The Rationale of the Maximum Principle

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Example

Maximize $\quad V = \int_0^1 -u^2 dt$

Subject to $\quad \dot{y} = y + u$

$y(0) = 1, \quad y(1) = 0$

- With fixed endpoints, we need no transversality condition in this problem.
- **Step I** - Since the Hamiltonian function is nonlinear:

$H = -u^2 + \lambda(y + u)$

- and since u is unconstrained, we can apply the first-order condition:
- $\frac{\partial H}{\partial u} = -2u + \lambda = 0$ Therefore $\boldsymbol{u(t) = \frac{1}{2}\lambda(t)}$

Example

- Since

$$\frac{\partial^2 H}{\partial u^2} = -2$$

- $u(t)$ solution does maximize rather than minimize H.
- But since this solution is expressed in terms of $\lambda(t)$, we must find the latter path before $u(t)$ becomes determinate.
- **Step II -** From the costate equation of motion

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = -\lambda(t) \Rightarrow \frac{\dot{\lambda}}{\lambda} = -1 \Rightarrow \int \frac{\dot{\lambda}}{\lambda} dt = -\int 1 dt$$

$$ln\lambda = -t + c \Rightarrow \lambda(t) = e^{-t+c} \Rightarrow \lambda(t) = ke^{-t}$$

- Where $k = e^c$ [k arbitrary]. Therefore $\boldsymbol{u(t) = \frac{1}{2}\lambda(t) = \frac{1}{2}ke^{-t}}$

Example

- **Step III -** The equation of motion for y is $\boldsymbol{\dot{y} = y + u}$.
- We can rewrite this equation as:

$$\dot{y} - y = \frac{1}{2}ke^{-t}$$

- This is a first-order linear differential equation with a variable coefficient and a variable term, of the type

$$\frac{dy}{dt} + u(t)y = w(t)$$

- here with $u(t) = -1$ and $w(t) = \frac{1}{2}ke^{-t}$
- Via a standard formula, its solution can be found as follows

Example

$$y(t) = e^{-\int -1dt}\left(c + \int \frac{1}{2} k e^{-t} e^{\int -1dt} dt\right)$$

$$= e^{t}\left(c + \int \frac{1}{2} k e^{-t} e^{-t} dt\right) = e^{t}\left(c + \frac{1}{2} k \int e^{-2t} dt\right)$$

$$= e^{t}\left(c - \frac{1}{4} k e^{-2t}\right) = ce^{t} - \frac{1}{4} k e^{-t} \qquad [c \text{ arbitrary}]$$

- **Step IV** - The boundary conditions $y(0) = 1$ and $y(1) = 0$ are directly applicable, and they give definite values for c and k:

$$y(0) = c - \frac{1}{4}k = 1; \quad y(1) = ce - \frac{1}{4} k e^{-1} = 0 \Rightarrow ce^{2} - \frac{1}{4}k = 0 \Rightarrow k = 4ce^{2}$$

$$c - \frac{1}{4} 4ce^{2} = 1 \Rightarrow c(1 - e^{2}) = 1 \Rightarrow c = \frac{1}{(1-e^{2})}; \quad k = 4ce^{2} \Rightarrow k = \frac{4e^{2}}{(1-e^{2})}$$

Example

- Therefore:

$$y^*(t) = ce^t - \frac{1}{4}ke^{-t} = \underbrace{\frac{1}{(1-e^2)}}_{c}e^t - \underbrace{\frac{e^2}{(1-e^2)}}_{k/4}e^{-t}$$

$$\lambda^*(t) = ke^{-t} = \frac{4e^2}{(1-e^2)}e^{-t}$$

$$u^*(t) = \frac{1}{2}ke^{-t} = \frac{2e^2}{(1-e^2)}e^{-t}$$

- The search for the $u^*(t)$, $y^*(t)$, and $\lambda^*(t)$ paths in the present problem turns out to be an intertwined process. This is because, unlike the simplest problem of optimal control, where the transversality condition $\lambda(t) = 0$ may enable us to get a definite solution of the costate path $\lambda^*(t)$ at an early stage.

The Constancy of the Hamiltonian in Autonomous Problems

- The example discussed previously share the common feature that the problems are "autonomous;" that is, **the functions in the integrand and f in the equation of motion do not contain t** as an explicit argument.
- An important consequence of this feature is that **the optimal Hamiltonian** - the Hamiltonian evaluated along the optimal paths of y, u, and λ - **will have a constant value over time**.
- To see this, let us first examine the time derivative of the Hamiltonian $H(t, y, u, \lambda)$ in the general case:

(41) $$\frac{dH(t,y,u,\lambda)}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial \lambda}\dot{\lambda}$$

The Constancy of the Hamiltonian in Autonomous Problems

- When H **is maximized, we have** $\partial H/\partial u = 0$ (for an interior solution) or $u = 0$ (for a boundary solution). Thus the third term on the right drops out.
- Moreover, **the maximum principle also stipulates that** $\dot{y} = \partial H/\partial \lambda$ **and** $\dot{\lambda} = -\partial H/\partial y$. So the second and fourth terms on the right exactly cancel out.
- The net result is that H^*, the Hamiltonian evaluated along the optimal paths of all variables, **satisfies the equation**

  (42) $\frac{dH}{dt} = \frac{\partial H}{\partial t}$
- This result holds generally, for both autonomous and nonautonomous problems.

The Constancy of the Hamiltonian in Autonomous Problems

- In the special case of an **autonomous problem**, **since $\boldsymbol{t}$ is absent** from the F and f functions as an explicit argument, the Hamiltonian must not contain the t argument either.
- Consequently, we have $\boldsymbol{\partial H^* / \partial t = 0}$, so that
- (43) $\frac{dH^*}{dt} = 0$ or $\boldsymbol{H^*}$ **is constant** [for autonomous problems]
- This result is of practical use in an autonomous problem with a horizontal terminal line.
- The transversality condition $\boldsymbol{[H]_{t=T} = 0}$ is normally expected to hold at the terminal time only. But if the Hamiltonian is a constant in the optimal solution, **then it must be zero for all $\boldsymbol{t}$**.

The Calculus of Variations and Optimal Control Theory Compared

- Before, we state a special problem of optimal control as:

Maximize $\quad V = \int_0^T F(t, y, u)dt$

Subject to $\quad \dot{y} = u$

$y(0) = A, \quad y(T) = free \quad (A, T\ given)$

- By substituting the equation of motion into the integrand function:

Maximize $\quad V = \int_0^T F(t, y, \dot{y})dt$

Subject to $\quad y(0) = A, \quad y(T) = free \quad (A, T\ given)$

- For this problem, the Hamiltonian function is

The Calculus of Variations and Optimal Control Theory Compared

(44) $H = F(t, y, u) + \lambda u$

- Assuming this function to be differentiable with respect to u, we may list the following conditions by the maximum principle:

$$\frac{\partial H}{\partial u} = F_u + \lambda = 0 \quad \Longrightarrow \quad \lambda = -F_u$$

(45) $\dot{y} = \frac{\partial H}{\partial \lambda} = u \quad \Longrightarrow \quad F_{\dot{y}} = F_u \qquad [\dot{y} = u]$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} = -F_y \qquad \text{and } \lambda(t) = 0$$

- From (45), we have

(46) $\lambda = -F_{\dot{y}}$

The Calculus of Variations and Optimal Control Theory Compared

- Differentiation of (46) with respect to t yields

(47) $\dot{\lambda} = -\frac{d}{dt}F_{\dot{y}}$

- In addition, since $\dot{\lambda} = -F_y$, (47) may be expressed as

(48) $F_y - \frac{d}{dt}F_{\dot{y}} = 0$

- which is identical with the **Euler equation**.
- Further differentiation of the $\partial H/\partial u$ expression in (45) yields

(49) $\frac{\partial^2 H}{\partial u^2} = F_{uu} = F_{\dot{y}\dot{y}} \leq 0$

- Equation (49) is the Legendre necessary condition for maximum in the Calculus of Variations.

The Calculus of Variations and Optimal Control Theory Compared

- Thus the maximum principle is perfectly consistent with the conditions of the calculus of variations.
- For a control problem with a vertical terminal line, the transversality condition is $\lambda(T) = 0$.
- By (46) $\left(\lambda = -F_{\dot{y}}\right)$, this may be written as $\left[-F_{\dot{y}}\right]_{t=T} = 0$, or, equivalently,

  (50) $\left[F_{\dot{y}}\right]_{t=T} = 0$
- This is precisely the transversality condition in the calculus of variations in the vertical terminal line problem.

More on Optimal Control

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An Economic Interpretation of the Maximum Principle

- Consider **a firm** that seeks to **maximize its profits** over the time interval $[\mathbf{0}, \boldsymbol{T}]$.
- There is a single **state variable**, capital stock $\boldsymbol{K}$.
- And there is a single **control variable** $\boldsymbol{u}$, representing some **business decision the firm has to make at each moment of time** (such as its advertising budget or inventory policy).
- The **firm starts out** at time 0 **with capital** $\boldsymbol{K_0}$, but the **terminal capital stock is left open**.
- At any moment of time, **the profit of the firm depends on the amount of capital** it currently holds as well as **on the policy** $\boldsymbol{u}$ it currently selects.

An Economic Interpretation of the Maximum Principle

- It follows that the optimal control problem is to

Maximize $\Pi = \int_0^T \pi(t, K, u)dt$

(1) Subject to $\dot{K} = f(t, K, u)$

$\mathrm{K}(0) = K_0,$ $\mathrm{K}(T) = free$ $(K_0, T\ given)$

- The maximum principle places conditions on three types of variables:

1. Control.
2. State.
3. Costate.

The control variable u and the state variable K have already been assigned their economic meanings. **What about the costate variable $\boldsymbol{\lambda}$?**

An Economic Interpretation of the Maximum Principle

- Remember that:

$$\mathcal{V} = \int_0^T \underbrace{\left[H(t,y,u,\lambda) + y(t)\dot{\lambda}(t)\right]}_{\Omega_1} dt - \underbrace{\lambda(T)y_T}_{\Omega_2} + \underbrace{\lambda(0)y_0}_{\Omega_3}$$

- Thefore:

$$(2)\quad \Pi^* = \int_0^T \left[H(t,K^*,u^*,\lambda^*) + K^*(t)\,\dot{\lambda}^*(t)\right]dt - \lambda^*(T)K^*(T) + \lambda^*(0)K_0$$

- Partial differentiation of Π^* with respect to the (given) initial capital and the (optimal) terminal capital yields

$$(3)\quad \frac{\partial \Pi^*}{\partial K_0} = \lambda^*(0) \qquad \text{and} \qquad \frac{\partial \Pi^*}{\partial K^*(T)} = -\lambda^*(T)$$

An Economic Interpretation of the Maximum Principle

- Thus, $\lambda^*(0)$, the optimally determined initial costate value, is a measure of the sensitivity of the optimal total profit Π^* to the given initial capital.
- If we had one more (infinitesimal) unit of capital initially, Π^* would be larger by the amount $\lambda^*(0)$.
- Therefore, the latter expression can be taken as the imputed value or shadow price of a unit of initial capital.
- In the other partial derivative in (3), the terminal value of the optimal costate path, $\lambda^*(T)$ is seen to be the negative of the rate of change of Π^* with respect to the optimal terminal capital stock.
- If we wished to preserve one more unit (use up one less unit) of capital stock at the end of the planning period, then we would have to sacrifice our total profit by the amount $\lambda^*(T)$.

The Hamiltonian and the Profit Prospect

- The Hamiltonian of problem (1) is

  (4) $H = \pi(t, K, u)dt + \lambda(t)\, f(t, K, u)$

- The **first component on the right is simply the profit function at time t, based on the** current capital and the current policy decision taken at that time.
- In the **second component** of (4), the $f(t, K, u)$ function **indicates the rate of change of (physical) capital, K, corresponding to policy u.**
- When the $\boldsymbol{f}$ **function is multiplied by** the shadow price, $\boldsymbol{\lambda}(\boldsymbol{t})$, it is converted to a **monetary value**.
- Hence, the second component of the Hamiltonian represents the "**rate of change of capital value corresponding to policy $\boldsymbol{u}$**."

The Hamiltonian and the Profit Prospect

- Unlike the first term, which relates to the current-profit effect of u, the **second term** can be viewed as **the future-profit effect of $\boldsymbol{u}$**, since the objective of capital accumulation is to pave the way for the production of profits for the firm in the future.
- In sum, then, **the Hamiltonian** represents the overall **profit prospect of the various policy decisions**, with both **the immediate** and **the future** effects taken into account.
- The maximum principle requires the maximization of the Hamiltonian with respect to u. What this means is that the firm must try at each point of time to maximize the overall profit prospect by the proper choice of u.

The Hamiltonian and the Profit Prospect

- To see this more clearly, examine the weak version of the "$Max\ H$" condition:

(5) $\frac{\partial H}{\partial u} = \frac{\partial \pi}{\partial u} + \lambda(t)\frac{\partial f}{\partial u} = 0$

- it is rewritten into the form

(6) $\frac{\partial \pi}{\partial u} = -\lambda(t)\frac{\partial f}{\partial u}$

- This condition shows that **the optimal choice $\boldsymbol{u}^*$ must balance** any **marginal increase in the current profit** made possible by the policy [the left-hand-side expression in (6)] **against the marginal decrease in the future profit** that the policy will induce via the change in the capital stock [the right-hand-side expression in (6)].