4300337 - Introdução à Relatividade

Solucionário da 3° Lista de Exercícios

Questão 01.

Por definição,

$$h_{00} = -\frac{2\Phi}{c^2} = \frac{M}{R}\frac{2G}{c^2}.$$

Aqui, estou usando para constantes $G \approx 6,7 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ e $c \approx 3 \cdot 10^8 m/s$ de forma que

$$\frac{2G}{c^2} \approx 1,5 \cdot 10^{-27} m/kg \approx 1,5 \cdot 10^{-28} cm/g.$$

Então temos

(a) $M/R \approx 0,85 \cdot 10^{-10} g/cm$, $h_{00} \approx 1,3 \cdot 10^{-38}$.

(b) Neste caso (usando $g \approx 10 m/s^2$)

$$h_{00} = \frac{2}{c^2}\frac{MG}{R} = \frac{2gR}{c^2} \approx 1,4 \cdot 10^9.$$

(c) $M/R \approx 2,9 \cdot 10^{22} g/cm$, $h_{00} \approx 4,4 \cdot 10^{-6}$.

(d) $M/R \approx 2 \cdot 10^{27} g/cm$, $h_{00} \approx 3 \cdot 10^{-1}$.

Questão 02.

(a) Considere que em um sistema de coordenadas x'^{μ}, em que $S' = g'_{\mu\nu} D'^{\mu\nu} = g'_{\mu\nu} A'^{\mu} B'^{\nu}$. Usando as regras de transformações vetoriais e tensoriais

$$\begin{aligned} g'_{\mu\nu} A'^{\mu} B'^{\nu} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x'^{\nu}}{\partial x^{\beta}} B^{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g_{\alpha\beta} A^{\alpha} B^{\beta} = \\ &= \frac{\partial x^{\alpha}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta}} g_{\alpha\beta} A^{\alpha} B^{\beta} = g_{\alpha\beta} A^{\alpha} B^{\beta} \to g_{\mu\nu} A^{\mu} B^{\nu}. \end{aligned}$$

Desta forma, $S' = S$, e concluímos que S é invariante por mudança de coordenadas.

(b) Temos que

$$S = g_{\mu\nu} D^{\mu\nu} = g_{\mu\nu} A^{\mu} B^{\nu} = A_{\mu} B^{\mu},$$

e é direto que

$$A_{\mu} B^{\mu} = g_{\mu\nu} A^{\nu} B^{\mu} = A^{\nu} g_{\mu\nu} B^{\mu} = A^{\nu} B_{\nu} \to S = A^{\mu} B_{\mu}.$$

Questão 03.

Do cálculo em múltiplas variáveis, o elemento de volume de um espaço $n-$dimensional de coordenadas x^α se transforma por uma mudança de coordenadas $x^\alpha \to y^\alpha$ de acordo com

$$\mathrm{d}V = \mathrm{d}^n x = |J(x \to y)| \mathrm{d}^n y,$$

onde o termo J multiplicando $\mathrm{d}^n y$ é o determinante jacobiano de x^α para y^α, isto é

$$J(x \to y) = \frac{\partial(x^1, \cdots, x^n)}{\partial(y^1, \cdots, y^n)} = \det\left[\frac{\partial x^\alpha}{\partial y^\beta}\right].$$

Dada a métrica de um espaço-tempo $g^x_{\mu\nu}$ para coordenadas x^α e $g^y_{\mu\nu}$ para y^α, essas métricas se relacionam pela lei de transformação tensorial

$$g^y_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g^x_{\alpha\beta}.$$

Do ponto de vista matricial, podemos tomar o determinante dessa expressão de forma que

$$\det\left(g^y_{\mu\nu}\right) = \det\left(\frac{\partial x^\alpha}{\partial y^\mu}\right)\det\left(\frac{\partial x^\beta}{\partial y^\nu}\right)\det\left(g^x_{\alpha\beta}\right) =$$
$$g^y = J(x \to y) J(x \to y) g^y = J(x \to y)^2 g^x$$

onde aqui eu defini $g^x \equiv \det\left(g^x_{\mu\nu}\right)$ e $g^y \equiv \det\left(g^y_{\mu\nu}\right)$. Então é direto que

$$J(x \to y) = \pm\sqrt{\frac{g^x}{g^y}}.$$

Usando então a relação da transformação entre $\mathrm{d}^n x$ e $\mathrm{d}^n y$,

$$\mathrm{d}^n x = \sqrt{\frac{g^x}{g^y}}\mathrm{d}^n y = \sqrt{\frac{-g^x}{-g^y}}\mathrm{d}^n y \to \sqrt{-g^x}\mathrm{d}^n x = \sqrt{-g^y}\mathrm{d}^n y,$$

Como o resultado vale para quaisquer transformações de coordenadas, então concluímos que

$$\mathrm{d}V_n \equiv \sqrt{-g^x}\mathrm{d}^n x \text{ é invariante sobre transformações de coordenadas.}$$

Se vale para n arbitrário, vale para $n = 4$, então o o volume próprio $\mathrm{d}V_4$ é invariante no espaço-tempo.

Questão 04.

É direto que a métrica do sistema $\{u, v\}$ é

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & v^2 \end{bmatrix}, \qquad\qquad g_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{v^2} \end{bmatrix},$$

isto é, a métrica neste sistema é diagonal e não depende da coordenada u.

(b) Usando o índice 0 para v e 1 para u.

$$\Gamma^1_{10} = \Gamma^1_{01} = \frac{1}{v}, \qquad \Gamma^1_{11} = 0, \qquad \Gamma^1_{00} = 0,$$
$$\Gamma^0_{10} = \Gamma^0_{01} = 0, \qquad \Gamma^0_{11} = v, \qquad \Gamma^0_{00} = 0.$$

(a) Uma forma direta de verificar se essa métrica corresponde a um espaço-tempo de Minkowski é verificando para um vetor V^μ arbitrário que $\Delta V^\mu = 0$, onde

$$\Delta V^\mu = \frac{1}{2} \oint R^\mu{}_{\nu\alpha\beta} V^\nu \, \mathrm{d}x^\alpha \, \mathrm{d}y^\beta \, .$$

Num cálculo direto, usando as propriedades de anti-simetria do tensor de Riemman, é possível verificar que ele é todo nulo, então

$$\Delta V^\mu = 0,$$

assim concluímos que essa métrica corresponde a um espaço-tempo de Minkowski. Da regra de transformação entre métricas (lembrando que tanto em coordenadas cartesianas quanto nesse sistema $\{u, v\}$, a métrica é diagonal), temos

$$\begin{aligned} \left(\frac{\partial t}{\partial v}\right)^2 - \left(\frac{\partial x}{\partial v}\right)^2 &= 1 \\ \left(\frac{\partial x}{\partial u}\right)^2 - \left(\frac{\partial t}{\partial u}\right)^2 &= v^2 \end{aligned} \tag{1}$$

Usando a relação trigonométrica hiperbólica fundamental, onde, para qualquer η, $\cosh^2 \eta - \sinh^2 \eta = 1$, podemos supor inicialmente que

$$\begin{aligned} \frac{\partial t}{\partial v} &= \cosh u \to t = v \cosh u \\ \frac{\partial x}{\partial v} &= \sinh u \to x = v \sinh u, \end{aligned} \tag{2}$$

essa definição respeita as duas condições impostas a partir da regra de transformação da métrica e são de fato a transformação procurada.

(c) A lagrangiana de uma partícula livre relativística é $(c = 1)$

$$L = m\sqrt{-g_{\mu\nu} u^\mu u^\nu} = m \sqrt{\left(\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)^2 - v^2 \left(\frac{\mathrm{d}u}{\mathrm{d}\tau}\right)^2}. \tag{3}$$

Para cada x^μ, as equações de Euler-Lagrange para o problema são

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \tag{4}$$

onde o momento canônico é dado por

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu}. \tag{5}$$

Como u é uma variável cíclica, vê-se que $\mathrm{d}p_u = 0$ e $\mathrm{d}p_v \neq 0$

Questão 05.

(a) Dado um vetor V^α nu sistema de coordenadas x^μ, temos

$$D_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\sigma} V^\sigma. \tag{6}$$

Se esse vetor é dado por V'^α num sistema y^μ, então, pela regra de transformação de vetores contravariantes,

$$V^\alpha = \frac{\partial x^\alpha}{\partial y^\beta} V'^\beta \tag{7}$$

e usamos a definição

$$\Gamma^a_{bc} = \frac{\partial x^a}{\partial z^d} \frac{\partial^2 z^d}{\partial x^b \partial x^c}. \tag{8}$$

onde z^μ são as coordenadas do referencial local. Teremos

$$\begin{aligned}
\partial_\mu V^\alpha &= \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \left(\frac{\partial x^\alpha}{\partial y^\beta} V'^\beta \right) = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\beta} V'^\beta + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial^2 x^\alpha}{\partial y^\nu \partial y^\beta} V'^\beta + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \\
&= \frac{\partial^2 x^\alpha}{\partial x^\mu \partial y^\beta} V'^\beta + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \frac{\partial^2 x^\alpha}{\partial y^\beta \partial x^\mu} V'^\beta + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \frac{\partial}{\partial y^\beta} \left(\frac{\partial x^\alpha}{\partial x^\mu} \right) + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \\
&= \frac{\partial}{\partial y^\beta} \left(\delta^\alpha_\mu \right) + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu}
\end{aligned}$$

Teremos para o símbolo de Christofel

$$\begin{aligned}
\Gamma^\alpha_{\mu\sigma} &= \frac{\partial x^\alpha}{\partial z^\lambda} \frac{\partial^2 z^\lambda}{\partial x^\mu \partial x^\sigma} = \frac{\partial y^\xi}{\partial z^\lambda} \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial}{\partial x^\mu} \left(\frac{\partial z^\lambda}{\partial x^\sigma} \right) = \frac{\partial y^\xi}{\partial z^\lambda} \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial}{\partial y^\kappa} \left(\frac{\partial z^\lambda}{\partial y^\epsilon} \frac{\partial y^\epsilon}{\partial x^\sigma} \right) = \\
&= \frac{\partial y^\xi}{\partial z^\lambda} \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \left(\frac{\partial^2 z^\lambda}{\partial y^\kappa \partial y^\epsilon} \frac{\partial y^\epsilon}{\partial x^\sigma} + \frac{\partial z^\lambda}{\partial y^\epsilon} \frac{\partial^2 y^\epsilon}{\partial y^\kappa \partial x^\sigma} \right) = \frac{\partial y^\xi}{\partial z^\lambda} \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial^2 z^\lambda}{\partial y^\kappa \partial y^\epsilon} \frac{\partial y^\epsilon}{\partial x^\sigma} + \frac{\partial z^\lambda}{\partial y^\epsilon} \frac{\partial y^\xi}{\partial z^\lambda} \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\epsilon}{\partial x^\mu \partial x^\sigma} \\
&= \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\epsilon}{\partial x^\sigma} \Gamma'^\xi_{\kappa\epsilon} + \delta^\xi{}_\epsilon \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\epsilon}{\partial x^\mu \partial x^\sigma} = \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\epsilon}{\partial x^\sigma} \Gamma'^\xi_{\kappa\epsilon} + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\xi}{\partial x^\mu \partial x^\sigma}
\end{aligned}$$

Dessa forma

$$\begin{aligned}
\Gamma^\alpha_{\mu\sigma} V^\sigma &= \left(\frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\epsilon}{\partial x^\sigma} \Gamma'^\xi_{\kappa\epsilon} + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\xi}{\partial x^\mu \partial x^\sigma} \right) \left(\frac{\partial x^\sigma}{\partial y^\beta} V'^\beta \right) = \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \frac{\partial y^\epsilon}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial y^\beta} \Gamma'^\xi_{\kappa\epsilon} V'^\beta + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial^2 y^\xi}{\partial x^\sigma \partial x^\mu} V'^\beta = \\
&= \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \delta^\epsilon_\beta \Gamma'^\xi_{\kappa\epsilon} V'^\beta + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial x^\sigma}{\partial y^\beta} \frac{\partial^2 y^\xi}{\partial x^\sigma \partial x^\mu} V'^\beta = \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \Gamma'^\xi_{\kappa\epsilon} V'^\epsilon + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\xi}{\partial y^\beta \partial x^\mu} V'^\beta = \\
&= \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \Gamma'^\xi_{\kappa\epsilon} V'^\epsilon + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial^2 y^\xi}{\partial x^\mu \partial y^\beta} V'^\beta = \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \Gamma'^\xi_{\kappa\epsilon} V'^\epsilon + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial}{\partial x^\mu} \left(\frac{\partial y^\xi}{\partial y^\beta} \right) V'^\beta = \\
&= \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \Gamma'^\xi_{\kappa\epsilon} V'^\epsilon + \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial}{\partial x^\mu} \left(\delta^\xi_\beta \right) V'^\beta = \frac{\partial x^\alpha}{\partial y^\xi} \frac{\partial y^\kappa}{\partial x^\mu} \Gamma'^\xi_{\kappa\epsilon} V'^\epsilon.
\end{aligned}$$

Dessa forma, renomeando índices $\kappa \to \alpha$ e $\xi \to \beta$ temos

$$\Gamma^\alpha_{\mu\sigma} V^\sigma = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \Gamma'^\beta_{\nu\epsilon} V'^\epsilon \tag{9}$$

Assim

$$D_\mu V^\alpha = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial V'^\beta}{\partial y^\nu} + \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \Gamma'^\beta_{\nu\epsilon} V'^\epsilon = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} \left(\frac{\partial V'^\beta}{\partial y^\nu} + \Gamma'^\beta_{\nu\epsilon} V'^\epsilon \right) = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial x^\alpha}{\partial y^\beta} D'_\nu V'^\beta \tag{10}$$

onde um outro sistema de coordenadas y^μ

$$D'_\mu V'^\alpha = \partial'_\mu V'^\alpha + \Gamma'^\alpha_{\mu\sigma} V'^\sigma. \tag{11}$$

(b) É direto que

$$D_\mu(\delta^\alpha_\beta) = \partial_\mu(\delta^\alpha_\beta) + \delta^\nu_\beta \Gamma^\alpha_{\mu\nu} - \delta^\alpha_\nu \Gamma^\nu_{\mu\beta} = \Gamma^\alpha_{\mu\beta} - \Gamma^\alpha_{\mu\beta} = 0. \tag{12}$$

(c) É direto que

$$0 = D_\mu(\delta^\alpha_\nu) = D_\mu(g^{\alpha\beta} g_{\beta\nu}) = D_\mu(g^{\alpha\beta}) g_{\beta\nu} + g^{\alpha\beta} D_\mu(g_{\beta\nu}) = D_\mu(g^{\alpha\beta}) g_{\beta\nu} \tag{13}$$

Como em geral $g_{\beta\nu} \neq 0$, então conclui-se

$$D_\mu(g^{\alpha\beta}) = 0. \tag{14}$$

Questão 06.

(a) Dado do exercício é $20,2km$. Usando o raio da terra como $\approx 6,5 \cdot 10^3 km$ e a massa da Terra como $\approx 6 \cdot 10^{24} kg$, obtemos

$$\frac{t_2}{t_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 + Gm/c^2 r_2}{1 - Gm/c^2 r_1}} \approx 1 - \frac{Gm}{c^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
$$\rightarrow \frac{\Delta t}{t_1} = \frac{Gm}{c^2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \approx 6,2 \cdot 10^{-10}.$$

Então, em $t_1 = 1$ ano, o atraso é de aproximadamente $\Delta t = t_2 - t_1 = 19,029 ms$.

(b) $c\Delta t \approx 5700 km$ ao ano.

Questão 07.

(a) Temos o segunite

$$R_{\mu\nu\alpha\beta} = g_{\sigma\mu} R^{\sigma}{}_{\nu\alpha\beta} = g_{\sigma\mu}\left(\partial_\alpha \Gamma^\sigma_{\beta\nu} - \partial_\beta \Gamma^\sigma_{\alpha\nu} + \Gamma^\sigma_{\alpha\lambda}\Gamma^\lambda_{\beta\nu} - \Gamma^\sigma_{\beta\lambda}\Gamma^\lambda_{\alpha\nu}\right).$$

onde

$$\Gamma^a_{bc} = \frac{g^{ad}}{2}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) \tag{15}$$

Para os primeiros dois termos, (usando $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$) temos

$$g_{\sigma\mu}\partial_\alpha \Gamma^\sigma_{\beta\nu} = \partial_\alpha(g_{\sigma\mu}\Gamma^\sigma_{\beta\nu}) - \Gamma^\sigma_{\beta\nu}\partial_\alpha g_{\sigma\mu} = \frac{1}{2}\partial_\alpha(\partial_\beta g_{\mu\nu} + \partial_\nu g_{\mu\beta} - \partial_\mu g_{\beta\nu}) - \Gamma^\sigma_{\beta\nu}\partial_\alpha g_{\sigma\mu} \tag{16}$$

e analogamente

$$g_{\sigma\mu}\partial_\beta \Gamma^\sigma_{\alpha\nu} = \frac{1}{2}\partial_\beta(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha}) - \Gamma^\sigma_{\alpha\nu}\partial_\beta g_{\sigma\mu} \tag{17}$$

Para os outros dois termos, usando a mesma propriedade da métrica acima, temos

$$g_{\sigma\mu}\Gamma^\sigma_{\alpha\lambda}\Gamma^\lambda_{\beta\nu} = \frac{1}{2}(\partial_\alpha g_{\mu\lambda} + \partial_\mu g_{\alpha\lambda} - \partial_\lambda g_{\mu\alpha})\Gamma^\lambda_{\beta\nu} \tag{18}$$

e

$$g_{\sigma\mu}\Gamma^\sigma_{\beta\lambda}\Gamma^\lambda_{\alpha\nu} = \frac{1}{2}(\partial_\beta g_{\mu\lambda} + \partial_\lambda g_{\beta\lambda} - \partial_\lambda g_{\mu\beta})\Gamma^\lambda_{\alpha\nu} \tag{19}$$

Usando essas definições, e reorganizando os termos, é direto que (usando propriedade de troca de índice livre de $\sigma \to \lambda$)

$$\begin{aligned}
R_{\mu\nu\alpha\beta} =& + \frac{1}{2}\partial_\alpha(\cancelto{0}{\partial_\beta g_{\mu\nu}} + \partial_\nu g_{\mu\beta} - \partial_\mu g_{\beta\nu}) - \cancelto{\frac{1}{2}\Gamma^\sigma_{\beta\nu}\partial_\alpha g_{\sigma\mu}}{\Gamma^\sigma_{\beta\nu}\partial_\alpha g_{\sigma\mu}} + \\
& - \frac{1}{2}\partial_\beta(\cancelto{0}{\partial_\alpha g_{\mu\nu}} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}) + \cancelto{-\frac{1}{2}\Gamma^\sigma_{\alpha\nu}\partial_\beta g_{\sigma\mu}}{\Gamma^\sigma_{\alpha\nu}\partial_\beta g_{\sigma\mu}} + \\
& + \frac{1}{2}(\cancelto{0}{\partial_\alpha g_{\mu\lambda}} + \partial_\mu g_{\alpha\lambda} - \partial_\lambda g_{\mu\alpha})\Gamma^\lambda_{\beta\nu} \\
& - \frac{1}{2}(\cancelto{0}{\partial_\beta g_{\mu\lambda}} + \partial_\mu g_{\beta\lambda} - \partial_\lambda g_{\mu\beta})\Gamma^\lambda_{\alpha\nu} = \\
=& \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right) + \\
& - \frac{1}{2}(\partial_\alpha g_{\lambda\mu} - \partial_\mu g_{\alpha\lambda} + \partial_\lambda g_{\mu\alpha})\Gamma^\lambda_{\beta\nu} + \frac{1}{2}(\partial_\beta g_{\lambda\mu} - \partial_\mu g_{\beta\lambda} + \partial_\lambda g_{\mu\beta})\Gamma^\lambda_{\alpha\nu} = \\
=& g_{\sigma\lambda}\left(\frac{g^{\sigma\epsilon}}{2}(\partial_\beta g_{\mu\epsilon} + \partial_\mu g_{\beta\epsilon} - \partial_\epsilon g_{\beta\mu})\Gamma^\lambda_{\alpha\nu} - \frac{g^{\sigma\epsilon}}{2}(\partial_\alpha g_{\nu\epsilon} + \partial_\nu g_{\alpha\epsilon} - \partial_\epsilon g_{\alpha\nu})\Gamma^\lambda_{\beta\nu}\right) + \\
& + \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right) = \\
=& g_{\sigma\lambda}\left(\Gamma^\lambda_{\beta\mu}\Gamma^\lambda_{\alpha\nu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\lambda_{\beta\nu}\right) + \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right)
\end{aligned}$$

(b) É direto que

$$\begin{aligned}
R_{\mu\nu\alpha\beta} &= g_{\sigma\lambda}\left(\Gamma^\lambda_{\beta\mu}\Gamma^\lambda_{\alpha\nu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\lambda_{\beta\nu}\right) + \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right) = \\
&= -g_{\sigma\lambda}\left(\Gamma^\lambda_{\alpha\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\lambda_{\beta\mu}\Gamma^\lambda_{\alpha\nu}\right) - \frac{1}{2}\left(g_{\alpha\mu,\nu\beta} - g_{\alpha\nu,\mu\beta} + g_{\nu\beta,\alpha\mu} - g_{\mu\beta,\alpha\nu}\right) = -R_{\mu\nu\beta\alpha},
\end{aligned}$$

e

$$\begin{aligned}
R_{\mu\nu\alpha\beta} &= g_{\sigma\lambda}\left(\Gamma^{\lambda}_{\beta\mu}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\lambda}_{\alpha\mu}\Gamma^{\lambda}_{\beta\nu}\right) + \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right) = \\
&= -g_{\sigma\lambda}\left(\Gamma^{\lambda}_{\beta\nu}\Gamma^{\lambda}_{\alpha\mu} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\lambda}_{\beta\mu}\right) - \frac{1}{2}\left(g_{\alpha\mu,\nu\beta} - g_{\alpha\nu,\mu\beta} + g_{\nu\beta,\alpha\mu} - g_{\mu\beta,\alpha\nu}\right) = -R_{\nu\mu\alpha\beta}.
\end{aligned}$$

(c) A partir das propriedades de simetria do símbolo Christoffel, da métrica e da ordem de derivadas

$$\begin{aligned}
R_{\mu\nu\alpha\beta} &= g_{\sigma\lambda}\left(\Gamma^{\lambda}_{\beta\mu}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\lambda}_{\alpha\mu}\Gamma^{\lambda}_{\beta\nu}\right) + \frac{1}{2}\left(g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta} + g_{\mu\beta,\alpha\nu} - g_{\nu\beta,\alpha\mu}\right) = \\
&= g_{\sigma\lambda}\left(\Gamma^{\lambda}_{\nu\alpha}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\lambda}_{\nu\beta}\Gamma^{\lambda}_{\mu\alpha}\right) + \frac{1}{2}\left(g_{\beta\mu,\nu\alpha} - g_{\beta\nu,\mu\alpha} + g_{\nu\alpha,\beta\mu} - g_{\mu\alpha,\beta\nu}\right) = R_{\alpha\beta\mu\nu}.
\end{aligned}$$

(d) Temos, também usando propriedades do Símbolo de Christoffel, da métrica e das derivadas

$$\begin{aligned}
R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = \quad & g_{\sigma\lambda}\left(\cancel{\Gamma^{\lambda}_{\beta\mu}\Gamma^{\lambda}_{\alpha\nu}} - \cancel{\Gamma^{\lambda}_{\alpha\mu}\Gamma^{\lambda}_{\beta\nu}}\right) + \frac{1}{2}\left(\cancel{g_{\alpha\nu,\mu\beta}} - \cancel{g_{\alpha\mu,\nu\beta}} + \cancel{g_{\mu\beta,\alpha\nu}} - \cancel{g_{\nu\beta,\alpha\mu}}\right) + \\
& + g_{\sigma\lambda}\left(\cancel{\Gamma^{\lambda}_{\alpha\mu}\Gamma^{\lambda}_{\nu\beta}} - \cancel{\Gamma^{\lambda}_{\nu\mu}\Gamma^{\lambda}_{\alpha\beta}}\right) + \frac{1}{2}\left(\cancel{g_{\nu\beta,\mu\alpha}} - \cancel{g_{\nu\mu,\beta\alpha}} + \cancel{g_{\mu\alpha,\nu\beta}} - \cancel{g_{\beta\alpha,\nu\mu}}\right) + \\
& + g_{\sigma\lambda}\left(\cancel{\Gamma^{\lambda}_{\nu\mu}\Gamma^{\lambda}_{\beta\alpha}} - \cancel{\Gamma^{\lambda}_{\beta\mu}\Gamma^{\lambda}_{\nu\alpha}}\right) + \frac{1}{2}\left(\cancel{g_{\beta\alpha,\mu\nu}} - \cancel{g_{\beta\mu,\alpha\nu}} + \cancel{g_{\mu\nu,\beta\alpha}} - \cancel{g_{\alpha\nu,\beta\mu}}\right) = 0
\end{aligned}$$

(e) Essas propriedades de simetria e anti-simetria do tensor de Riemman fornecem vínculos que nos permitem calcular o número de componentes livres do tensor. Se a dimensão de um espaço é d, a princípio o número de componentes de $R_{\mu\nu\alpha\beta}$ é d^4.

(1) $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta}$: Essa condição implica que apenas $\binom{d}{2}^2$ são não nulas.

(2) $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$: Essa condição implica que eu tenho vínculo sobre $\binom{k}{2}$ onde $k = \binom{d}{2}$.

(3) $R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$: Essa condição implica que eu tenho $\binom{d}{2}$ componente não independente.

Dessa forma, o número total de componentes livres são

$$\binom{d}{2}^2 - \binom{k}{2} - \binom{d}{4} = \frac{d^2(d^2-1)}{12} \tag{20}$$

Para $d = 4$, o número de componentes independentes são 20.

Questão 08.

(a) O tensor de Riemman depende da segunda derivada da métrica que, por conta da curvatura, é em geral, não nula. Caso o tensor de Riemman fosse nulo numa região localmente inercial do espaço-tempo, ele seria nulo em todo espaço.

(b) Numa região local do espaço-tempo, os símbolos de Christoffel somem, mas as suas derivadas não, então é possível mostrar a identidade.