, consider the case of a **horizontal terminal line.**
- In that case, the firm has a **prespecified terminal capital level**, say K_T, but is **free to choose the time** to reach the target.
- The transversality condition

(11) $[H]_{t=T} = 0$

- simply means that, at the terminal time, the sum of the current and future profits pertaining to that point of time must be zero.
- In other words, the firm should not attain K_T at a time when the sum of immediate and future profits (the value of H) is still positive; rather, **it should attain $\boldsymbol{K_T}$ at a time when the sum of immediate and future profits has been squeezed down to zero.**

The Current-Value Hamiltonian

- In economic applications of optimal control theory, the integrand function F often contains a discount factor $e^{-\rho t}$.
- Such an F function can in general be expressed as

(12) $F(t, y, u) = G(t, y, u)e^{-\rho t}$

- So that the optimal control problem is to

Maximize $\quad \mathrm{V} = \int_0^T G(t, y, u)e^{-\rho t} dt$

(13) Subject to $\quad \dot{y} = f(t, y, u)$

and boundary conditions

The Current-Value Hamiltonian

- By the standard definition, the Hamiltonian function takes the form

  (14) $H = G(t, y, u)e^{-\rho t} + \lambda(t)f(t, y, u)$

- But since the maximum principle calls for the differentiation of H with respect to u and y, and **since the presence of the discount factor adds complexity to the derivatives**, it may be desirable to **define a new Hamiltonian that is free of the discount factor**.
- Such a Hamiltonian is called the **current-value Hamiltonian**, where the term "current-value" (as against "present-value") serves to convey the **"undiscounted" nature** of the new Hamiltonian.
- The concept of the current-value Hamiltonian necessitates the companion concept of the **current-value Lagrange multiplier**.

The Current-Value Hamiltonian

- Let us therefore first define a new (current-value) Lagrange multiplier m:

  (15) $m(t) = \lambda(t)e^{\rho t}$ [implying $\boldsymbol{\lambda(t) = m(t)e^{-\rho t}}$]
- Then the current-value Hamiltonian, denoted by H_c, can be written as

  (16) $H_c = He^{\rho t} = G(t, y, u) + m(t)f(t, y, u)$
- As intended, $\boldsymbol{H_c}$ **is now free of the discount factor**.
- Note that (16) implies:

  (16') $H = H_c e^{-\rho t}$

The Maximum Principle Revised

- **The first condition** in the maximum principle is to **maximize $\boldsymbol{H}$ with respect to $\boldsymbol{u}$ at every point of time**.
- When we **switch to** the current-value Hamiltonian $\boldsymbol{H_c = He^{\rho t}}$, **the condition is essentially unchanged** except for the substitution of H_c for H. This is because the exponential term $\boldsymbol{e^{\rho t}}$ **is a constant** for any given t.
- **The particular $\boldsymbol{u}$ that maximizes $\boldsymbol{H}$ will therefore also maximize $\boldsymbol{H_c}$.** Thus the revised condition is simply

  (17) $\underbrace{Max}_{u} H_c$ for all the $t \in [0, T]$

- The **equation of motion for the state variable** originally appears in the canonical system as $\boldsymbol{\dot{y} = \partial H / \partial \lambda}$.

The Maximum Principle Revised

- Since $\boldsymbol{\partial H/\partial \lambda = f(t, y, u) = \partial H_c/\partial m}$ [see (14) and (16)], **the condition $\dot{\boldsymbol{y}} = \boldsymbol{\partial H/\partial \lambda}$ should be revised to**

(18) $\dot{y} = \frac{\partial H_c}{\partial m}$ **[equation of motion for $\boldsymbol{y}$]**

- **To revise** the equation of motion for the costate variable**, $\dot{\boldsymbol{\lambda}} = -\boldsymbol{\partial H/\partial y}$, we shall transform each side of this equation into an expression involving the new Lagrange Multiplier $\boldsymbol{m}$**, and then equate the two results.

- For the left-hand side, **by differentiating (15) $\boldsymbol{m(t) = \lambda(t)e^{\rho t}}$**:

(19) $\dot{m} = \dot{\lambda}e^{\rho t} + \rho\lambda e^{\rho t} = \dot{\lambda}e^{\rho t} + \rho m$ ➡ $\dot{\boldsymbol{\lambda}} = \dot{\boldsymbol{m}}\boldsymbol{e^{-\rho t} - \rho m e^{-\rho t}}$

The Maximum Principle Revised

- Using the definition of H in (16') $\boldsymbol{H = H_c e^{-\rho t}}$, we can rewrite the right-hand side of (18) as

(20) $-\frac{\partial H}{\partial y} = -\frac{\partial H_c}{\partial y} e^{-\rho t}$

- Equating (19) and (20):

$\dot{m} e^{-\rho t} - \rho m e^{-\rho t} = -\frac{\partial H_c}{\partial y} e^{-\rho t}$ ➡ $\dot{m} - \rho m = -\frac{\partial H_c}{\partial y}$

(21) $\dot{m} = -\frac{\partial H_c}{\partial y} + \rho m$ **[equation of motion for $\boldsymbol{m}$]**

- Note that, **compared with the original equation of motion for $\boldsymbol{\lambda}$, the new one for $\boldsymbol{m}$ involves an extra term, $\boldsymbol{\rho m}$**.

The Maximum Principle Revised

- It remains to examine the transversality conditions. We shall do this for the vertical terminal line and the horizontal terminal line only.
- For the former, we can deduce that [remember that $\boldsymbol{\lambda(t) = m(t)e^{-\rho t}}$]

(22) $\lambda(T) = 0 \Longrightarrow [me^{-\rho t}]_{t=T} = 0$

(22’) $m(T)e^{-\rho T} = 0$

- Similar reasoning shows that for the horizontal terminal line problem [remember that $\boldsymbol{H = H_c e^{-\rho t}}$]:

(23) $[H]_{t=T} = 0 \Longrightarrow [H_c e^{-\rho t}]_{t=T} = 0$

(23’) $[H_c]_{t=T} e^{-\rho T} = 0$