26/03/2020

$$\nabla^2\phi = 0 \quad , \quad \vec{V} = \nabla\phi$$

Far-field ⇒ Perturbation velocity vanishes

$$\lim_{\vec{r}\to\infty} \vec{V} = 0$$

One is left with undisturbed Flow

Body velocity $\vec{U}$

fluid has induced velocity $\vec{V}$

Lab. Frame ⇒ inertial

$$F(\vec{x}, t)\Big|_B = 0 \qquad \text{Level Surface: } \hat{n}\Big|_B = \frac{\nabla F}{\|\nabla F\|}\Bigg|_B$$

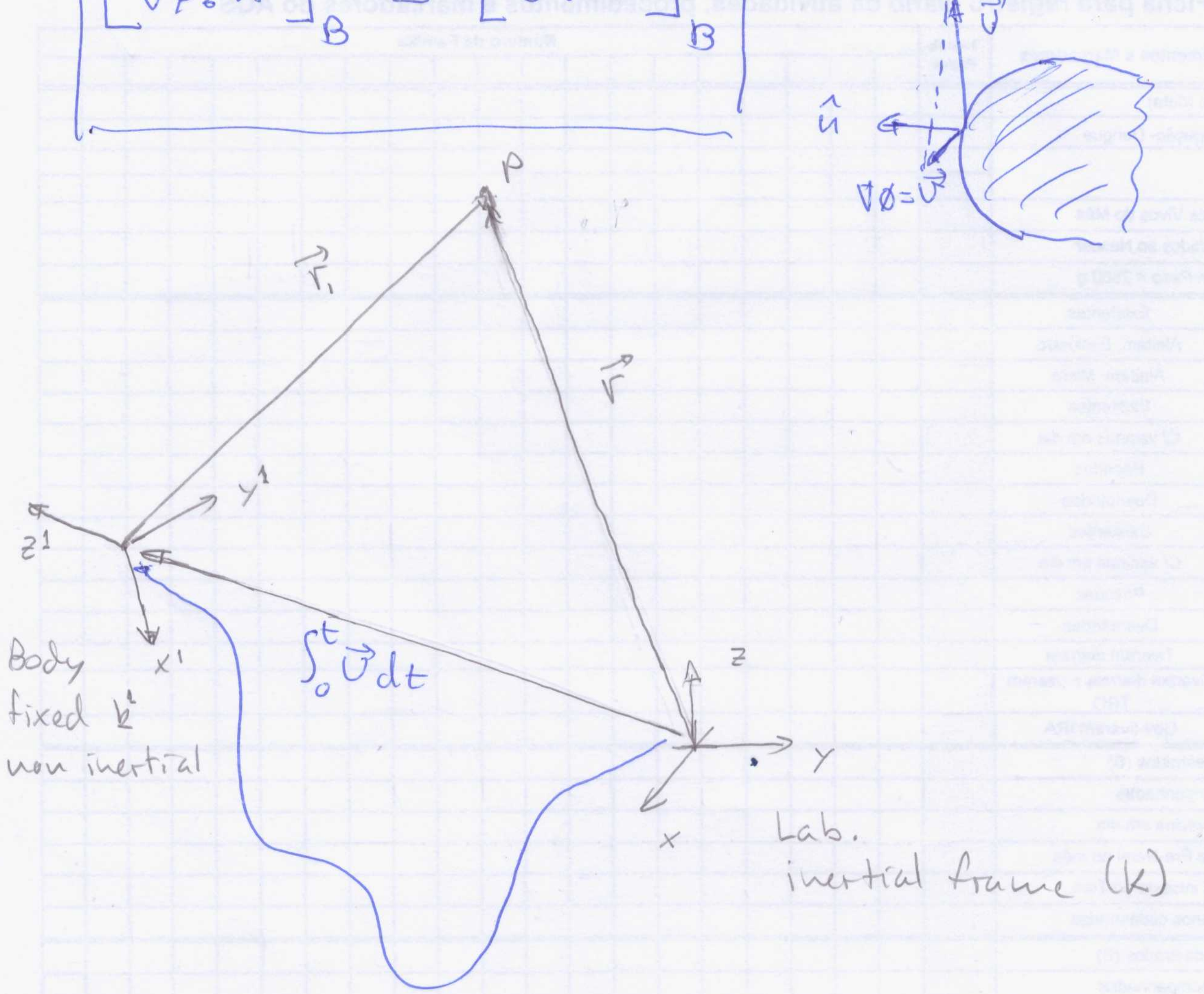
$$\text{Lagrange: } \frac{DF}{Dt} = 0$$

$$\text{Euler: } \frac{\partial F}{\partial t} + \underset{-\vec{U}}{\nabla\phi}\cdot\nabla F = 0$$

$$\frac{DF}{Dt}\Big|_B = 0 \Rightarrow \cancel{\frac{\partial F}{\partial x^i}\frac{\partial x^i}{\partial t}} + \nabla\phi\cdot\nabla F = 0$$

$$\nabla F$$

$$\boxed{\left[\nabla\phi\cdot\hat{n}\right]_B = \left[\vec{U}\cdot\hat{n}\right]_B}$$



$$P(\vec{r},t) = P_\infty - \rho\left[\frac{\partial\phi}{\partial t} + \frac{\|\nabla\phi\|^2}{2}\right]$$

$$\vec{r}^{(1)} = \vec{r} - \int_0^t \vec{U}(\tau)\,d\tau \quad , \quad t_1 = t$$

$$\begin{cases} \phi_1(\vec{r}^1, t_1) = \phi(\vec{r},t) \\ F_1(\vec{r}^1, t_1) = F(\vec{r},t) \\ P_1(\vec{r}^1, t_1) = P(\vec{r},t) \end{cases}$$

Scalar functions

Let's assume that at the time we are considering now, at this moment, both frames coincide:

$t_1 = t$

$$\nabla_1(\circ\circ\circ) = \nabla(\circ\circ\circ) \; ; \; \nabla_1^2(\circ\circ\circ) = \nabla^2(\circ\circ\circ) \; ; \; \frac{\partial t}{\partial t_1} = 1$$

$$\frac{\partial(\circ\circ\circ)}{\partial t_1} = \frac{\partial(\circ\circ\circ)}{\partial t} + U \cdot \nabla(\circ\circ\circ) \Rightarrow \frac{\partial(\circ\circ\circ)}{\partial t} = \frac{\partial(\circ\circ\circ)}{\partial t_1} - \vec{U} \cdot \nabla_1(\circ\circ\circ)$$

$$P_\infty - P = \rho \left[\overbrace{\frac{\partial \phi}{\partial t_1} - \vec{U} \cdot \nabla_1 \phi}^{\partial\phi/\partial t} + \frac{\| \nabla_1 \phi \|^2}{2} \right]$$

$$\boxed{\frac{P_\infty - P}{\rho} = \frac{\partial \phi}{\partial t} - \vec{U} \cdot \vec{q}_1 + \frac{\| \vec{q}_1^{\,2} \|}{2}}$$

$$C_p = \frac{-2}{U_\infty^2} \left[\frac{\partial \phi}{\partial t_0} - \vec{U} \cdot \vec{q} + \frac{\| \vec{q} \|^2}{2} \right]$$

$$\vec{F} = -\oiint_{S_B} p\,\hat{n}\,ds = \oiint_{S_B} \rho\,\frac{\partial\phi}{\partial t}\,\hat{n}\,ds + \oiint_{S_B} \rho\left[\frac{q^2}{2} - \vec{U}\cdot\vec{q}\right]\hat{n}\,ds$$

$$\vec{F} = \frac{\partial}{\partial t}\oiint_{S_B} \rho\,\phi\,\hat{n}\,ds + \oiint_{S_B} \rho\left[\frac{q^2}{2} - \vec{U}\cdot\vec{q}\right]\hat{n}\,ds$$

$$\vec{U}\times(\hat{n}\times\vec{q}) = (\vec{U}\cdot\vec{q})\,\hat{n} - (\vec{U}\cdot\hat{n})\,\vec{q}$$

$$= (\vec{U}\cdot\vec{q})\,\hat{n}_{B} - (\vec{q}\cdot\hat{n})_{B}\,\vec{q}$$

Pure translation

$$\vec{F} = \frac{\partial}{\partial t}\oiint_{S_B} \rho\,\phi\,\hat{n}\,ds + \oiint_{S_B} \rho\left[\frac{q^2}{2}\hat{n} - (\vec{q}\cdot\hat{n})\,\vec{q}\right]ds - \rho\,\vec{U}\times\oiint_{S_B}(\hat{n}\times\vec{q})\,d$$

$$\oiint_{S_B}\left[\frac{(\vec{q}\cdot\vec{q})}{2}\hat{n} - (\vec{q}\cdot\hat{n})\,\vec{q}\right]ds + \underbrace{\iint_{\substack{\Sigma \\ \text{far field} \\ \vec{r}\to\infty}}\left[\frac{(\vec{q}\cdot\vec{q})}{2}\hat{n} - (\vec{q}\cdot\hat{n})\,\vec{q}\right]ds}_{\text{vanishes as } \lim_{\vec{r}\to\infty}\vec{q} = 0} =$$

$$= \iint_{\substack{D \\ \text{Flow}}}\left[\vec{q}\cdot\nabla\vec{q} - \vec{q}\cdot\nabla\vec{q}\right]d\forall = 0$$

$$\oint_{S_B} \left[\frac{q^2}{2}\hat{u} - (\vec{q}\cdot\hat{u})\vec{q} \right] ds = 0$$

$$\boxed{\vec{F} = \frac{\partial}{\partial t}\oiint_{S_B} \rho\phi\hat{u}\, ds - \rho\vec{U}\times\oiint_{S_B}(\hat{u}\times\vec{q})\, ds}$$

D'Allambert's Paradox

$$\phi = \phi_1 + \phi_2 + \phi_3$$

$$\nabla^2 \phi = 0 \Rightarrow \nabla^2 \phi_i = 0 \quad i = 1, 2, 3$$

$$\nabla^2 \phi_i = 0 \; ; \; \nabla \phi_i \cdot \hat{n}_B = \left(U_i n_i \right)_B$$

no summation convention!

Furthermore, we define unitary potentials, φ_i, of the form:

$$\phi_i = U_i \varphi_i \Rightarrow \nabla^2 \varphi_i = 0$$

$$\nabla \varphi_i \cdot \hat{n} = \frac{\partial \varphi_i}{\partial n} = n_i \quad \text{on } S_B$$

$$\phi = U_1 \varphi_1 + U_2 \varphi_2 + U_3 \varphi_3 = \vec{U} \cdot \hat{\varphi}$$

↳ General translating motion.

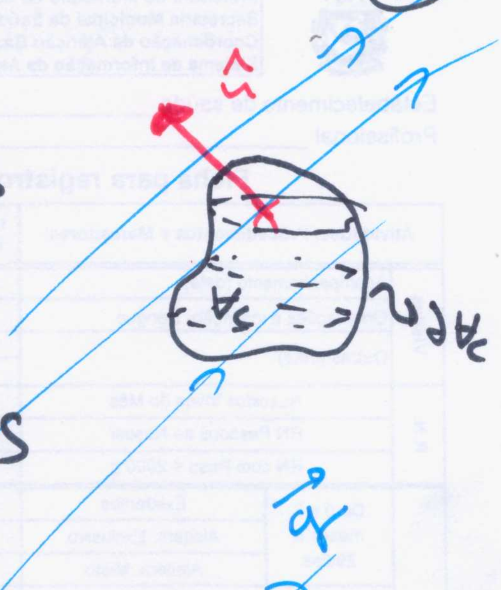
$$\vec{F}_e = \frac{d}{dt}\left(m_B \vec{U} - \oiint_{S_B} \rho \phi \hat{n} \, ds \right) = \left(m \delta_{ik} + m_{ik} \right) \frac{dU_k}{dt}$$

$$m_{ik} = m_{ki} = - \oiint_{S_B} \rho \varphi_k n_i \, ds = - \oiint \rho \varphi_k \frac{\partial \varphi_i}{n} ds$$

T. Gauss

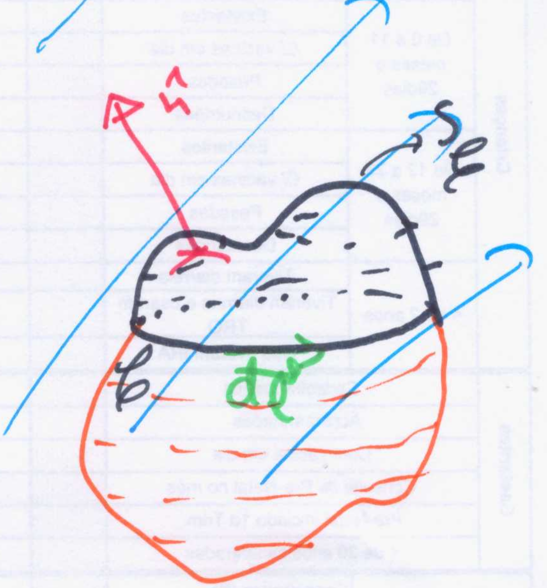
$$\oiint_{\partial \forall_c} \vec{q}\cdot\hat{u}\, ds = \oiiint_{\forall_c} \nabla\cdot\vec{q}\, ds$$

$$\oiint_{\partial \forall_c} \hat{n}\times\vec{q}\, ds = \oiiint_{\forall_c} \nabla\times\vec{q}\, ds$$



Stokes Theorem:

$$\oint_{\ell} \vec{q}\cdot\vec{dl} = \oiint_{S_\ell} (\nabla\times\vec{q})\cdot\hat{n}\, ds$$



Circulation definition:

$$\Gamma \equiv \oint_{\ell} \vec{q}\cdot\vec{dl} = \oiint_{S_\ell} (\nabla\times\vec{q})\cdot\hat{n}\, ds$$

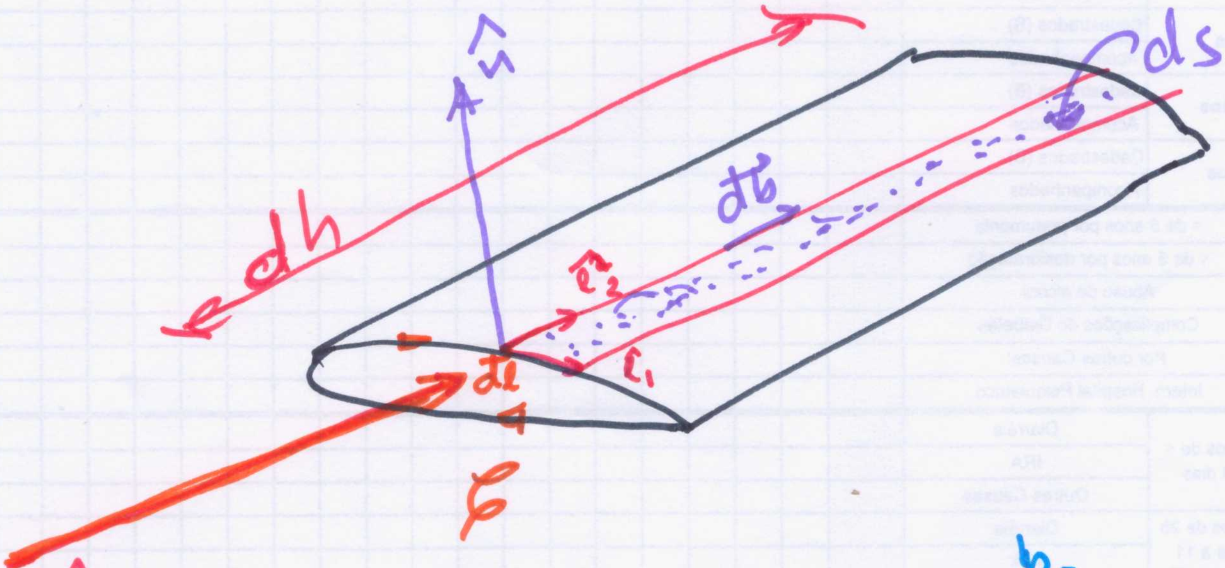
notice that through Gauss' Theorem this is related to $\oiint \hat{n}\times\vec{q}\, ds$

$$\vec{F}=\frac{d}{dt}\oiint_{S_B}\rho\phi\hat{n}\,ds-\rho\vec{U}\times\overbrace{\oiint_{S_B}(\hat{n}\times\vec{q})\,ds}^{\text{Ⓘ}}$$

$$\vec{I}\equiv\oiint_{S_B}(\hat{n}\times\vec{q})\,ds$$

$$\hat{\ell}\cdot\vec{I}=\hat{\ell}\cdot\oiint_{S_B}(\hat{n}\times\vec{q})\,ds=\int_{h_1}^{h_2}\left[\oint_{\ell}\vec{q}\cdot d\vec{l}\right]dh=$$

$$=\int_{h_1}^{h_2}\Gamma_{\ell}(h)\,dh$$



$$\rho\vec{U}\times\oiint\hat{n}\times\vec{q}\,ds=\rho\vec{U}\times\int^{b_2}\Gamma(b)\,d\vec{b}$$

(16/04/2020)

$\vec{z} = x + iy \qquad i = \sqrt{-1}$

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} \Rightarrow e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^{k}}{k!}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{(2k+1)} y^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{2k}}{2k!} + i \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{2k+1}}{(2k+1)!}$$

$$e^{iy} = \cos(y) + i \sin(y)$$

$$\cancel{\#} \quad z = r e^{i\theta} = r(\cos\theta + i \sin\theta)$$

$$r = |z| = +\sqrt{x^{2}+y^{2}}$$

$$\theta = \arg(z)$$

Analytic Functions

$$f(z) = u(x,y) + iv(x,y)$$

Cauchy-Riemann Conditions
$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial u}{\quad} = -\dfrac{\partial v}{\quad} \end{array} \right.$$

$f'(z) = u_x + i v_x$

$f'(z) = v_y - i u$

$$f(z) = u(x,y) + i v(x,y)$$

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y \partial x} \\ \\ \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial xy} \end{array}\right\} \begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \nabla^2 u = 0 \\ \text{Similarly:} \\ \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \nabla^2 v = 0 \end{array}$$

Cauchy-Riemann in polar form:

$x = r\cos\theta$

$y = r\sin\theta$

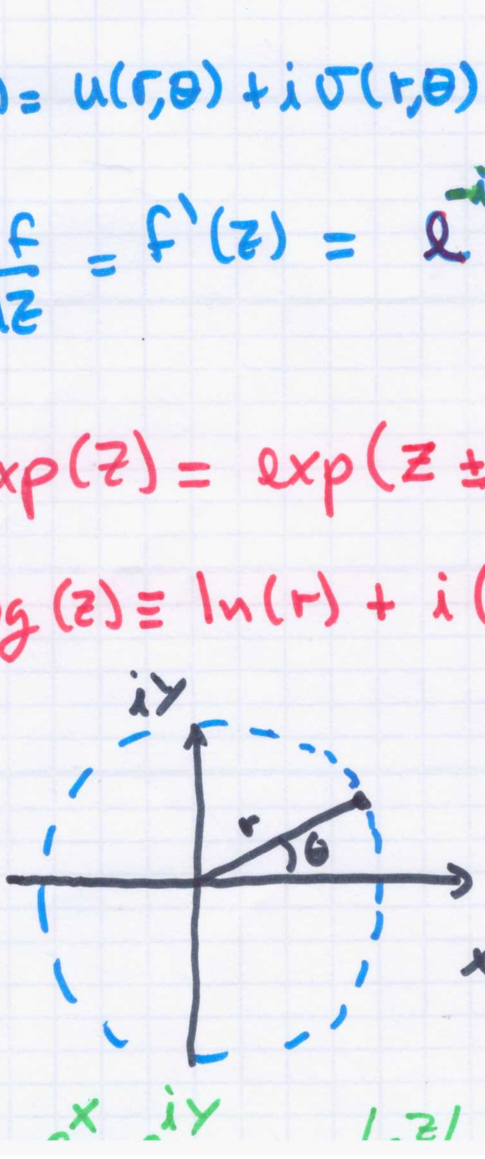
$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{cases}$$

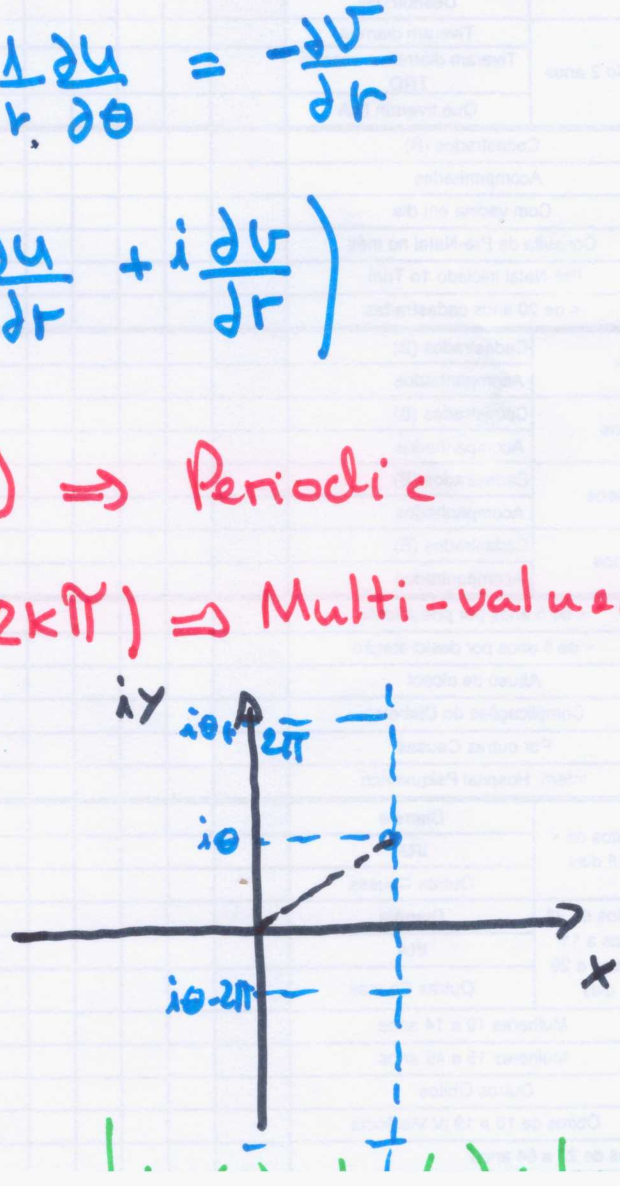
$f(z) = u(r,\theta) + i v(r,\theta)$

$$\frac{df}{dz} = f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$\exp(z) = \exp(z \pm k2\pi i) \Rightarrow$ Periodic

$\log(z) \equiv \ln(r) + i(\theta \pm 2k\pi) \Rightarrow$ Multi-valued





e^{z} e^{x} e^{iy} $|e^{z}|$ e^{x} $(e^{i\theta})$

$$\exp[\log(z)] = z$$

- $$\log[\exp(z)] = z \pm i2k\pi$$

Cylindrical Coordinates

$$\nabla(\ldots) = \frac{\partial(\ldots)}{\partial r}\hat{\imath}_r + \frac{1}{r}\frac{\partial(\ldots)}{\partial\theta}\hat{\imath}_\theta + \frac{\partial(\ldots)}{\partial z}\hat{\imath}_z$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Spherical Coordinates:

$$\nabla(\ldots) = \frac{\partial(\ldots)}{\partial r}\hat{\imath}_r + \frac{1}{r}\frac{\partial(\ldots)}{\partial\theta}\hat{\imath}_\theta + \frac{1}{r\sin\theta}\frac{\partial(\ldots)}{\partial\varphi}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) +$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

Point Source: (3-D)

$$\phi = -\frac{\sigma}{4\pi r}$$

$$\vec{r} = (x, y, z)^T$$

$$r = \|\vec{r}\|$$

$\sigma \Rightarrow$ intensity

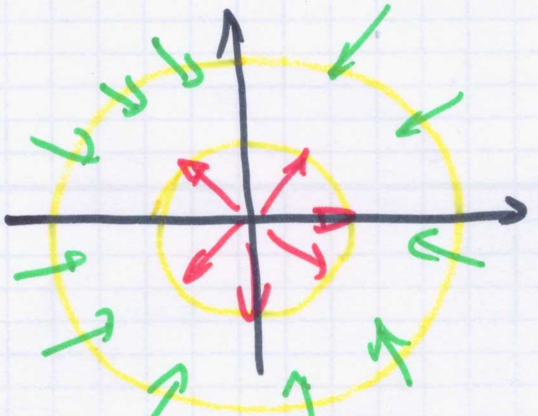
Laplace eq.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\sigma}{4\pi r^2}\right) = 0$$ okay

$$\vec{q} = -\frac{\sigma}{4\pi}\nabla\left(\frac{1}{r}\right) = \frac{\sigma\vec{r}}{4\pi r^3}$$

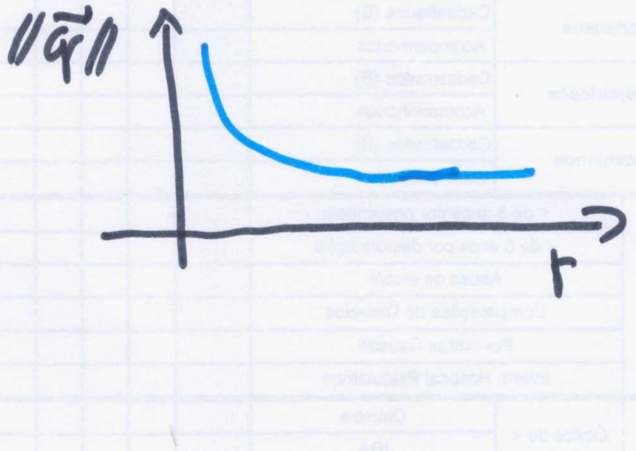
$$\boxed{r \neq 0 \quad (r > 0)}$$

$$\vec{q} = (q_r, q_\theta, q_\varphi)^T \Rightarrow \left(\frac{\sigma}{4\pi r^2}, 0, 0\right)^T$$

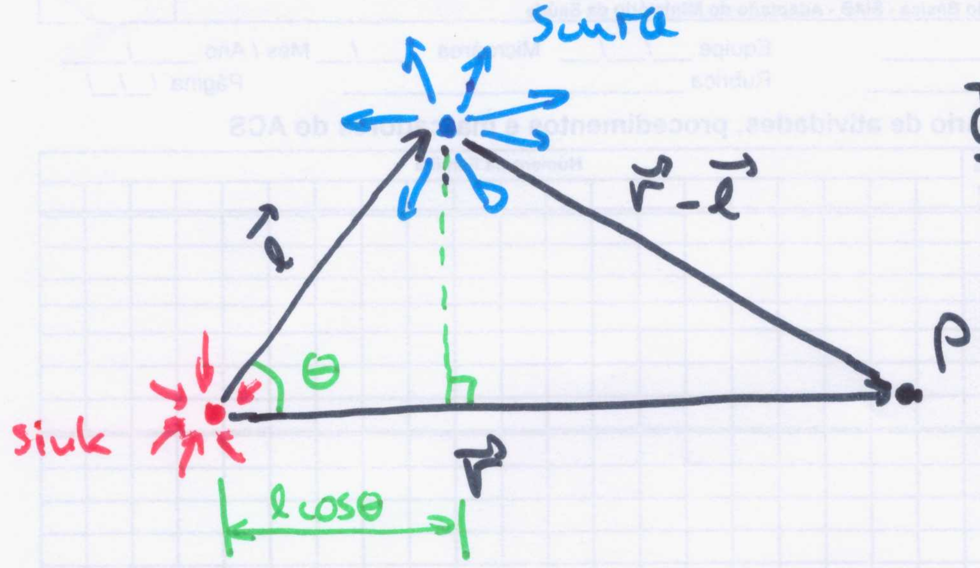


$\sigma > 0 \Rightarrow$ source

$\sigma < 0 \Rightarrow$ sink



Doublet.



$$\Phi_p = \frac{\sigma}{4\pi}\left(\frac{1}{\|\vec{r}\|} - \frac{1}{\|\vec{r}-\vec{\ell}\|}\right)$$

limit where

$\vec{\ell} \to 0$; $\|\vec{\ell}\| = \ell$

$\sigma \to \infty$

and $\sigma\ell \to \mu$

$$\Phi = \lim_{\substack{\ell\to 0\\ \sigma\to\infty\\ \sigma\ell\to\mu}} \frac{\sigma}{4\pi}\left(\frac{\|\vec{r}-\vec{\ell}\| - \|\vec{r}\|}{\|\vec{r}\|\,\|\vec{r}-\vec{\ell}\|}\right) = ?$$

$\lim_{\ell\to 0} \|\vec{r}\|\,\|\vec{r}-\vec{\ell}\| = r^2$ and $\lim_{\ell\to 0}\left\{\|\vec{r}-\vec{\ell}\| - \|\vec{r}\|\right\} = -\ell\cos\theta$

$$\|\vec{r}-\vec{\ell}\|^2 = \|\vec{r}\|^2 + \|\vec{\ell}\|^2 - 2\|\vec{r}\|\,\|\vec{\ell}\|\cos\theta$$

$$\|\vec{r}\| = r \;, \|\vec{\ell}\| = \ell$$

$$\|\vec{r}-\vec{\ell}\|^2 - \|\vec{r}\|^2 = \ell^2 - 2r\ell\cos\theta$$

$$\|\vec{r}-\vec{\ell}\| - \|\vec{r}\| = \frac{\ell^2 - 2r\ell\cos\theta}{\|\vec{r}-\vec{\ell}\| + \|\vec{r}\|}$$

[illegible] $\ll 1 \Rightarrow \|\vec{r}-\vec{\ell}\| - \|\vec{r}\| \simeq \underline{-2r\ell\cos\theta} =$

$$\lim_{\substack{l\to 0\\ \sigma\to\infty\\ \sigma_l\to\mu}} \Phi_p = \lim_{\substack{l\to 0\\ \sigma\to\infty\\ \sigma_l\to\mu}} -\frac{\sigma l\cos\theta}{4\pi r^2} = -\frac{\mu\cos\theta}{4\pi r^2}$$

$$\Phi_{\text{Dipole}} = \frac{-\mu\cos\theta}{4\pi r^2} = \frac{-\vec{\mu}\cdot\vec{r}}{4\pi r^3}$$

$$\vec{\mu}\cdot\vec{r} = \mu r\cos\theta$$

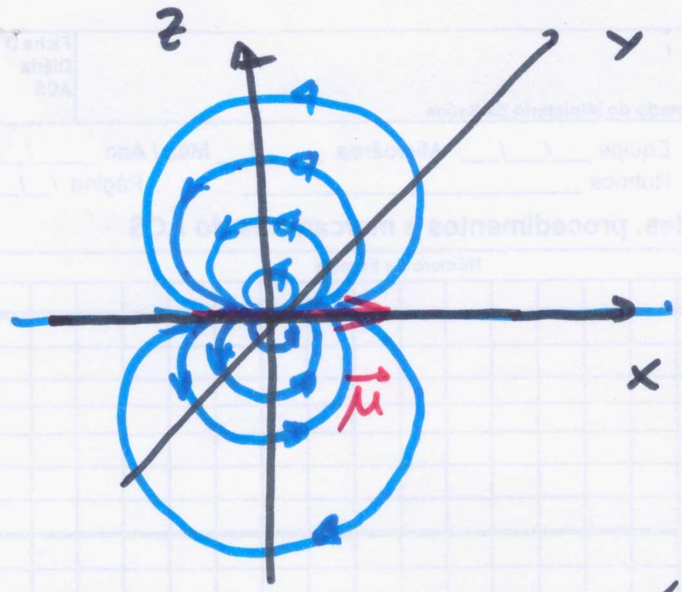
$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\mu\cos\theta}{2\pi r^3}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta\,\mu}{4\pi r^2}\right) =$$

$$[\sin^2\theta]' = 2\sin\theta\cos\theta$$

$$-\frac{\mu\cos\theta}{2\pi r^4} + \frac{\mu\cos\theta}{2\pi r^4} = 0$$

$$q_r = \frac{\partial\Phi}{\partial r} = \frac{\mu\cos\theta}{2\pi r^3}\;; \quad q_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{\mu\sin\theta}{4\pi r^3}$$

$$q_\varphi = \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi} = 0$$



Polynomials

$$\Phi = Ax + By + Cz$$

↳ 1st order

$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} = A = U_\infty \\ v = \dfrac{\partial \phi}{\partial y} = B = V_\infty \\ w = \dfrac{\partial \Phi}{\partial z} = C = W_\infty \end{cases}$$

$$\Phi = Ax^2 + By^2 + Cz^2 \Rightarrow \nabla^2 \phi = A + B + C = 0$$

$\Rightarrow$ for $B = 0$, $A = -C$

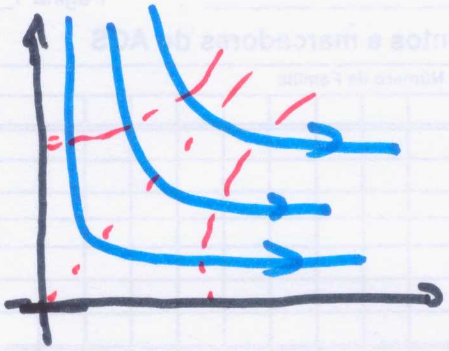
$$\Phi = A(x^2 - z^2) \Rightarrow \begin{cases} u = 2Ax \\ v = 0 \\ w = -2Az \end{cases}$$

Stream line function: $\left.\right\}$ $\vec{q} \times \vec{dl} = 0$

Stream line element: $\vec{dl}$

Stream line equation: $\boxed{\dfrac{dx}{} = \dfrac{dy}{} = \dfrac{dz}{}}$

$$\frac{dx}{2Ax} = \frac{dz}{-2Az} \Rightarrow xz = \text{constant along streamlines}$$



Corner Flow

2-D source : Cylindrical / polar coordinat

$q_\theta = 0$

Vorticity: $\omega_z = -\frac{1}{r}\left(\frac{\partial(r q_\theta)}{\partial r} - \frac{\partial(q_r)}{\partial \theta}\right) =$

$$= \frac{1}{r}\frac{\partial(q_r)}{\partial \theta} = 0 \Rightarrow q_r(r)$$

Continuity: $\nabla . \vec{q} = \frac{1}{r}\frac{\partial(r q_r)}{\partial r} = 0 \Rightarrow r . q_r = ct$

$$r . q_r = \frac{Q}{2\pi} = \frac{\Gamma}{2\pi}$$

Q ⇒ Volumetric flow rate : $Q = \Gamma = \int_0^{2\pi} q_r r d\theta = 2\pi q_r r$

$$\left.\begin{aligned} q_r &= \frac{\partial \phi}{\partial r} = \frac{\Gamma}{2\pi r} \\ q_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \end{aligned}\right\} \quad \Phi = \frac{\Gamma \ln(r)}{2\pi} + \cancelto{0}{C}$$

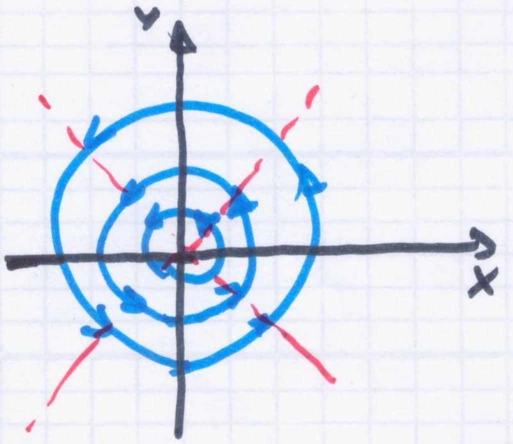
$$\boxed{\Phi = \underline{\Gamma} \ln(r)}$$

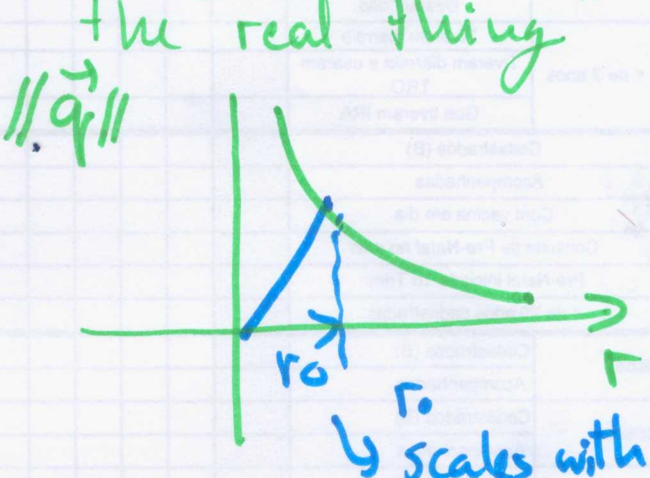
2-D Doublet (similar reasoning)

$$\vec{\mu} = (\mu, 0) \Longrightarrow \begin{cases} \Phi(r,\theta) = -\dfrac{\mu\cos\theta}{2\pi r} \\ q_r = \dfrac{\partial\Phi}{\partial r} = \dfrac{\mu\cos\theta}{2\pi r^2} \\ q_\theta = \dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta} = \dfrac{\mu\sin\theta}{2\pi r^2} \end{cases}$$

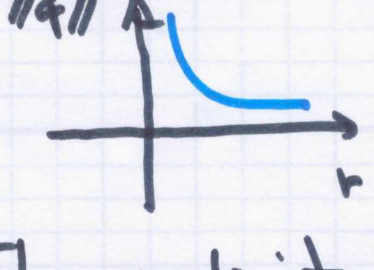
$$\boxed{\Phi = -\frac{\vec{\mu}\cdot\vec{r}}{2\pi r^2}}$$

Vortex (2-D point vortex)





r_0 ↳ scales with "viscosity" Kolmogorov



The vorticity is zero everywhere, except for a single point, at the origin in this particular case.

$$\omega_z = -\frac{1}{r}\left[\frac{\partial(rq_\theta)}{\partial r} - \frac{\partial q_r}{\partial\theta}\right] = -\frac{1}{r}\frac{\partial}{\partial r}(rq_\theta) = 0$$

$$r q_\theta = A$$

$$r \to \infty \Rightarrow q_\infty \to 0$$

$$\Gamma = \oint \vec{q} \cdot \vec{dl} = \int_0^{2\pi} q_\theta . r \, d\theta = 2\pi A$$

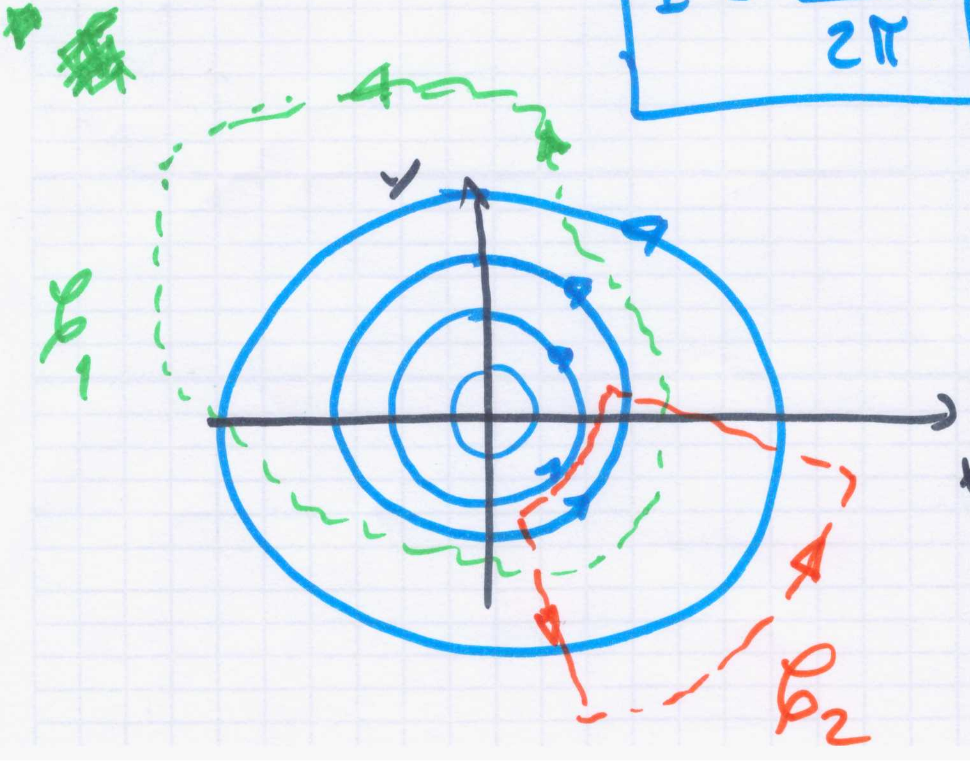
→ circle with radius "r" center at the origin and oriented counter_clockwise (positive)

$$A = \frac{\Gamma}{2\pi}$$

$$\left.\begin{cases} q_r = 0 \\ q_\theta = \dfrac{\Gamma}{2\pi r} \end{cases}\right\} \Rightarrow \phi = \int q_\theta . r \, d\theta + c$$

$$\phi = \frac{\Gamma \theta}{2\pi} + \cancel{c}^{\,0}$$

$$\boxed{\Phi = \frac{\Gamma \theta}{2\pi}}$$



$$\oint_{C_1} \vec{q} \cdot \vec{dl} = \Gamma$$

$$\oint_{C_2} \vec{q} \cdot \vec{dl} = 0$$

Complex Potential: (2-D)

$$F(z) \equiv \underbrace{\Phi(x,y)}_{\text{Potential function}} + i\,\underbrace{\Psi(x,y)}_{\text{Stream function}}$$

C.R : $\Phi_{,x} = \psi_{,y} = u$

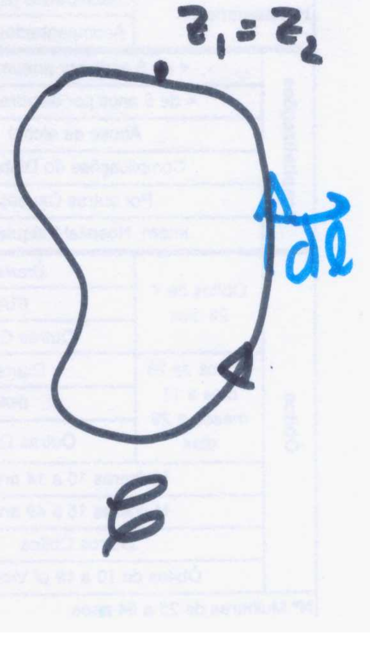
$\Phi_{,y} = -\psi_{,x} = v$

$$F'(z) = \frac{dF}{dz} = \phi_{,x} + i\,\psi_{,x} = \psi_{,y} - i\,\Phi_{,y} = u - iv$$

$F'(z) = u - iv = \overline{U}(z) \equiv W$ complex velocit

$$\boxed{\frac{dF}{dz} = u - iv = W = \overline{U}}$$

$$\Gamma = \oint_C \vec{q}\cdot d\vec{l} = \oint_C u\,dx + v\,dy = \oint_C d\Phi$$



$$Q = \oint_C (u\,dy - v\,dx) = \oint_C d\psi$$

$$\oint_{C} w(z)\,dz = \oint_{C} (u - iv)(dx + idy) =$$

$$= \oint_{C} (u\,dx + v\,dy) + i \oint_{C} (u\,dy - v\,dx) =$$

$$= \oint_{C} d\Phi + i \oint_{C} d\Psi = \Gamma + iQ$$

$$\oint_{C} w\,dz = \Gamma + iQ = F(z_2) - F(z_1)$$

$$\nabla\phi = \vec{u} \parallel \Psi\Big|_{\text{constant}} \qquad \nabla\phi \perp \nabla\Psi$$

$$\nabla\Phi = \vec{u} \perp \Phi\Big|_{\text{constant}}$$

$$\Psi\Big| \quad \perp \quad \Phi\Big|$$

Flow	$F(z)$	$W(z)$
Uniform	Az	$A \in \mathbb{C}$
Corner of Angle $\theta = \frac{\pi}{n}$	Az^n	Anz^{n-1}
Source at $z=0$ $Q = 2\pi A$ $A \in \mathbb{R}$	$A \log(z)$ $\frac{Q}{2\pi} \log(z)$	$\frac{A}{z}$ $\frac{Q}{2\pi z}$
Vortex at $z=0$ Γ	$-\frac{i\Gamma}{2\pi} \log(z)$	$\frac{i\Gamma}{2\pi z}$
Doublet at $z=0$ $\vec{\mu} \parallel \hat{x}$ direction μ [illegible]	$-\frac{\mu}{2\pi z}$	$\frac{\mu}{2\pi z^2}$

(16/04/2020)

①

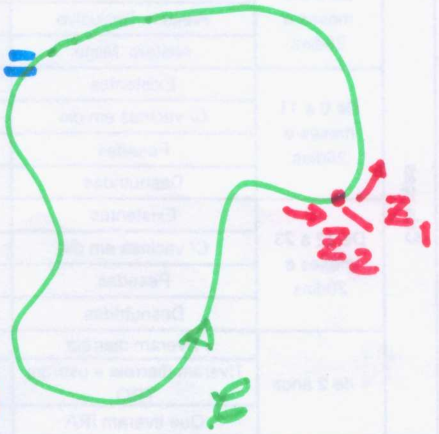
$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} \\[2ex] v = \dfrac{\partial \Phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

$$F = \Phi + i\Psi$$

$$\frac{dF}{dz} = W = u - iv$$

$$U = \overline{W} = u + iv$$

$$\oint_C W(z)dz = \oint_C (u-iv)(dx+idy) =$$

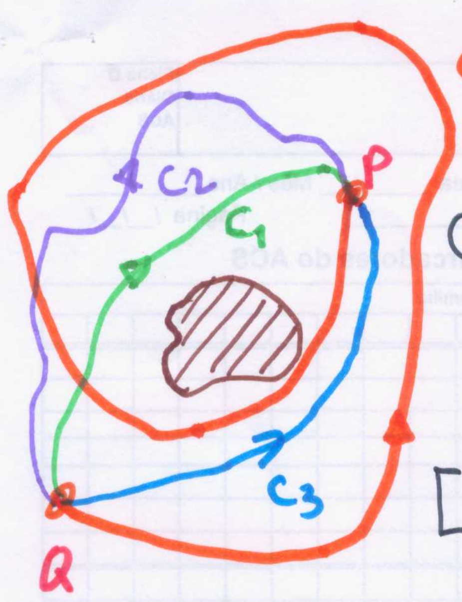


$$= \oint_C (udx + vdy) + i\int_C (udy - vdx)$$

$$= \oint_C d\phi + i\oint_C d\psi = \Gamma + i\,Q.$$

$$\Gamma + i\,Q = \oint_C W(z)dz = F(z_2) - F(z_1)$$

(2-D)



$$C_1 \cup C_2 : \int_{Q \atop C_1}^{P} d\phi - \int_{Q \atop C_2}^{P} = 0$$

$$[\Phi(P) - \Phi(Q)]_{C_1} = [\Phi(P) - \Phi(Q)]_{C_2}$$

$$\int_{Q \atop C_3}^{P} d\phi - \int_{Q \atop C_1}^{P} d\phi = \Gamma \Rightarrow \begin{cases} [\Phi(P) - \Phi(Q)]_{C_3} - [\Phi(P) - \Phi(Q)]_{C_1} = \Gamma \\ [\Phi(P) - \Phi(Q)]_{C_4} - [\Phi(P) - \Phi(Q)]_{C_3} = \Gamma \end{cases}$$

$C_3 \cup C_1$

$C_4 \cup C_3 \Rightarrow$

$\oplus$

$$[\Phi(P) - \Phi(Q)]_{C_4} - [\Phi(P) - \Phi(Q)]_{C_1} = 2\Gamma$$

$C_4 \cup C_1$

Then, on taking Q as a reference point, we can write:

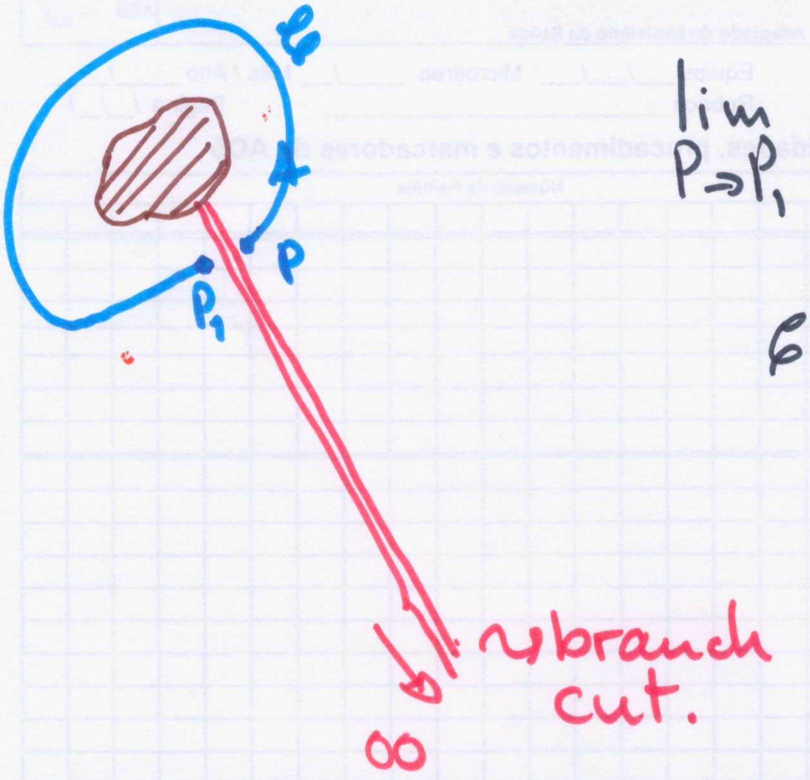
$$[\Phi(P)_{C_4} - \Phi(P)_{C_1}] = 2\Gamma + [\Phi(Q)_{C_4} - \Phi(Q)_{C_1}]$$

Reference Point $\Rightarrow \Phi(Q)_{C_4} = \Phi(Q)_{C_1}$

therefore, we get:

$$\Phi(P)_{C_4} - \Phi(P)_{C_1} = 2\Gamma$$

③



$$\lim_{p\to p_1}\int_{p}^{p_1} d\Phi = \lim_{p\to p_1}\left[\Phi(p)-\Phi(p_1)\right]_{\mathcal{C}} = \Gamma$$

$\mathcal{C}$

$$F(z) = U_\infty z + \frac{A}{z} = U_\infty(x+iy) + \frac{A}{r}e^{-i\theta}$$

$$= \underbrace{\left(U_\infty x + \frac{A}{r}\cos\theta\right)}_{\Phi} + i\underbrace{\left(U_\infty y - \frac{A}{r}\sin\theta\right)}_{\Psi}$$

$$= \Phi + i\Psi$$

$\Psi = 0$ at the circle: $\Psi = U_\infty y - \frac{A}{r}\sin\theta = 0$

$$\Psi = U_\infty r\sin\theta - \frac{A}{r}\sin\theta = 0$$

$U_\infty r^2 = A$ $\qquad r = +\sqrt{\frac{A}{U_\infty}} = +\sqrt{\frac{\mu}{2\pi U_\infty}}$

$\mu = 2\pi A$

Cylinder with Magnus effect.

$$F(z) = U_\infty z + \frac{A}{z} - ib\log\left(\frac{z}{a}\right) =$$

$$= U_\infty z + \frac{\mu}{2\pi z} + \frac{i\Gamma}{2\pi}\log\left(\frac{z}{a}\right) \qquad \text{where } a \in \mathbb{R}$$

$$F(z) = \left[U_\infty x + \frac{\mu}{2\pi r}\cos\theta - \frac{\Gamma}{2\pi}(\theta \pm 2k\pi)\right] +$$

$$+ i\left[U_\infty y - \frac{\mu \sin\theta}{2\pi r} + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)\right] = \Phi + i\Psi$$

$$\Psi = 0 \Rightarrow \text{wall} \Rightarrow \Psi = U_\infty a\sin\theta - \mu\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{a}{a}\right)$$

$(r=a)$

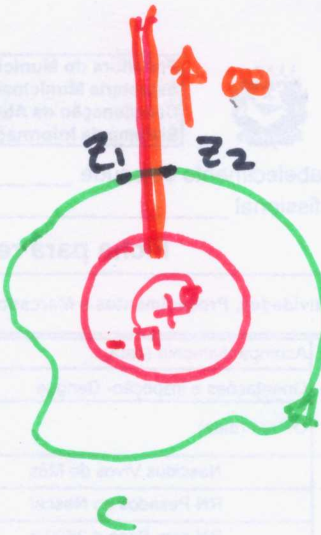
$$\Psi = 0 \Rightarrow \boxed{a = +\sqrt{\frac{\mu}{2\pi U_\infty}}}$$

same as before

Kutta-Joukowski Theorem:

$$\vec{F} = \rho\vec{U} \times \oint_{b_1}^{b_2} F(b)\,d\vec{b} \Rightarrow \boxed{\vec{L} = \rho U_\infty \Gamma}$$

$$W = \frac{dF}{dz} = U_\infty - \frac{\mu}{2\pi z^2} + \frac{i\Gamma a}{2\pi z}$$



$$W = U_\infty - \frac{\mu e^{-i2\theta}}{2\pi r^2} + \frac{i\Gamma a}{2\pi r} e^{-i\theta}$$

$$\boxed{U = \overline{W} = U_\infty - \frac{\mu e^{i2\theta}}{2\pi r^2} - \frac{i\Gamma a}{2\pi r} e^{i\theta}}$$

$$z_1 = x + iy = z_2 \qquad \begin{cases} z_1 = Re^{i\theta} \\ z_2 = Re^{i(\theta+2\pi)} \end{cases}$$

Circulation ou Γ:

$$\Gamma_T + iQ_T = \oint_C W(z)dz = F(z_2) - F(z_1) = \frac{i\Gamma}{2\pi}\left[\log\left(\frac{z_2}{a}\right) - \log\left(\frac{z_1}{a}\right)\right.$$

$$= \frac{i\Gamma}{2\pi}\log\left(\frac{z_2}{z_1}\right) = \frac{i\Gamma}{2\pi}\left\{\ln\left(\frac{R}{R}\right) + i\left(\theta + 2\pi - \theta\right)\right\}$$

$$= \frac{i\Gamma}{2\pi}\left\{i2\pi(1)\right\} = -\Gamma \Rightarrow \Gamma_T + iQ_T = -\Gamma + i0$$

Laurent Series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$$R_1 < |z - z_0| < R_2$$

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad \forall n \in \mathbb{Z}$$

two important cases:

$$[z^{-1}]' = -z^{-2} \Longrightarrow \int z^{-2}\,dz = -\frac{1}{z}$$

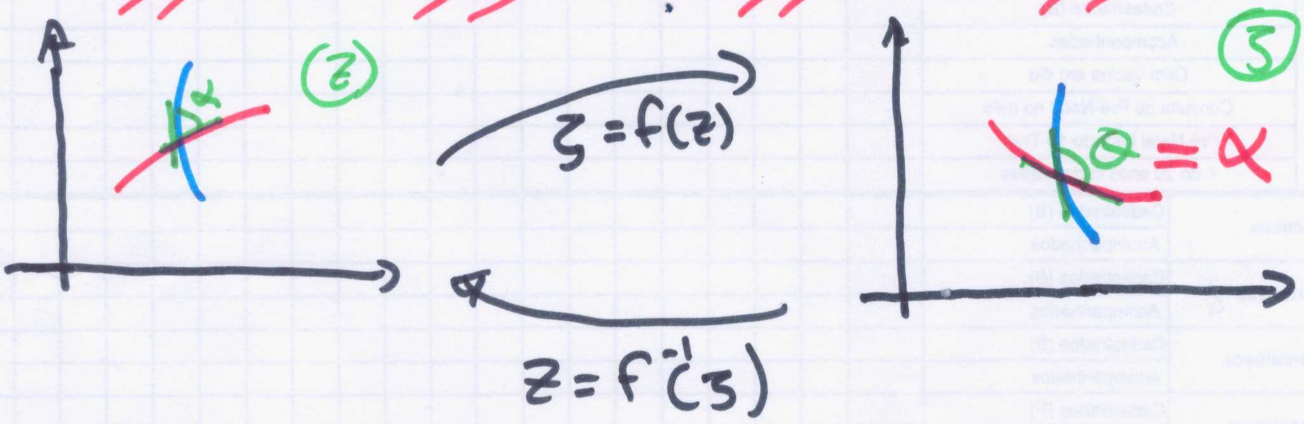
$$[z^{-2}]' = -2z^{-3} \Longrightarrow \int z^{-3}\,dz = -\frac{1}{2z^{2}}$$

* * *

$$W(z) = A_0 + \frac{A_1}{z} + \dots = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

$$F(z) = A_0 z + A_1 \log(z) - \sum_{n=2}^{\infty} \frac{A_n}{(n-1)z^{(n-1)}} + C$$

* * *



$$\delta z = \frac{d\zeta}{dz}\,\delta z \Longrightarrow \delta\zeta = \frac{df}{dz}\,\delta z$$

$$\begin{cases} |\delta\zeta| = \left|\frac{df}{dz}\right| \cdot |\delta z| \\ \arg(\delta\zeta) = \arg\left(\frac{df}{dz}\right) + \arg(\delta z) \end{cases}$$

⟹ local isotropic stretchin

local isotropic and finite rotation.

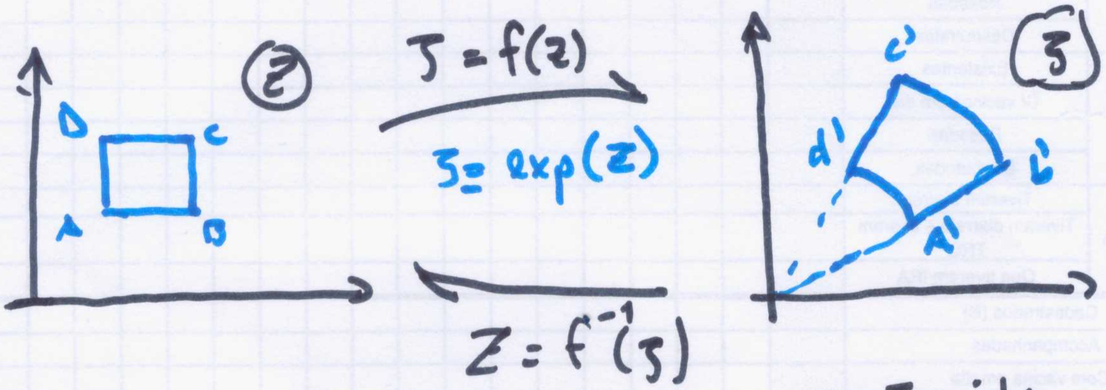
Conformal Mapping Preserves local angles

(23/04/2020) ①

$$W(z) = A_0 + \frac{A_1}{z} + \ldots = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

$$F(z) = A_0 z + A_1 \log(z) - \sum_{n=2}^{\infty} \frac{A_n}{(n-1) z^{(n-1)}} + \ldots$$

Conformal Mapping flows:



$z = x + iy$ $\qquad$ $\zeta = \xi + i\eta$

$$\left.\begin{array}{l} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{array}\right\} \Rightarrow \zeta = F(z) \rightleftharpoons z = F^{-1}(\zeta) \Rightarrow \left\{\begin{array}{l} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{array}\right.$$

$$F(z) = \Phi(x,y) + i\Psi(x,y) = \Phi(\xi,\eta) + i\Psi(\xi,\eta) \Rightarrow F(\zeta)$$

Scalar Function.

$$J = \left|\begin{array}{cc} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{array}\right|_{C.R.} = (\xi_{,x})^2 + (\xi_{,y})^2 = (\eta_{,x})^2 + (\eta_{,y})^2 = \left|\frac{d\zeta}{dz}\right|^2$$

so, whenever and wherever $J \neq 0$, we are good.

②

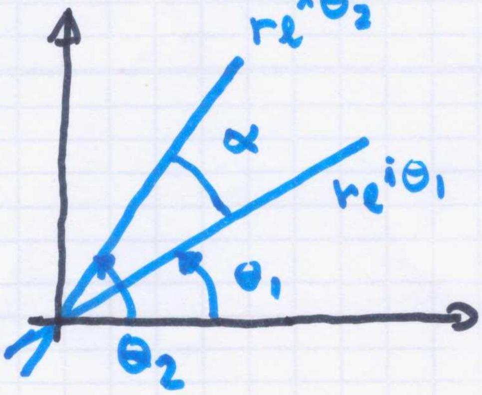
$$\frac{dF}{dz} = \frac{dF}{d\zeta}\frac{d\zeta}{dz} \iff \frac{dF}{d\zeta} = \frac{dF}{dz}\frac{1}{(d\zeta/dz)}$$

$$\nabla^2_{xy}(\dots) = \underbrace{\left|\frac{d\zeta}{dz}\right|^2}_{(J)} \nabla^2_{\xi\eta}(\dots)$$

$$\frac{dz}{d\zeta} = \frac{1}{d\zeta/dz} \quad ; \quad \tilde{W}(\zeta) = W[z(\zeta)]\frac{dz}{d\zeta}$$

At a critical point of the transformation, $\frac{d\zeta}{dz} = 0 \iff \frac{dz}{d\zeta} = \infty$

Therefore, we try to keep those points out of the region of interest, so as not to get $\tilde{W}(\zeta) \to \infty$. Unless we know that $W(z) = 0$ there: stagnation point.



$$\left.\begin{array}{l}\theta_2 - \theta_1 = \alpha \\ \zeta = z^n\end{array}\right\} \begin{array}{l}\delta\zeta = n z^{(n-1)} \delta z \\ \zeta_1 = r^n e^{i(n\theta_1)} \\ \zeta_2 = r^n e^{i(n\theta_2)}\end{array}$$

$$\underset{(z)}{\alpha} \xrightarrow{f(z)} \underset{(\zeta)}{n\alpha}$$

$$\Gamma_\zeta + i Q_\zeta = \oint_{C_\zeta} \tilde{W}(\zeta)\, d\zeta = \oint_{C_\zeta} \frac{W(z)}{(d\zeta/dz)} \frac{d\zeta}{dz}\, dz =$$

$$= \int_{C_z} W(z)\, dz = \Gamma + iQ$$

The requirement for uniform flow at $z=\infty$, further imposes the condition:

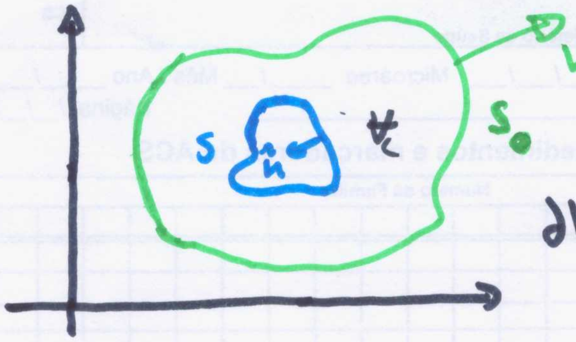
$$\zeta = \infty \Rightarrow \left\{ z = \zeta = \infty \quad \text{and} \quad \left.\frac{dz}{d\zeta}\right|_{\zeta=\infty} = 1 \right.$$

$$z(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{C_n}{\zeta^n} \Rightarrow \frac{dz}{d\zeta} = 1 - \sum_{n=1}^{\infty} n \frac{C_n}{\zeta^{(n+1)}}$$

Blazius Relations for steady 2-D Flows

$$X - iY = i\frac{\rho}{2} \oint_{C_0} [W(z)]^2\, dz$$

$$M = -\frac{\rho}{2} \operatorname{Re}\left\{ \oint_{C_0} [W(z)]^2 z\, dz \right\}$$



$$\partial\forall_c = S \cup S_o$$

$$\cancelto{0}{\sum \vec{F}} = \iiint_{\forall_c} \frac{\partial}{\partial t}(\rho\vec{U})\,d\forall + \oiint_{S_o} \rho\vec{U}(\vec{U}\cdot\hat{n})\,ds + \cancelto{0}{\oiint_{S} \rho\vec{U}(\vec{U}\cdot\hat{n})\,ds} +$$

$$+ \oiint_{S_o} p\hat{n}\,ds + \underbrace{\oiint_{S} p\hat{n}\,ds}_{\text{Force interaction between fluid and body}}$$

Bernoulli:

$$p = \underbrace{p_T}_{\text{constant}} - \frac{\rho U^2}{2} - \cancelto{0 \to \text{far field}}{\rho\frac{\partial\Phi}{\partial t}}$$

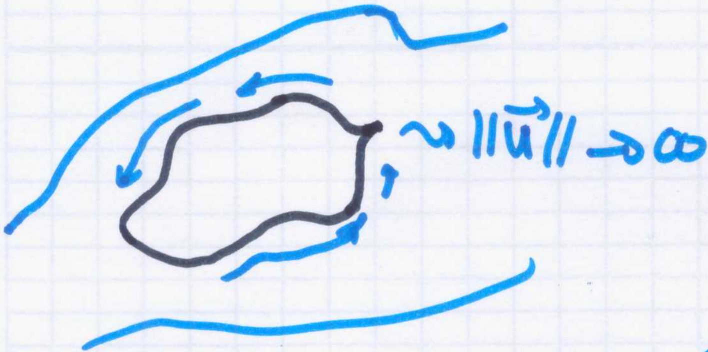
$$\oiint_{S} p\hat{n}\,ds = \vec{F} = -\cancelto{\text{steady}}{\iiint_{\forall_c} \rho\frac{\partial\vec{U}}{dt}} - \iint_{S_o} \rho\vec{U}(\vec{U}\cdot\hat{n})\,ds + \oiint_{S_o} \frac{\rho U^2}{2}\hat{n}\,ds$$

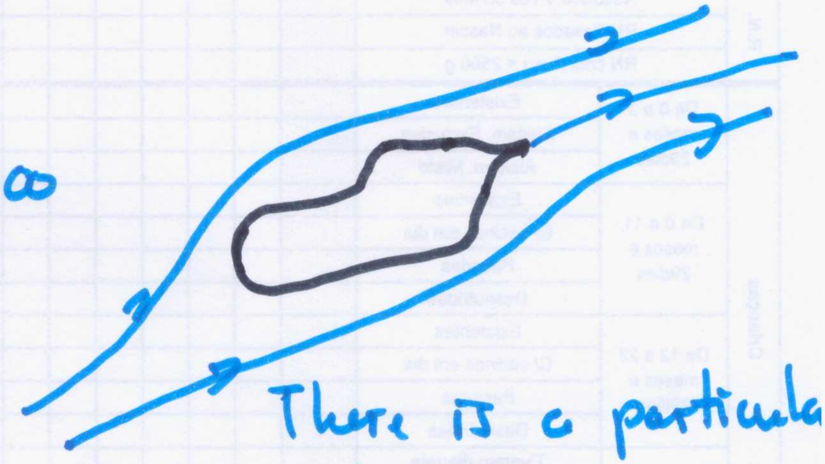
$$\cancelto{0}{\sum \vec{M}} = \cancelto{0}{\iiint_{\forall_c} \frac{\partial}{\partial t}(\rho\vec{r}\times\vec{U})\,d\forall} + \oiint_{S_o} \rho\vec{r}\times\vec{U}\,\vec{U}\cdot\hat{n}\,ds + \cancelto{0}{\oiint_{S} \rho\vec{r}\times\vec{U}\,\vec{U}\cdot\hat{n}\,ds} +$$

$$+ \oiint_{S_o} \vec{r}\times\hat{n}\,p\,ds + \underbrace{\oiint_{S} \vec{r}\times\hat{n}\,p\,ds}_{\vec{M}}$$

⑤

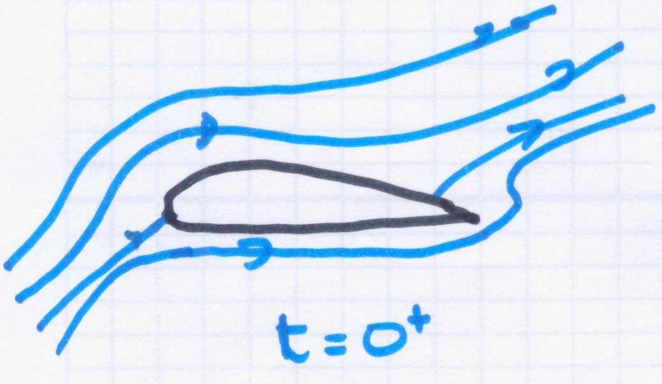
$$\oiint_S \vec{r}\times\hat{n}\,p\,ds = \vec{M} = -\oiint_{S_0} \rho\,\vec{r}\times\vec{U}(\vec{U}\cdot\hat{n})\,ds + \oiint_{S_0} \vec{r}\times\hat{n}\,\frac{\rho U^2}{2}\,ds$$



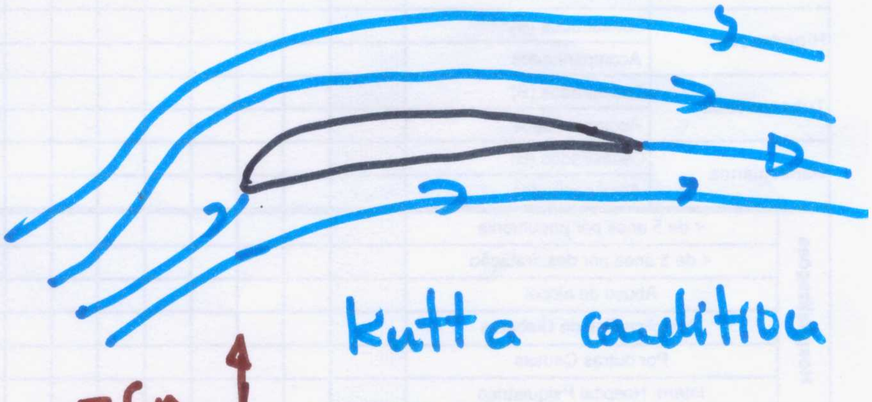


it cancels the "tangent" component at that point.

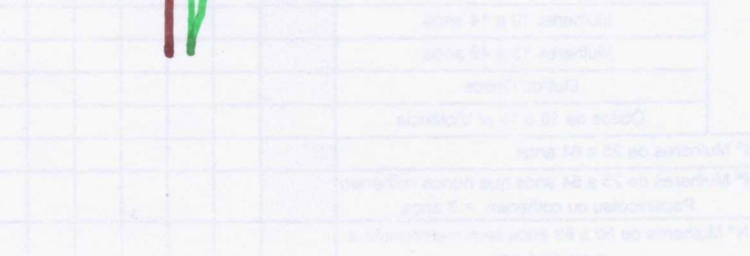
There is a particular value of Γ that makes the flow leave the discontinuity smoothly



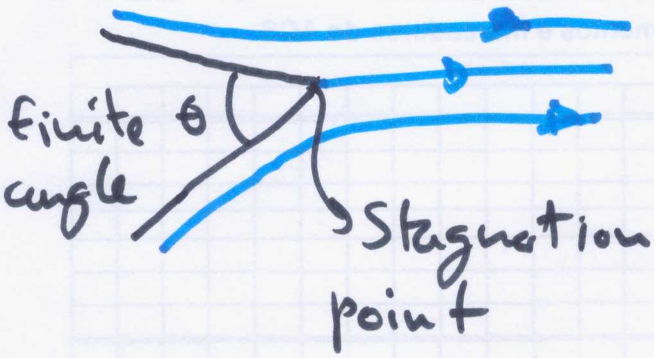
$t = 0^+$

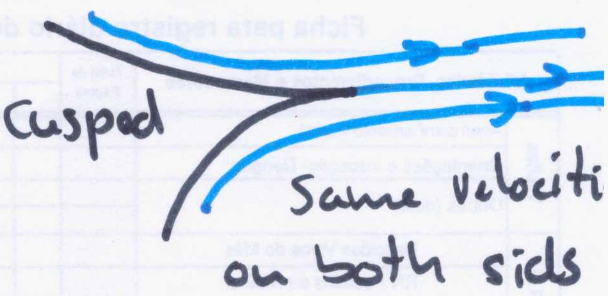


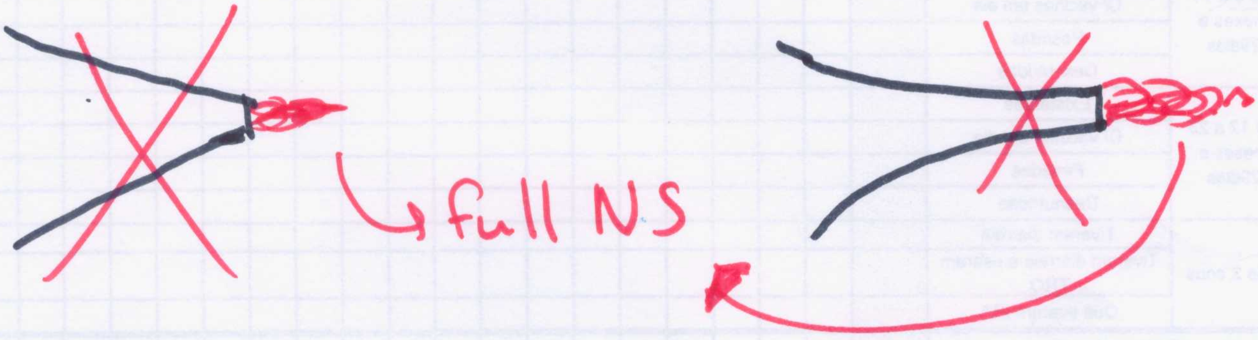
Kutta condition



T. E.





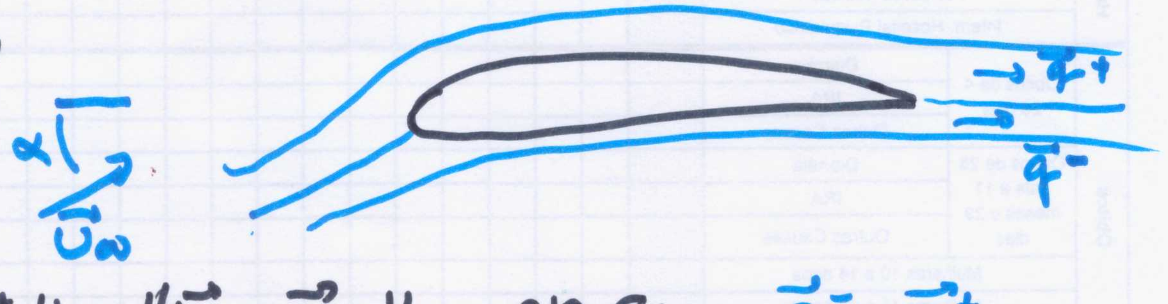


Steady discontinuities must meet the conditions

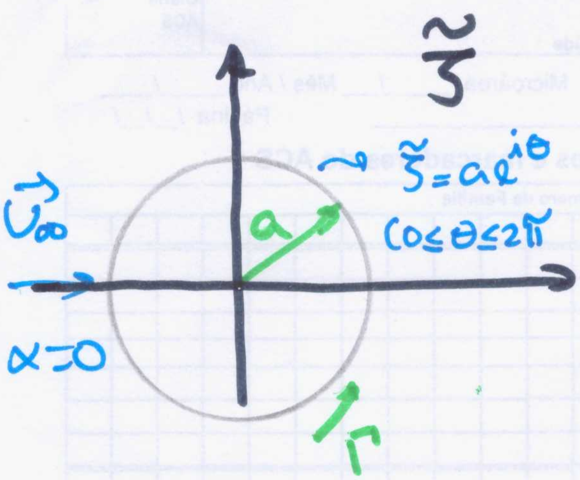
$$p^+ = p^- \quad \text{and} \quad \vec{V}^+ \cdot \hat{n} = \vec{V}^- \cdot \hat{n}$$

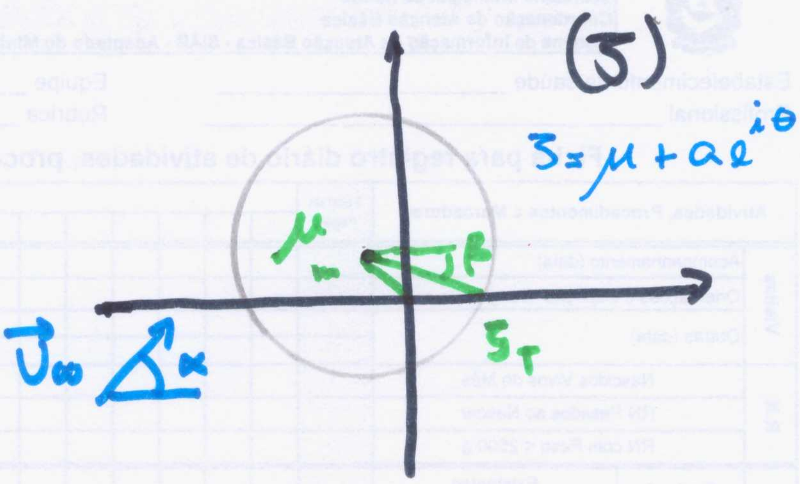
Bernoulli: $p^+ + \frac{\rho(U^+)^2}{2} = P_0 = P^- + \frac{\rho(U^-)^2}{2} \Rightarrow \|\vec{U}^-\| = \|\vec{U}^+\|$

2-D Flow



$$\|\vec{U}_\infty + \vec{q}^+\| = \|\vec{U}_\infty + \vec{q}^-\| \Rightarrow \text{2-D flow } \vec{q}^- = \vec{q}^+$$





$$F(\tilde{\zeta}) = U_\infty \tilde{\zeta} + \frac{U_\infty a^2}{\tilde{\zeta}} + \frac{i\Gamma}{2\pi} \log\left(\frac{\tilde{\zeta}}{a}\right)$$

$$\zeta = \tilde{\zeta} e^{i\alpha} + \mu \Rightarrow \tilde{\zeta} = (\zeta - \mu) e^{-i\alpha}$$

$$\mu = m e^{i\delta}, \zeta_T = \mu + a e^{-i\beta} .$$

$$F(\zeta) = U_\infty (\zeta - \mu) e^{-i\alpha} + \frac{U_\infty a^2 e^{i\alpha}}{(\zeta - \mu)} + \frac{i\Gamma}{2\pi} \log\left(\frac{\zeta - \mu}{a e^{i\alpha}}\right)$$

Kutta condition:

$$W(z)\Big|_{z_T} = W(\zeta)\Big|_{\zeta_T} \frac{1}{\left(\frac{dz}{d\zeta}\right)_{\zeta_T}} \qquad \begin{cases} W(\zeta_T) = 0 \\ \left.\frac{dz}{d\zeta}\right|_{\zeta_T} = 0 \end{cases}$$

$$W(\zeta) = \frac{dF}{d\zeta} = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(\zeta-\mu)^2} + \frac{i\Gamma}{2\pi} \frac{a e^{i\alpha}}{(\zeta-\mu)} \frac{1}{a e^{i\alpha}} \quad (8$$

$$W(\zeta) = V_\infty e^{-i\alpha} + \frac{i\Gamma}{2\pi(\zeta-\mu)} - \frac{V_\infty a^2 e^{i\alpha}}{(\zeta-\mu)^2}$$

Then for the kutta condition to hold, we need

$$W(\zeta_T) = 0 \quad = \quad (\zeta_T - \mu) = a e^{-i\beta}$$

On imposing this condition, we get

$$\Gamma = 4\pi a V_\infty \sin(\alpha + \beta)$$

Transformation: $Z = \zeta + \frac{(\zeta_T)^2}{\zeta}$

ζ_T is real ($\zeta_T \in \mathbb{R}$)

(30/04/20)

Kutta Condition $W(\zeta_T)=0$

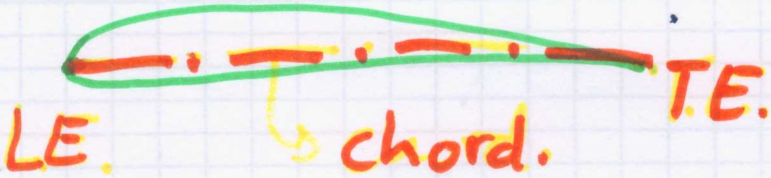
Critical Point of Joukowski transformation

$$\Gamma = 4\pi a U_\infty \sin(\alpha+\beta)$$

$$L = \rho U_\infty \Gamma = 4\pi \rho U_\infty^2 a \sin(\alpha+\beta)$$

$$C_L = 8\pi\left(\frac{a}{c}\right)\sin(\alpha+\beta) \Longleftarrow C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 c}$$

c = chord.



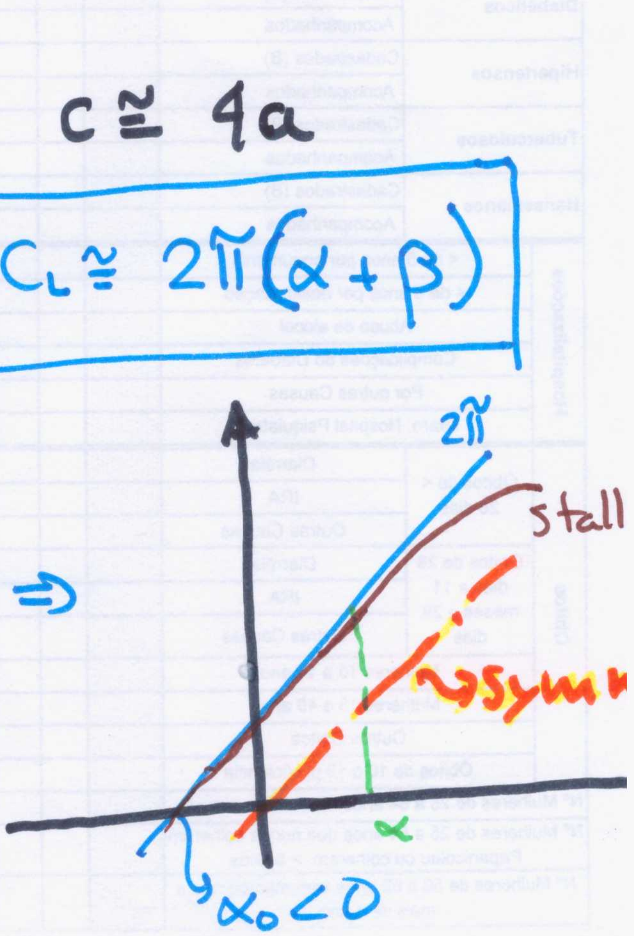
For Joukowski airfoils: $c \cong 4a$

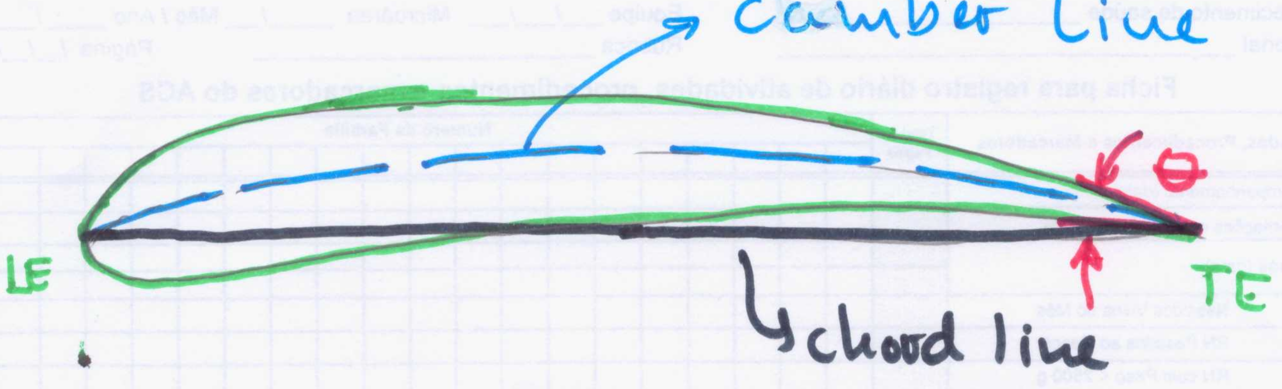
$$C_L = 2\pi \sin(\alpha+\beta) \Longrightarrow \boxed{C_L \cong 2\pi(\alpha+\beta)}$$

α and β in radians.

for $\alpha = -\beta \Rightarrow C_L = 0 \Rightarrow$

$$\alpha_0 = \alpha\Big|_{L0} = -\beta$$





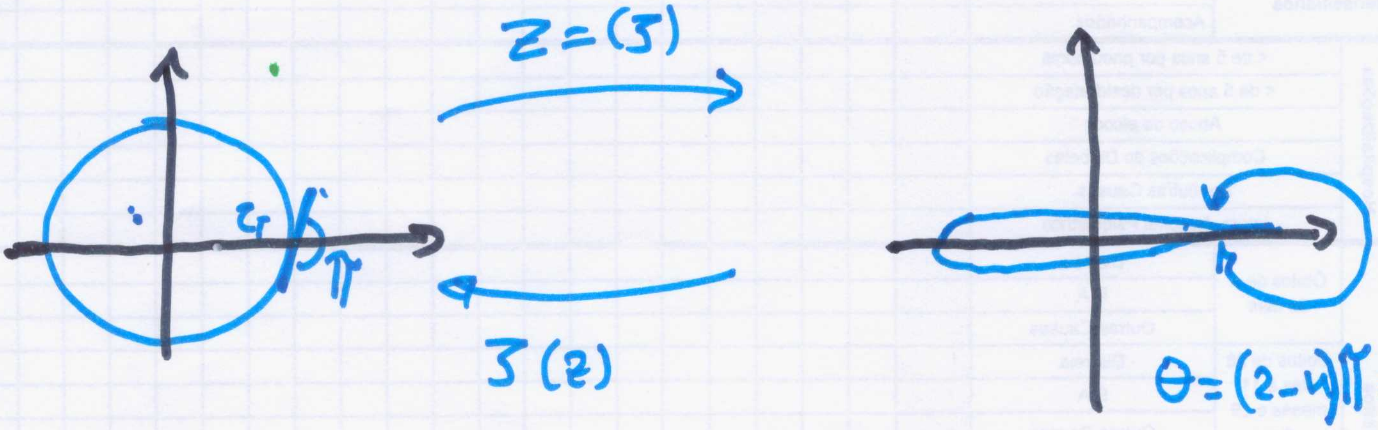
Joukowski transformation

TE: $z_t = \zeta_T = C \in \mathbb{R}$

$$z = \zeta + \frac{C^2}{\zeta}$$

$$\frac{dz}{d\zeta} = 1 - \frac{C^2}{\zeta^2} \Rightarrow \frac{dz}{d\zeta}\Big|_{\zeta_T} = 1 - \frac{\zeta_T^2}{\zeta_T^2} = 0$$

$$\frac{d^2z}{d\zeta^2} = 2\frac{C^2}{\zeta^3} \Rightarrow \frac{d^2z}{d\zeta^2}\Big|_{\zeta_T} = \frac{2}{\zeta_t} \neq 0 \Rightarrow \boxed{n=2}$$



$\theta = (2-n)\pi =$

$\theta = (2-n)\pi = 0$

Cuspid T.E.

$\boxed{\theta = 0}$

Joukowski Airfoil:

$$\boxed{C_l = 2\pi(\alpha + \beta)}, \quad \alpha_0 = -\beta$$

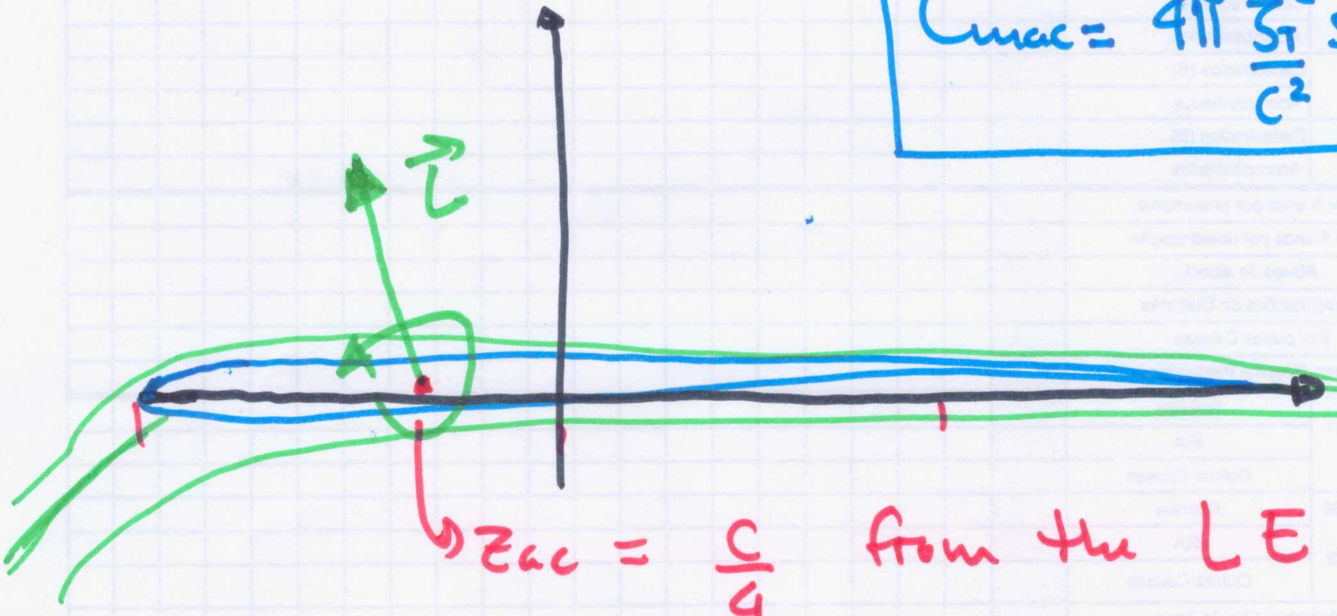
Aerodynamic Center coordinate

$$\boxed{z_{AC} \equiv \mu - \frac{\zeta_T^2}{a} e^{i\beta}} \quad ; \mu = m e^{i\delta}$$

$$M_{AC} = -2\pi \rho U_\infty^2 \zeta_T^2 \sin(2\beta)$$

For thin airfoils, z_{AC} is very close to the quarter-choord point

$$\boxed{C_{mac} = 4\pi \frac{\zeta_T^2}{c^2} \sin(2\alpha_0)}$$



$z_{ac} = \frac{c}{4}$ from the LE.

④

Katz and Plotkin, Low speed Aerodynamics.

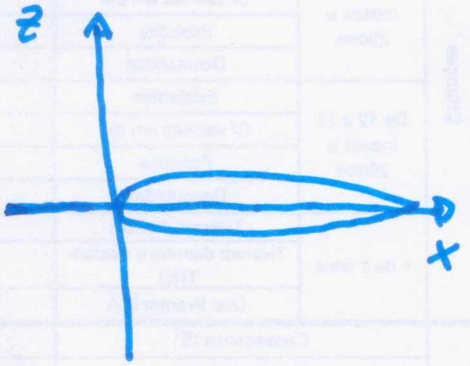
Chapter 5

Small disturbance Flow over 2-D airfoils.

wing upper and lower surfaces

$$z = \eta_u(x, y)$$

$$z = \eta_l(x, y)$$



thickness function:

$$\eta_T = \frac{1}{2}(\eta_u - \eta_l)$$

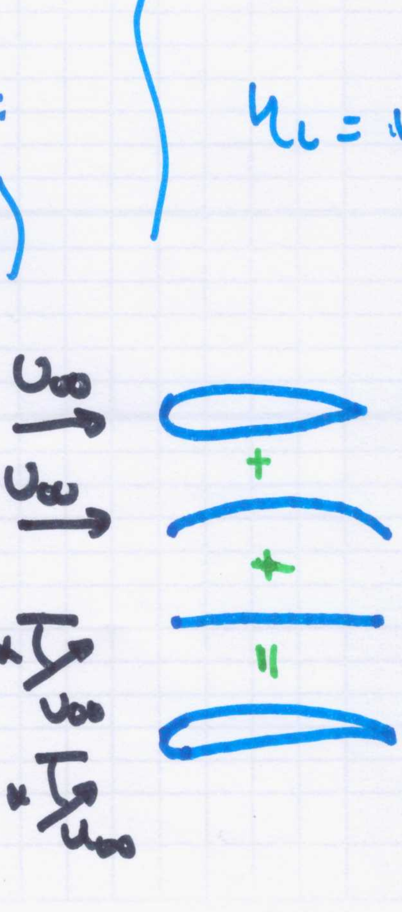
camber function:

$$\eta_c = \frac{1}{2}(\eta_u + \eta_l)$$

$$\left.\right\} \quad \eta_u = \eta_c + \eta_t$$

$$\eta_l = \eta_c - \eta_T$$

Superimposing solutions:



thickness Φ_1

camber Φ_2

flat plate at a.o.a. α Φ_3

$$\left.\right\} \Phi = \Phi_1 + \Phi_2 + \Phi_3$$

$$z = \eta(x,y) \Rightarrow F(x,y,z) = z - \eta(x,y) = 0$$

Outward normal

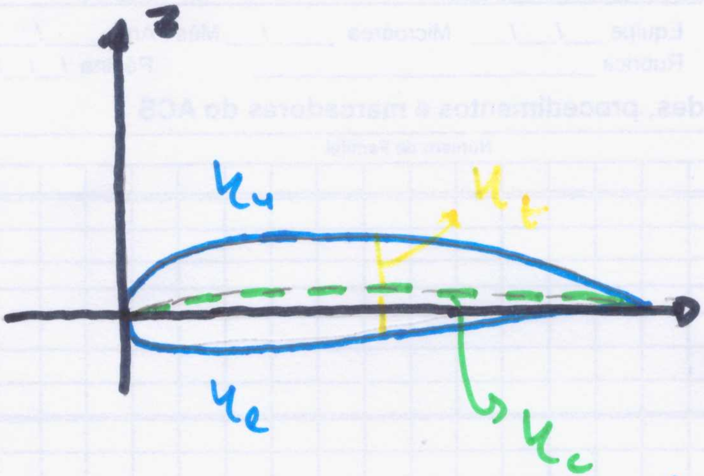
$$\hat{n} = \frac{\nabla F}{\|\nabla F\|} \;;$$

$$\hat{n} = \frac{1}{\|\nabla F\|}\left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1\right)$$

upward surface

on the lower side, simply $\hat{n}_L = -\hat{n}$

$$F(x,y,z) = z - \eta(x,y) = 0$$



Velocity potential:

$$\Phi = W_\infty z + U_\infty x + \phi$$

Wall B.c.

$$\nabla\Phi \cdot \hat{n} = \nabla\Phi \cdot \frac{\nabla F}{\|\nabla F\|} = 0$$

$$\overset{①}{\cancel{\frac{\partial F}{\partial z}}} \cdot \frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial x}\left(U_\infty + \frac{\partial \Phi}{\partial x}\right) + \frac{\partial \eta}{\partial y}\frac{\partial \Phi}{\partial y} - W_\infty$$

$$\vec{Q}_\infty = (U_\infty, V_\infty, W_\infty) \;, \quad \|\vec{Q}_\infty\| = Q_\infty$$

$$\frac{|\partial_x \phi|}{Q_\infty}, \frac{|\partial_y \phi|}{Q_\infty}, \frac{|\partial_z \phi|}{Q_\infty} \ll 1$$

Slender body hypothesis: $|\partial_x \eta|, |\partial_y \eta| \ll 1$

Fails at L.E.

⑥

$V_\infty \simeq 0$

$$\left|\frac{W_\infty}{U_\infty}\right| = \tan\alpha = \alpha << 1 \; : \qquad \begin{array}{l} W_\infty \cong Q_\infty \alpha \\ U_\infty \cong Q_\infty \end{array}$$

$$(\partial_x \eta)(\partial_x \Phi) \cong 0$$

$$(\partial_y \eta)(\partial_y \Phi) \cong 0$$

We are left with:

$$\left.\frac{\partial \phi}{\partial z}\right|_{(x,y,\eta)} = Q_\infty\left(\frac{\partial \eta}{\partial x} - \alpha\right)$$

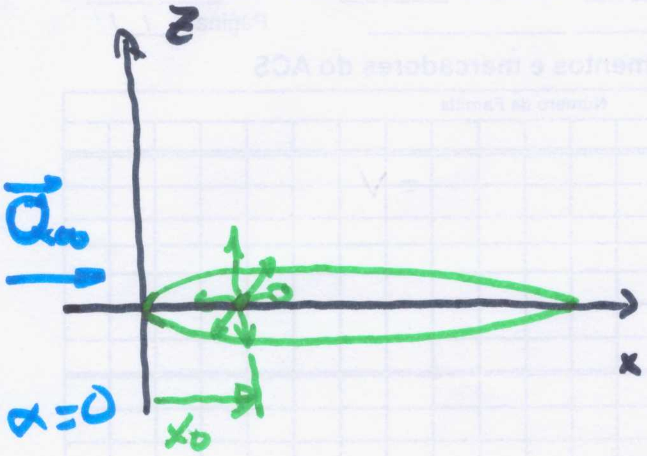
Taylor Series expansion $(|\eta| << 1)$

$$\left.\frac{\partial \Phi}{\partial z}\right|_{(x,y,z=\eta)} = \frac{\partial \Phi}{\partial z}(x,y,0) + \eta\frac{\partial^2 \Phi}{\partial z^2}(x,y,0) + O(\eta^3)$$

for very thin wings: $|\eta| << 1$, we make:

$$\boxed{\frac{\partial \Phi}{\partial z}(x,y,0) = Q_\infty\left(\frac{\partial \eta}{\partial x} - \alpha\right)}$$

(07/05/2020)



$$\nabla^2\Phi = 0$$

$$w(x,0^{\pm}) = \pm Q_\infty \frac{d\eta_t}{dx}$$

$$\Phi = \frac{\sigma_0}{2\pi}\ln(r) \quad , \quad r = \sqrt{(x-x_0)^2+z^2} \quad \Rightarrow \vec{q} \Rightarrow q_r = \frac{\sigma_0}{2\pi r}$$

Source/sink distribution:

$$\Phi(x,z) = \frac{1}{2\pi}\int_0^c \sigma(x_0)\ln\left[\sqrt{(x-x_0)^2+z^2}\right]dx_0$$

$$u(x,z) = \frac{1}{2\pi}\int_0^c \sigma(x_0)\frac{(x-x_0)}{(x-x_0)^2+z^2}\,dx_0$$

$$w(x,z) = \frac{1}{2\pi}\int_0^c \sigma(x_0)\frac{z}{(x-x_0)^2+z^2}\,dx_0$$

$$\lim_{z\to 0^+}\frac{1}{2\pi}\int_{x_{LE}}^{x_{TE}}\frac{\sigma(x_0)\,z\,dx_0}{(x-x_0)^2+z^2} = \lim_{z\to 0^+}\frac{\sigma(x)}{2\pi}\int_{-\infty}^{\infty}\frac{z\,dx_0}{(x-x_0)^2+z^2} = W(x,$$

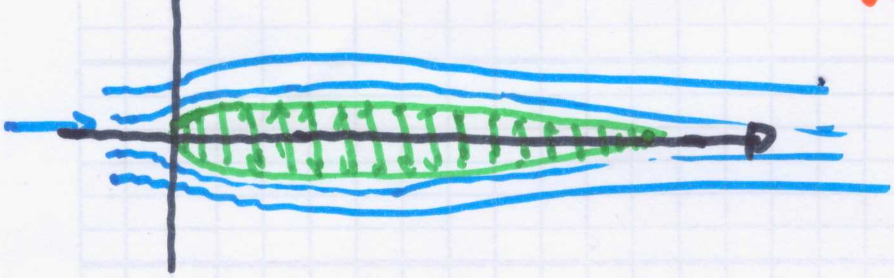
$$\lambda \equiv (x - x_0)/z \quad ; \quad d\lambda = -dx_0/z$$

$$W(x,0^+) = \lim_{z \to 0^+} \frac{\sigma(x)}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{1+\lambda^2} = \frac{\sigma(x)}{2\pi} \tan^{-1}(\lambda) \Big|_{-\infty}^{\infty}$$

$$W(x,0^+) = \frac{\sigma(x)}{2} \Rightarrow W(x,0^+_-) = \frac{\partial \Phi}{\partial z}(x,0^+_-) = \pm \frac{\sigma(x)}{2}$$

$$W^+ - W^- = \sigma(x) \quad \text{for} \quad x_{LE} \le x \le x_{TE} \quad \text{and} \quad z \to 0^+_-$$

$$\sigma = \frac{\partial \phi^+}{\partial z} - \frac{\partial \phi^-}{\partial z}$$



$$\boxed{\sigma(x) = 2Q_\infty \frac{d\eta_t}{dx}}$$

$$\phi(x,z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \ln\left[\sqrt{(x-x_0)^2 + z^2}\right] dx_0$$

$$u(x,z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{(x-x_0)}{(x-x_0)^2 + z^2} dx_0$$

$$W(x,z) = \frac{Q_\infty}{\pi} \int_0^c \frac{d\eta_t(x_0)}{dx} \frac{z \, dx_0}{(x-x_0)^2 + z^2}$$

$$C_p = -\frac{2u'}{U_\infty} = -\frac{2u'}{Q_\infty} \qquad \boxed{\alpha = 0}$$

$$\vec{F} = -\int_0^c P.\hat{n}\,ds$$

In the most General Case, we'd have: $F = z - \eta(x,y$

$$\hat{n} = \frac{1}{\|\nabla F\|}\left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1\right)$$

Then we would scale $F(x,y,z)$, so as to make $\|\nabla F\| = 1$ on $F(x,y,z) = 0$

and that would lead to

$$F_x = \oint_w \left[P_u \frac{\partial \eta_u}{\partial x} - P_l \frac{\partial \eta_l}{\partial x}\right] dx\,dy$$

$$F_y = \oint_w \left[P_u \frac{\partial \eta_u}{\partial y} - P_l \frac{\partial \eta_l}{\partial y}\right] dx\,dy$$

$$F_z = \oint_w (P_l - P_u)\,dx\,dy$$

$$D = F_x \cos\alpha + F_z \sin\alpha \cong F_x + F_z\,\alpha$$

$$L = -F_x \sin\alpha + F_z \cos\alpha \cong F_z - F_x\,\alpha$$

$$c_p = -\frac{2u(x,0)}{Q_\infty}$$

$$c_p = -\frac{2}{\pi}\int_0^c \frac{d\eta_t(x_0)}{dx}\,\frac{1}{(x-x_0)}\,dx_0 \qquad \alpha = 0$$

$$L = F_z = \int_0^c (P_l - P_u)\,dx_0 = 0$$

$$(c_{p_l} - c_{p_u}) = 0 \quad \text{for } \forall x \in [0, c]$$

owing to symmetry

Hence the symmetric thickness distribution does not contribute to Lift.

How about drag:

$$D = 2\int_0^c p_u \frac{d\eta_t}{dx}\,dx = -2\rho\frac{Q_\infty^2}{\pi}\int_0^c\int_0^c \frac{\eta'_t(x_0)\,\eta'_t(x)}{(x-x_0)}\,dx_0\,dx$$

$$\Delta = \int_0^c \eta'_t(x_0)\underbrace{\left[\int_0^c \frac{\eta'_t(x)\,dx}{(x-x_0)}\right]}_{}dx_0 = -\int_0^c \eta'_t(x)\underbrace{\left[\int_0^c \frac{\eta'_t(x_0)\,dx_0}{(x_0-x)}\right]}_{}dx$$

$$\Delta = \int_0^c \eta'_t(x_0)\,I(x_0)\,dx_0 = -\int_0^c \eta'_t(x)\,I(x)\,dx$$

$$\Delta = 0$$

Therefore, the symmetric thickness distribution does not contribute to Lift, nor to Drag, neither does it contributes to the pitching moment, owing to its symmetry.

* * *

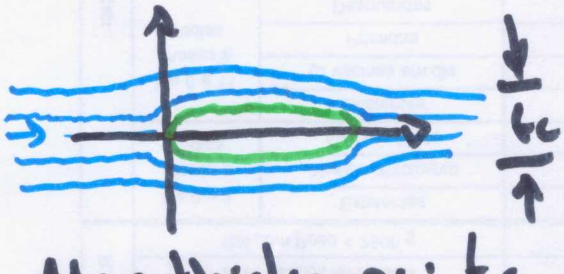
Glauert Integral:

$$\int_0^{\pi} \frac{\cos(n\theta_0)\, d\theta_0}{\cos\theta_0 - \cos\theta} = \frac{\pi \sin(n\theta)}{\sin\theta}$$

$n = 0, 1, 2, \dots$

* * *

Flow past an ellipse at zero a.o.a. $(\alpha = 0$



$$x = \frac{c}{2}(1 - \cos\theta) \Rightarrow dx = \frac{c}{2}\sin\theta\, d\theta$$

$$\frac{x}{c} = \tilde{x} = \frac{1 - \cos\theta}{2} \Rightarrow d\tilde{x} = \frac{\sin\theta}{2} d\theta$$

Max thickness: tc

Contour of the ellipse: $\dfrac{(x - c/2)^2}{\left(\frac{c}{2}\right)^2} + \dfrac{\eta^2}{\left(\frac{tc}{2}\right)^2} = 1$

$\eta(x) = z$

⑥

$$\eta = \pm t\sqrt{x(c-x)}$$

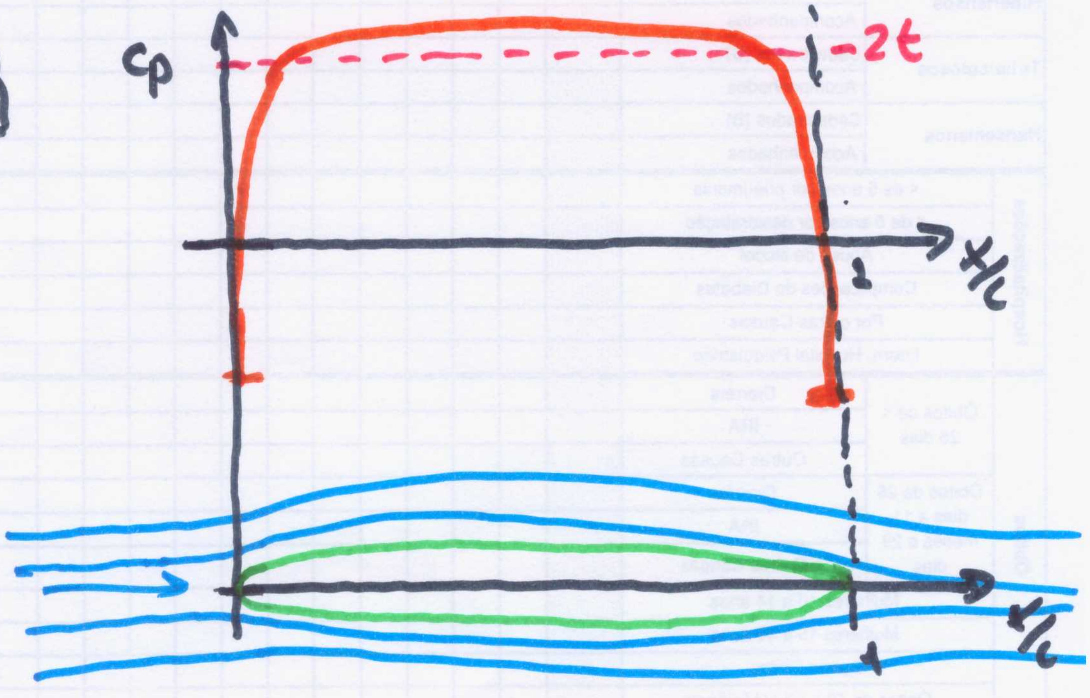
$$\frac{d\eta}{dx} = \pm \frac{t}{2}\frac{c-2x}{\sqrt{x(c-x)}}$$

then, and only then, we cast $\frac{d\eta}{dx}$ in terms of θ:

$$\frac{d\eta_t}{dx} = \frac{t}{2}\frac{c-c(1-\cos\theta)}{\sqrt{\frac{c}{2}(1-\cos\theta)\left[c-\frac{c}{2}(1-\cos\theta)\right]}} = t\frac{\cos\theta}{\sin\theta}$$

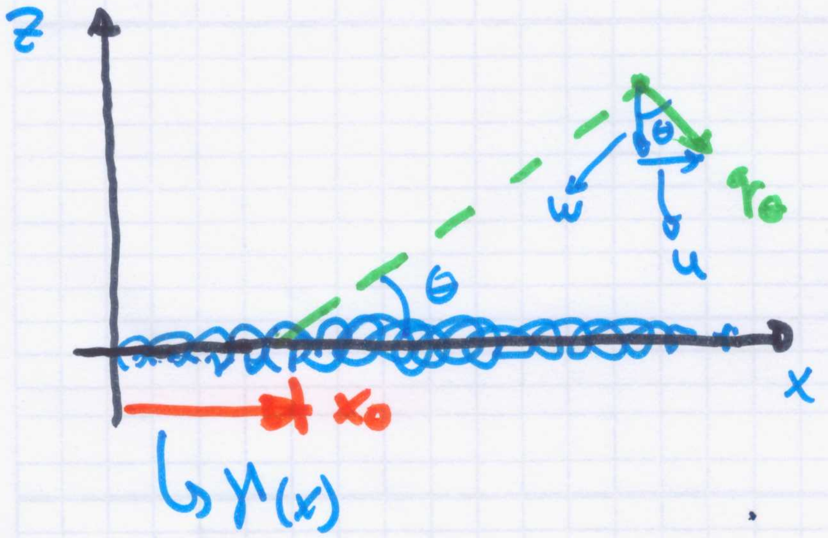
$$u(x,0) = \frac{t\,Q_\infty}{\pi}\int_0^{\pi}\frac{\cos\theta_0}{\cos\theta_0-\cos\theta}\,d\theta_0 = t\,Q_\infty$$

$$\boxed{C_p = -2t}$$



The Lift Problem — zero thickness airfoil at an a.o.a ($\alpha \neq 0$)

$$\frac{\partial \phi}{\partial z}(x, 0^{\pm}) = Q_\infty \left(\frac{d\psi_c}{dx} - \alpha \right)$$



$$\Phi_{\gamma(x_0)} = -\frac{\gamma(x_0)}{2\pi}\theta$$

$$\Phi\Big|_{\gamma(x_0)} = -\frac{\gamma(x_0)}{2\pi}\tan^{-1}\left(\frac{z}{x-x_0}\right.$$

$$q_\theta = -\frac{\gamma_0}{2\pi}, \quad q_r = 0$$

$$r = \sqrt{(x-x_0)^2 + z^2}$$

$$u = \frac{\partial \phi_x}{\partial x} = \frac{\gamma_0}{2\pi} \frac{z}{(x-x_0)^2 + z^2}$$

$$w = \frac{\partial \Phi_x}{\partial z} = -\frac{\gamma_0}{2\pi} \frac{(x-x_0)}{(x-x_0)^2 + z^2}$$

at $z = 0 \Rightarrow w = -\frac{\gamma_0}{2\pi(x-x_0)}$

$$\Phi(x,z) = -\frac{1}{2\pi}\int_0^c \gamma(x_0)\tan^{-1}\left(\frac{z}{x-x_0}\right)dx_0$$

$$u(x,z) = \frac{1}{2\pi}\int_0^c \gamma(x_0)\frac{z}{(x-x_0)^2 + z^2}dx_0 \quad \Bigg| \quad w(x,z) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(x_0)(x-x_0)\,dx_0}{(x-x_0)^2 + z^2}$$

A function like: $f(x,z) = \dfrac{z}{\pi\left[(x-x_0)^2+z^2\right]}$

behaves as a Dirac delta in the limit

$$\lim_{z\to 0}\int_{-\infty}^{\infty} g(x)\, f(x,z)\,dx = \int_{-\infty}^{\infty} g(x)\,\delta(x-x_0) = g(x_0)$$

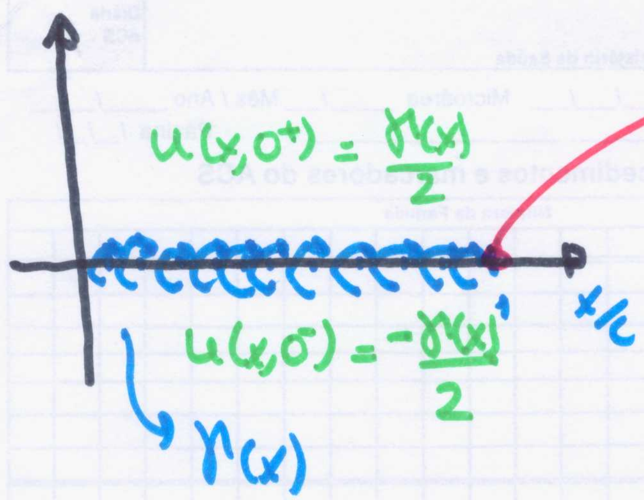
$g(x) \in C_c^{\infty}$

↳ infinitelly differentiable with compact support ($g \neq 0$ over a finite interval over the x axis)

$$u(x,0\pm) = -\lim_{z\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma(x_0)\,\frac{z\;dx_0}{(x-x_0)^2+z^2} = \frac{\pm\gamma(x)\pi}{2\pi}$$

$$\boxed{u(x,0\pm) = \pm\frac{\gamma(x)}{2}}$$

$$w(x,0) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(x_0)\,dx_0}{(x-x_0)}$$



$$\boxed{\text{T.E. } \gamma\left(\frac{x}{c}=1\right) = 0}$$

Kutta condition

$$\frac{\partial\phi}{\partial z}(x,0) = w(x,0) = Q_\infty\left(\frac{d\eta_c}{dx} - \alpha\right)$$

$$\boxed{-\frac{1}{2\pi}\int_0^c \gamma(x_0)\,\frac{dx_0}{(x-x_0)} = Q_\infty\left(\frac{d\eta_c}{dx} - \alpha\right) \qquad \forall x \mid 0 < x < c}$$

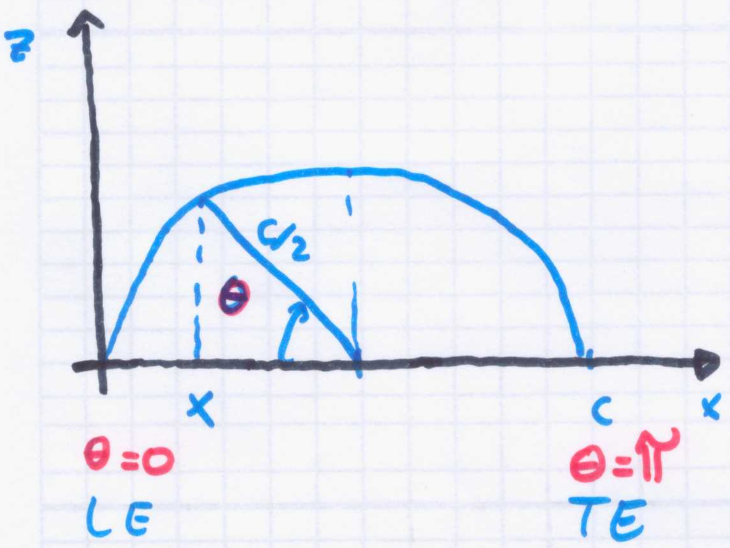
$$C_p = -\frac{2u'}{Q_\infty} \Rightarrow C_p = \mp\frac{\gamma}{Q_\infty} \Rightarrow \boxed{\Delta C_p = \frac{2\gamma}{Q_\infty}}$$

$$\Delta C_p\Big|_{TE} = 0 \Rightarrow \gamma\left(\frac{x}{c}=1\right) = 0 \Rightarrow \underline{\text{Kutta}}$$

19/05/2020

$$\Delta C_p = \frac{2\gamma}{Q_\infty} \Rightarrow \Delta P = P_l - P_u = \rho Q_\infty \gamma(x)$$

$$L = \rho Q_\infty \Gamma \Leftarrow \Gamma \int_0^c \gamma(x)\,dx$$



$$x = \frac{c}{2}(1 - \cos\theta)$$

$$0 \leq \theta \leq \pi$$

$$dx = \frac{c}{2}\sin\theta\, d\theta$$

$$\theta = \cos^{-1}\left(1 - 2x/c\right)$$

$$-\frac{1}{2\pi}\int_0^\pi \gamma(\theta_0)\,\frac{\sin\theta_0\, d\theta_0}{(\cos\theta - \cos\theta_0)} = Q_\infty\left[\frac{d\eta_c(\theta)}{dx} - \alpha\right]$$

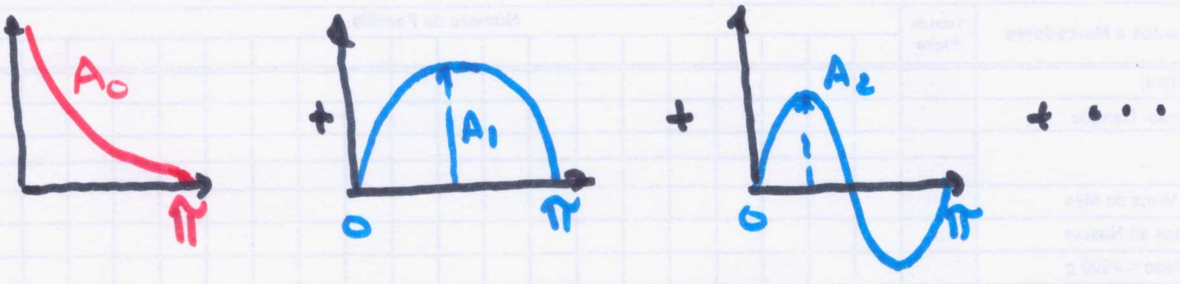
Kutta condition: $\gamma(\pi) = 0$

$$\sum_{n=1}^{\infty} A_n \sin(n\theta)$$

We need an additional term to capture the leading edge suction peak.

$$A_0 \cot\left(\frac{\theta}{2}\right) = A_0\,\frac{1 + \cos\theta}{\sin\theta}$$

$$\gamma(\theta) = 2Q_\infty \left[A_0 \frac{(1+\cos\theta)}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta) \right]$$



$$-\frac{1}{2\pi}\int_0^\pi 2Q_\infty\left[A_0\frac{(\overset{\cos(0\theta_0)=1}{1}+\cos\theta_0)}{\sin\theta_0}+\sum_{n=1}^{\infty}A_n\sin(n\theta_0)\right]\frac{\sin\theta_0\, d\theta_0}{(\cos\theta_0-\cos\theta)}=$$

$$= Q_\infty\left[\frac{dy_c(\theta)}{dx}-\alpha\right]$$

Reminder of Glauert's integral:

$$\int_0^\pi \frac{\cos(n\theta_0)\, d\theta_0}{\cos\theta_0-\cos\theta} = \frac{\pi\sin(n\theta)}{\sin\theta} \quad ; n=0,1,2,\dots$$

$$-\frac{1}{\pi}A_0\int_0^\pi \frac{\cos(0\theta_0)+\cos\theta_0}{\cos\theta_0-\cos\theta}\, d\theta_0 = -\frac{A_0}{\pi}(0+\pi) = -A_0$$

$$\sin(n\theta_0)\sin\theta_0 = \frac{1}{2}\left[\cos[(n-1)\theta_0]-\cos[(n+1)\theta_0]\right]$$

$n = 1, 2, 3 \dots$

$$-\frac{1}{\pi}\int_0^\pi \frac{A_n\sin(n\theta_0)\sin\theta_0}{\cos\theta_0-\cos\theta} = -\frac{A_n}{2\pi}\int_0^\pi\frac{[\cos[(n-1)\theta_0]-\cos[(n+1)\theta_0]]\, d\theta_0}{\cos\theta_0-\cos\theta}$$

$$\boxed{-A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{d\eta_c(\theta)}{dx} - \alpha}$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{d\eta_c}{dx}(\theta)\, d\theta \quad ; \quad n=0$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{d\eta_c}{dx}(\theta)\cos(n\theta)\, d\theta \quad ; \quad n = 1, 2, 3, \dots$$

$$\gamma(\theta) = 2Q_\infty\left[\frac{A_0(1+\cos\theta)}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)\right]$$

$$x = \frac{c}{2}(1-\cos\theta) \quad , \quad \begin{cases} x=0 \Rightarrow \theta=0 \Rightarrow LE \\ x=c \Rightarrow \theta=\pi = TE \end{cases}$$

Kutta $\gamma(\pi)=0$

$$dx = \frac{c}{2}\sin\theta\, d\theta$$

$$w(\theta) = Q_\infty\left(\frac{d\eta_c(\theta)}{dx} - \alpha\right) \Rightarrow \frac{w}{Q_\infty} = -A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)$$

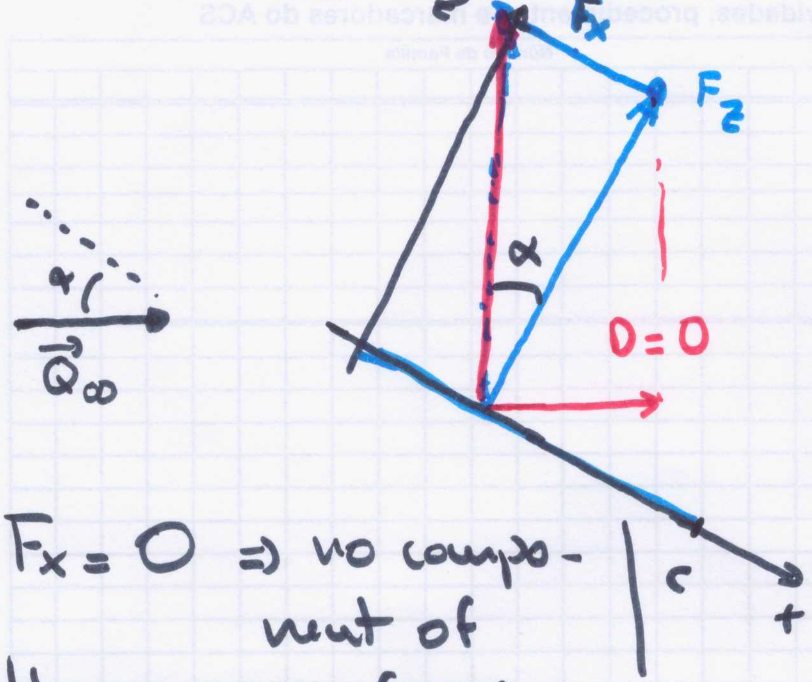
(downwash)

1) A_0 is the only coefficient that depends on α

2) A_0 gives a <u>constant</u> contribution to downwash.

$$\Delta C_p = \frac{2\gamma}{Q_\infty} \Rightarrow \Delta P(x) = \rho Q_\infty \gamma(x)$$

$\alpha << 1$



$$F_z = \int_0^c \Delta P(x)dx = \int_0^c \rho Q_\infty \gamma(x)\, dx$$

$$\boxed{F_z = \rho Q_\infty \Gamma}$$

$F_x = 0 \Rightarrow$ no component of the pressure forces (which act in the normal direction) along the chord.

$$L \cos\alpha = F_z$$

$$\boxed{L = F_z = \rho Q_\infty \Gamma}$$

As a result of this, we would get

$D = F_z \sin\alpha \cong F_z\, \alpha \Rightarrow$ Problem !!

From Kutta-Joukowsky theorem, we expect

$L = \rho Q_\infty \Gamma$, but $D = 0$

However, one can show that as the curvature radius of the L.E. goes to zero along with (local) thickness, the L.E. becomes a singularity with infinite acceleration of the flow. Hence there appears a suction force there.

And this suction force amounts to exactly $F_{x_{LE}} = -\rho Q_\infty \Gamma \alpha$

therefore, the end result for the Net Drag becomes:

$$D = \rho Q_\infty \Gamma \alpha - \rho Q_\infty \Gamma \alpha = 0$$

So we get the expected result $\begin{cases} L = \rho Q_\infty \Gamma \\ D = 0 \end{cases}$

— // — // — // —

Forces and Moments.

$$\Gamma = \int_0^c \gamma(x)dx = \int_0^\pi \gamma(\theta)\frac{c}{2}\sin\theta d\theta$$

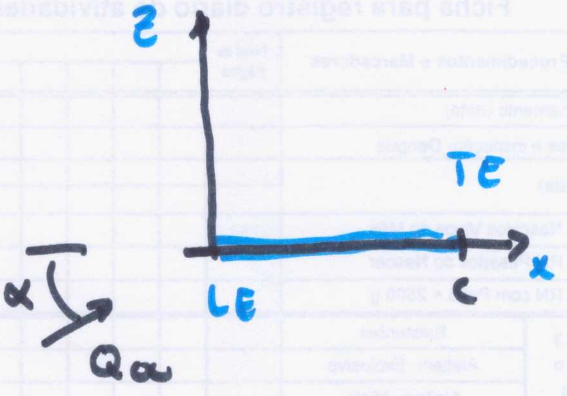
$$\Gamma = 2Q_\infty \int_0^\pi \left[A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)\right]\frac{c}{2}\sin\theta d\theta$$

$$\int_0^\pi (1+\cos\theta)d\theta = \pi$$

$$\int_0^\pi \sin(n\theta)\sin(\theta)d\theta = \begin{cases} \pi/2 & n=1 \\ 0 & n\neq 1 \end{cases}$$

$$\Gamma = Q_\infty c \pi \left(A_0 + \frac{A_1}{2}\right) \Rightarrow L = \rho Q_\infty^2 c \pi \left(A_0 + \frac{A_1}{2}\right)$$

$$\boxed{C_L = 2\pi\left(A_0 + \frac{A_1}{2}\right)}$$



Pitching moment:

1) with respect to the L.E. ⇒ origin

$$M_0 = -\int_0^c \Delta P x \, dx = \rho Q_\infty \int_0^\pi \gamma(\theta) \frac{c^2}{4}(1 - \cos\theta)\sin\theta \, d\theta =$$

$$= -\rho Q_\infty \left[\frac{c}{2} \overbrace{\int_0^\pi \gamma(\theta) \frac{c}{2} \sin\theta \, d\theta}^{\Gamma} - \frac{c^2}{4} \int_0^\pi \gamma(\theta) \cos\theta \sin\theta \, d\theta \right]$$

$$M_0 = -\frac{c}{2} L + \frac{\rho Q_\infty^2 c^2}{2} \left\{ A_0 \int_0^\pi (1 + \cos\theta)\cos\theta \, d\theta + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{A_n}{2} \underbrace{\int_0^\pi \sin(n\theta)\sin(2\theta)\, d\theta}_{} \right\}$$

Only n = 2 survives this integral.

$$M_0 = -\frac{c}{2}L + \frac{\rho Q_\infty^2 c^2}{2}\left\{A_0\left[-\cancelto{0}{(\sin\theta)_0^\pi} + \int_0^\pi \cos^2\theta \, d\theta\right] + \frac{A_2\pi}{4}\right\}$$

$$M_0 = -\frac{cL}{2} + \frac{\rho Q_\infty^2 c^2}{2}\left(\frac{A_0\pi}{2} + \frac{A_2\pi}{4}\right)$$

$$\boxed{M_0 = -\frac{cL}{2} + \frac{\rho Q_\infty^2 c^2}{4}\left(A_0\pi + \frac{A_2\pi}{2}\right)}$$

$$\cos^2\theta = \frac{\cos(2\theta) + 1}{2} \Rightarrow \int_0^\pi \cos^2\theta \, d\theta = \frac{\pi}{2}$$

$$\boxed{M_0 = -\rho Q_\infty^2 \frac{c^2}{4}\pi\left(A_0 + A_1 - \frac{A_2}{2}\right)}$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{d\eta_c}{dx}(\theta) \, d\theta \quad ; \quad A_n = \frac{2}{\pi}\int_0^\pi \frac{d\eta_c}{dx}(\theta)\cos(n\theta) \, d$$

$$C_L = 2\pi\left(A_0 + \frac{A_1}{2}\right) \Rightarrow C_L = 2\pi\Bigg(\underset{\substack{\uparrow \\ \text{AOA.}}}{\alpha} \underbrace{- \frac{1}{2}\int_0^\pi \frac{d\eta_c(\theta)}{dx} \, d\theta + \frac{A_1}{2}}_{\substack{\text{Camber contribution} \\ \text{(independent of } \alpha)}}\Bigg)$$

$$\boxed{C_L = 2\pi(\alpha - \alpha_{L0})}$$

$$\boxed{\alpha_{L0} = -\frac{1}{\pi}\int_0^\pi \frac{d\eta_c(\theta)}{dx}(\cos\theta - 1) \, d\theta}$$

$C_{L_\alpha} = \frac{dC_L}{d\alpha} = 2\pi$ $\qquad C_L = 2\pi(\alpha - \alpha_{L0})$

$$C_L = 2\pi\left(A_0 + \frac{A_1}{2}\right)$$

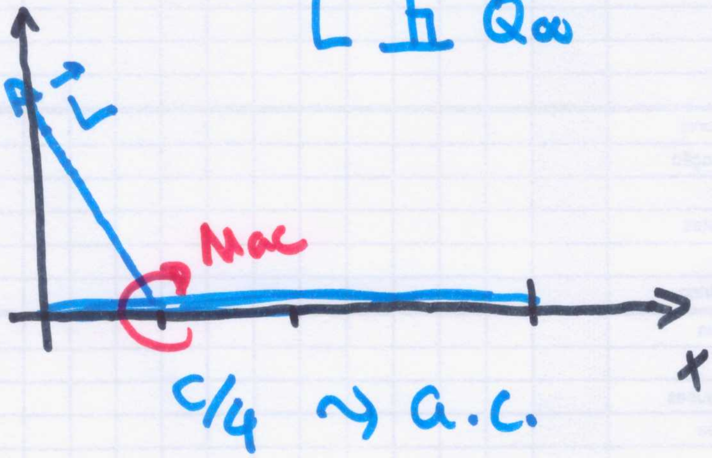
$$\boxed{C_{m_0} = \frac{M_0}{\frac{\rho Q_\infty^2}{2}\, c^2} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)}$$

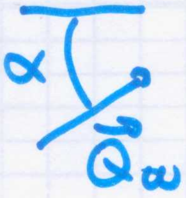
$$\boxed{C_{m_0} = -\frac{C_L}{4} + \underbrace{\frac{\pi}{4}(A_2 - A_1)}_{C_{m_{c/4}}}}$$

Pitching momen
sign conventio
nose up ⇒ ⊕
nose down ⇒ ⊖

$$\boxed{C_{m_{c/4}} = C_{m_{ac}} = \frac{\pi}{4}(A_2 - A_1)}$$

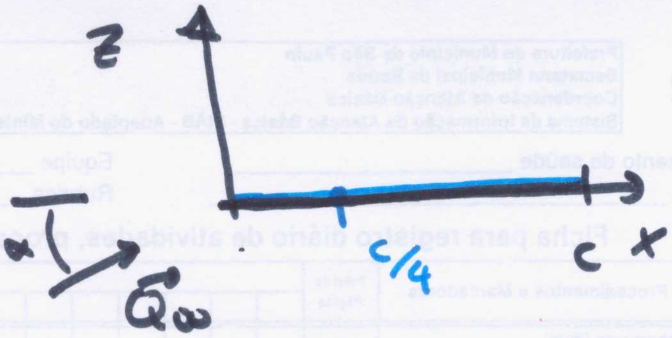
$$\vec{L} \perp \vec{Q}_\infty$$





Flat plate

$\alpha \neq 0$



$\eta_c(x) = 0 \quad \forall x \,\Big|\, 0 \le x \le c$

$\frac{d\eta_c}{dx} = 0 \qquad A_0 = \alpha$

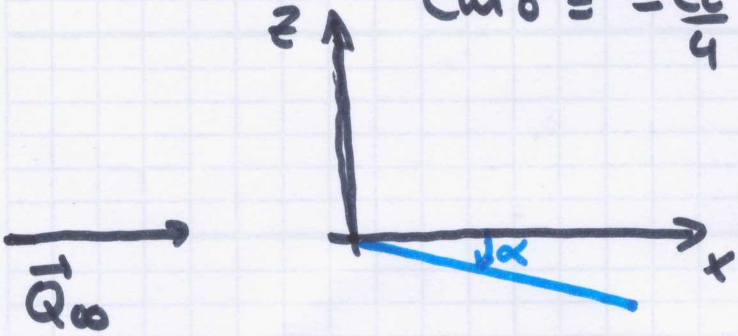
$A_n = 0 \;\; \forall n$

$$\begin{cases} A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{d\eta_c}{dx}(\theta)\,d\theta \\ A_n = \frac{2}{\pi}\int_0^\pi \frac{d\eta_c(\theta)}{dx}\cos(n\theta)\,d\theta \end{cases}$$

$\Gamma = Q_\infty \pi c \alpha \qquad C_L = 2\pi\alpha$

$C_{m_{ac}} = 0$

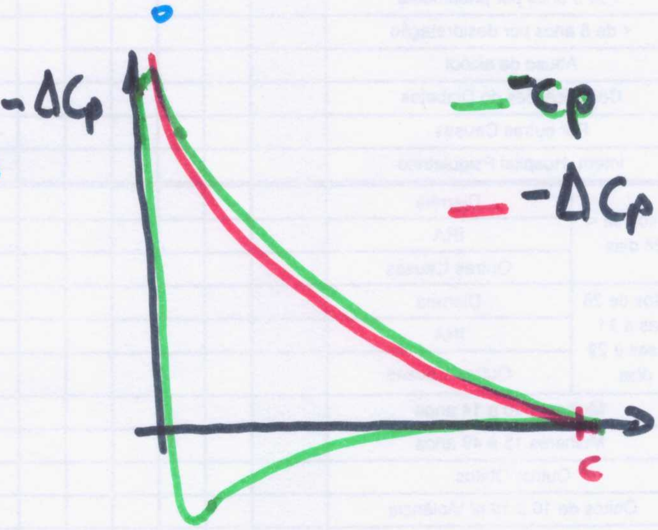
$C_{m_0} = -\frac{C_L}{4} = -\frac{\pi\alpha}{2}$



$\Rightarrow \alpha = 0,\; \eta_c = -\alpha x$

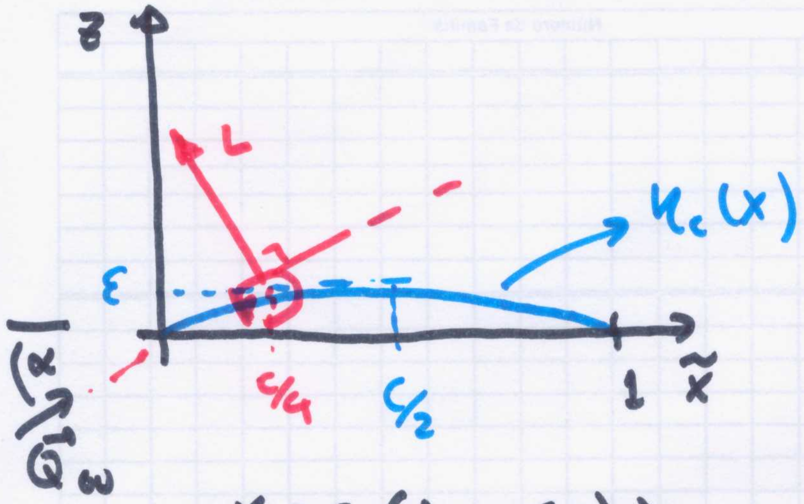
$\frac{d\eta_c}{dx} = -\alpha \qquad A_0 = \cancel{\alpha}_{0} - \frac{1}{\pi}\int_0^\pi (-\alpha)\,d\theta = +\alpha$

$\Delta C_p = \frac{2\gamma}{Q_\infty} = 4\alpha \underbrace{\frac{(1+\cos\theta)}{\sin\theta}}_{\cot(\theta)}$



$$\eta_c(x) = 4\epsilon \frac{x}{c}\left[1-\frac{x}{c}\right] = 4\varepsilon\tilde{x}(1-\tilde{x})$$

$(\varepsilon \ll 1)$



$$\frac{d\eta_c}{dx} = \frac{4\varepsilon}{c}\left[1-\frac{2x}{c}\right]$$

$$\frac{d\eta_c}{dx}(\theta) = \frac{4\varepsilon}{c}\cos\theta$$

$$\left.\begin{aligned} x &= \frac{c}{2}(1-\cos\theta) \\ dx &= \frac{c}{2}\sin\theta\, d\theta \end{aligned}\right\}$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{4\varepsilon}{c}\cos\theta\, d\theta$$

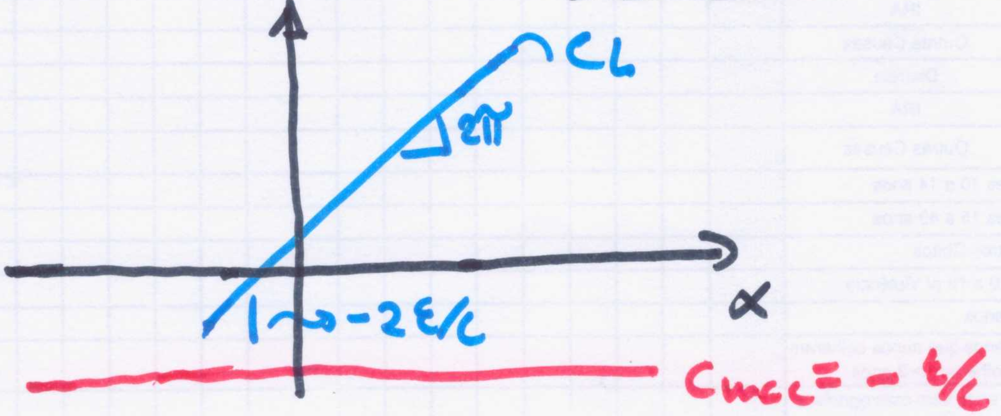
$$A_n = \frac{2}{\pi}\int_0^\pi \frac{4\varepsilon}{c}\cos\theta\cos(n\theta)\, d\theta$$

$$A_0 = \alpha$$

$$A_1 = \frac{4\varepsilon}{c}, \quad A_2 = A_3 = \dots\dots = 0$$

$$C_L = 2\pi\left(\alpha + 2\frac{\varepsilon}{c}\right) \Rightarrow \boxed{\alpha_{L0} = -2\frac{\varepsilon}{c}}$$

$$C_{mac} = -\frac{\pi}{\cancel{4}}\cancel{4}\frac{\varepsilon}{c} \Rightarrow \boxed{C_{mac} = -\frac{\varepsilon}{c}}$$



28/05/2020 ①

$$\gamma(\theta) = 2Q_\infty\left[A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty}A_n\sin(n\theta)\right]$$

Kutta condition: $\gamma(\theta)\Big|_{\theta=\pi} = \gamma(x)\Big|_{x=c} = 0$ (T.E.)

Glauert's Integral: $$\int_0^\pi \frac{\cos(n\theta_0)}{\cos\theta_0-\cos\theta}\,d\theta_0 = \frac{\pi\sin(n\theta)}{\sin\theta}$$

$n = 0, 1, 2 \ldots$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz_c(\theta)}{dx}\,d\theta \qquad\Bigg|\qquad C_l = 2\pi(A_0 + A_1/2)$$

$$C_l = 2\pi(\alpha - \alpha_{L0})$$

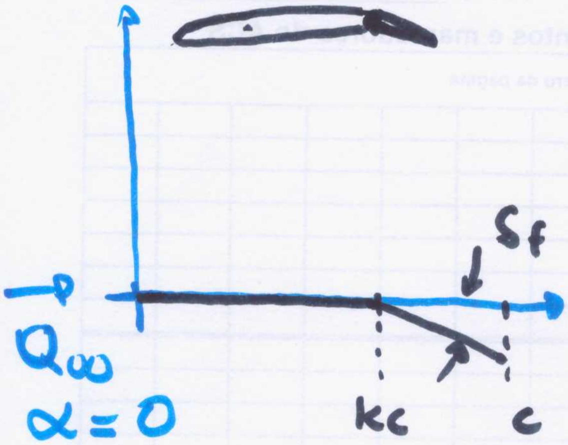
$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dz_c(\theta)}{dx}\cos(n\theta)\,d\theta \qquad\Bigg|\qquad C_{m_{ac}} = \frac{\pi}{4}(A_2 - A_1)$$

$$x_{ac} = \frac{c}{4}, \quad \alpha_{L0} = -\frac{1}{\pi}\int_0^\pi \frac{dz_c(\theta)}{dx}[\cos\theta - 1]\,d\theta$$

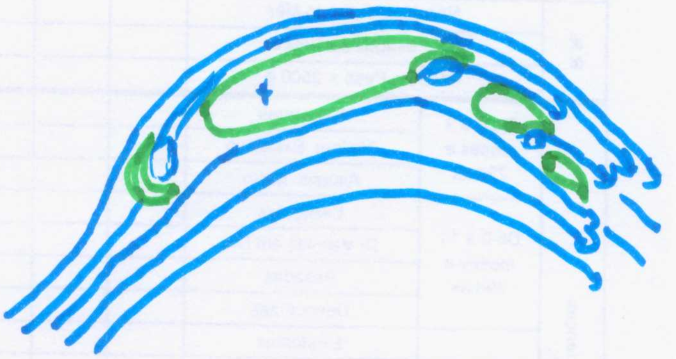
$$x = \frac{c}{2}(1-\cos\theta) \qquad \underset{\substack{\uparrow\\ \text{LE}}}{0} \le \theta \le \underset{\substack{\uparrow\\ \text{TE}}}{\pi}$$

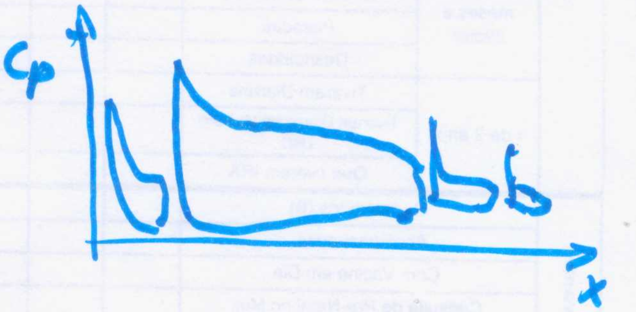
$$dx = \frac{c}{2}\sin\theta\,d\theta$$

Flapped Airfoil



Not the same thing as Hyperlifting devices:





$$\delta_f << 1$$

$$\alpha_{L0} = -\frac{1}{\pi}\int_0^\pi \frac{dy_c}{dx}(\theta)\underline{(\cos\theta - 1)}\,d\theta$$

Flaps afects camber y_c:
the factor $(\cos\theta - 1)$ is larger at $\theta \cong \pi$ than it would be at $\theta \cong 0$

$$\frac{dy_c}{dx} = 0 \quad \text{for} \quad 0 \leq x < kc \quad \Big| \; k \in \mathbb{R}$$

$$\frac{dy_c}{dx} = -\delta_f \quad \text{for} \quad k_c < x \leq c \quad \Big| \; 0 < k < 1$$

$$\theta_k \Rightarrow x = k\cancel{c} = \frac{\cancel{c}}{2}(1 - \cos\theta_k)$$

$$\boxed{1 - 2k = \cos\theta_k}$$

$$\frac{dy_c}{dx}(\theta) = 0 \quad \text{for } 0 \le \theta < \theta_k$$

$$\alpha_{L0} = -\frac{1}{\pi}\int_0^{\pi} \frac{dy_c}{dx}(\theta)\,(\cos\theta - 1)\,d\theta$$

$$= -\frac{\delta_f}{\pi}\int_0^{\pi}[\cos(\theta) - 1]\,d\theta = -\frac{\delta_f}{\pi}\Big[\sin\theta - \theta\Big]_{\theta_k}^{\pi}$$

$$\boxed{\alpha_{L0} = \frac{\delta_f}{\pi}\left\{\sin\theta_k + (\pi - \theta_k)\right\}} \quad \Big|\ \theta_k \Rightarrow \text{rad.}$$

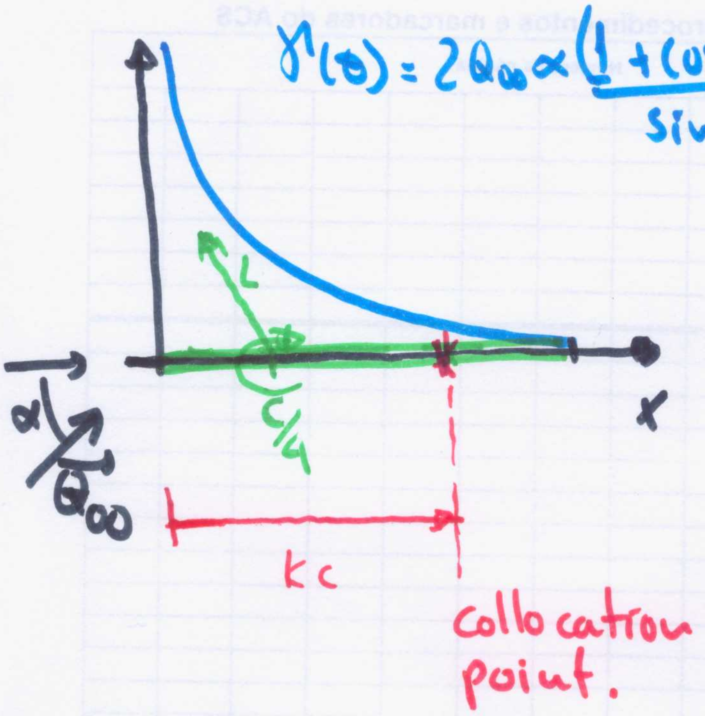
On substituting the above result for its counterpart in the expressions for C_l and C_{mac}, we get:

$$\frac{d\Delta C_l}{d\delta_f} = \left[2(\pi - \theta_k) + 2\sin(\theta_k)\right] \Rightarrow \Delta C_L = \frac{d\Delta C_L}{d\delta_f}\,\delta_f$$

$$\frac{d\Delta C_{mac}}{d\delta_f} = \left[\frac{\sin(2\theta_k)}{4} - \frac{\sin(\theta_k)}{2}\right] \Rightarrow \Delta C_{mac} = \frac{d\Delta C_{mac}}{d\delta_f}$$

$$\frac{d\Delta\alpha_{L0}}{d\delta_f} = \frac{1}{\pi}\left[\sin(\theta_k) + (\pi - \theta_k)\right] \Rightarrow \Delta\alpha_{L0} = \frac{d\Delta\alpha_{L0}}{d\delta_f}\,\delta$$

Lumped - vortex element



$$\gamma(\theta) = 2Q_\infty \alpha \frac{1+\cos\theta}{\sin\theta} \;;\; \Gamma = 2Q_\infty \alpha \int_0^\pi \frac{(1+\cos\theta)}{\cancel{\sin\theta}} \frac{c}{2} \cancel{\sin\theta}\, d\theta$$

$$\Gamma = \pi c Q_\infty \alpha$$

$$W(x,0) = \frac{-\gamma_0}{2\pi(x-x_0)} \;;\; x \neq x_0$$

Now, for our lumped vortex, $\gamma_0 = \Gamma$ and it sits at $x = c/4$

To find out kc, i.e. the point where tangency is met, we make:

$$W(kc,0) = \frac{-\Gamma}{2\pi c\left(k-\frac{1}{4}\right)} = Q_\infty \left[\cancelto{0}{\frac{dy_c}{dx}} - \alpha\right]$$

$$\frac{-\Gamma}{2\pi c\left(k-\frac{1}{4}\right)} = -Q\alpha \quad ; \quad \Gamma = \pi c Q_\infty \alpha$$

$$k = \frac{3}{4} \Rightarrow \boxed{kc = x = \frac{3c}{4}}$$

Collocation Point

Generalized Kutta - Joukowsky theorem

(Katz and Plotkin → P. 146 - 150)

Blazius : $X - iZ = \frac{i\rho}{2}\oint_c W^2 dy$

$$L = \rho Q_\infty \Gamma \left(1 + \frac{\vec{Q}_\infty \cdot \vec{q}_e}{Q_\infty^2}\right)$$

Γ → circulation of the vortex for which you are computing L

where $\vec{q}_e$ is the velocity "induced" by other vortices at the airfoil vortex location

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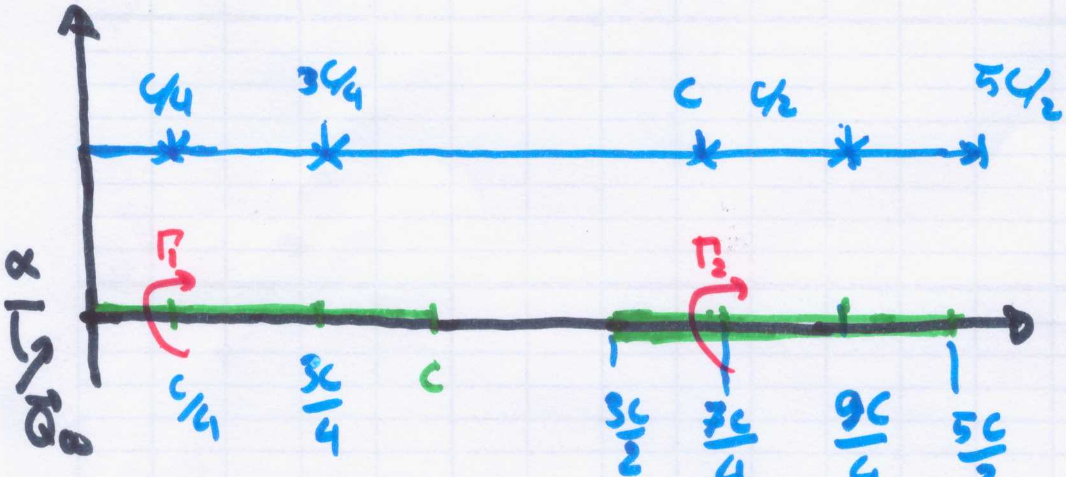
Tandem Airfoils



On assuming that $\alpha \ll 1$ the streamwise contribution "u" each vortex induces on the other is neglected. We are left with the vertical contribut only.

$$W_{1\to2} = -\frac{\Gamma_1}{2\pi c\left(\frac{7}{4}-\frac{1}{4}\right)} = -\frac{\Gamma_1}{3\pi c}$$

$$W_{2\to1} = -\frac{\Gamma_2}{2\pi c\left(\frac{1}{4}-\frac{7}{4}\right)} = +\frac{\Gamma_2}{3\pi c}$$

$$L_1 = \rho Q_\infty \Gamma_1 \left(1 + \frac{\alpha \Gamma_2}{3\pi c}\right)$$

$$L_2 = \rho Q_\infty \Gamma_2 \left(1 - \frac{\alpha \Gamma_1}{3\pi c Q_\infty}\right)$$

$\vec{W}\cdot\vec{Q}_\infty = W Q_\infty \cos\alpha = W Q_\infty \alpha$

tangency condition:

$$w = Q_\infty \left[\cancelto{0}{\frac{dy_c}{dx}} - \alpha \right]$$

(Nett)

$$w_1 = \frac{-\Gamma_1}{2\pi(c/2)} + \frac{\Gamma_2}{2\pi c} + Q_\infty \alpha = 0$$

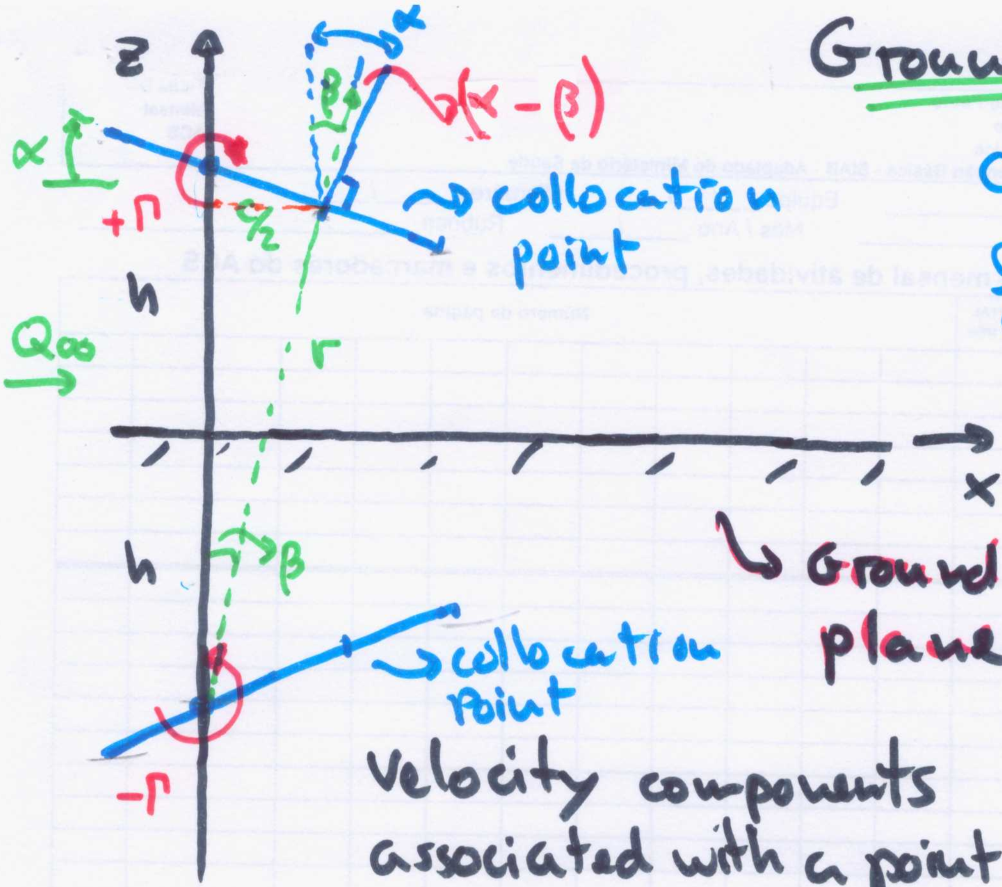
$$w_2 = \frac{-\Gamma_1}{2\pi 2c} + \frac{-\Gamma_2}{2\pi(c/2)} + Q_\infty \alpha = 0$$

$$\Gamma_1 = \frac{4}{3}\pi c Q_\infty \alpha \quad ; \quad \Gamma_2 = \frac{2}{3}\pi c Q_\infty \alpha$$

$$C_{\ell_1} = \frac{8\pi\alpha}{3}\left(1 + \frac{2\alpha^2}{9}\right)$$

$$C_{\ell_2} = \frac{4\pi\alpha}{3}\left(1 - \frac{4\alpha^2}{9}\right)$$

Ground Effect



Cosine Law

$$\frac{c^2}{4} = 4h^2 + r^2 - 2(2hr\cos\beta)$$

$$r\cos\beta = 2h - \frac{c}{2}\sin\alpha$$

Hence, we get:

$$\boxed{r^2 = \frac{c^2}{4} + 4h^2 - 2hc\sin\alpha}$$

Velocity components associated with a point vortex:

$$\begin{cases} u = \dfrac{\Gamma}{2\pi}\dfrac{(z-z_0)}{(z-z_0)^2+(x-x_0)^2} \\ w = -\dfrac{\Gamma}{2\pi}\dfrac{(x-x_0)}{(z-z_0)^2+(x-x_0)^2} \end{cases}$$

normal unit vector: $\hat{n} = (\sin\alpha, \cos\alpha)^T \cong (\alpha, 1)^T$

upper collocation point: $x = \frac{c}{2}\cos\alpha$, $z = -\frac{c}{2}\sin\alpha$

"Induced" velocities:

at upper vortex:

$$\vec{q}_I = \begin{pmatrix} -2\Gamma\dfrac{2h}{(2h)^2} \\ +\dfrac{\Gamma}{2\pi}\dfrac{0}{(2h)^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{\Gamma}{4\pi h} \\ 0 \end{pmatrix}$$

at upper collocation point

$$\vec{q}_{Ic} = \frac{-\Gamma}{2\pi}\begin{pmatrix} \dfrac{2h - c\sin\alpha/2}{r^2} \\ \dfrac{c\cos\alpha/2}{r^2} \end{pmatrix}$$

$$r = \sqrt{4h^2 + c^2/4 - 2ac\sin\alpha}$$

$\|\vec{q}_{1c}\| = \frac{+\Gamma}{2\pi r}$; $\vec{q}_{1c}\cdot\hat{n} = \frac{\Gamma}{2\pi r}\cos\left(\beta-\alpha+\frac{\pi}{2}\right)$

$\vec{q}_{1c}\cdot\hat{n} = -\frac{\Gamma}{2\pi r}\sin(\beta-\alpha)$ ~ velocity that is induced by the lower vortex on the upper collocation point

Velocity induced by the upper vortex on the upper collocation point is given by:

$\vec{q}_{1c1} = -\Gamma/2\pi(c/2) = -\frac{\Gamma}{\pi c}$ $\perp$ to the plate.

Tangency condition implies:

$$-\frac{\Gamma}{\pi c} + \vec{q}_{1c}\cdot\hat{n} + Q_\infty\alpha = 0$$

$$-\frac{\Gamma}{\pi c} - \frac{\Gamma}{2\pi r}\sin(\beta-\alpha) + Q_\infty\overset{\nearrow(\sin\alpha)}{\alpha} = 0$$

$$\Gamma = \pi Q_\infty c\sin\alpha\left[\frac{1-(c/2h)\sin\alpha+(c^2/16h^2)}{1-(c/4h)\sin\alpha}\right]$$

$$L = \rho Q_\infty \Gamma \left(1 - \frac{\vec{Q_\infty} \cdot \vec{q_c}}{Q_\infty^2} \right)$$

$$L = \rho Q_\infty \Gamma \left(1 - \frac{\Gamma}{4\pi Q_\infty h} \right)$$

Ground effect

On substituting the above result for Γ, and on taking it to the limit as c/h → 0, we get

$$L = \pi \rho Q_\infty^2 c \sin\alpha \left[1 - \frac{c}{2h} \sin\alpha + \frac{c^2}{16 h^2} (1 + \sin^2\alpha) + O\left(\frac{c^3}{h^3}\right) \right.$$

04/06/2020 ①

P. 4.6 and P4.7 (P. 228) Anderson's book :
Fundamentals of Aerodynamics

NACA 4012

$$\tilde{\eta}_c \begin{cases} = 0.25(0.8\tilde{x} - \tilde{x}^2) & \text{for } 0 \le \tilde{x} \le 0.4 \\ = 0.111(0.2 + 0.8\tilde{x} - \tilde{x}^2) & \text{for } 0.4 \le \tilde{x} \le 1 \end{cases}$$

where $\tilde{x} = x/c$ and $\hat{\eta}_c = \eta/c$

$$\tilde{x} = \frac{1 - \cos\theta}{2} \; ; \; d\tilde{x} = \frac{\sin\theta \, d\theta}{2} \; ; \; \begin{cases} \tilde{x} = 0 \Rightarrow \theta = 0 \\ \tilde{x} = 1 \Rightarrow \theta = \pi \end{cases}$$

$$A_0 = \alpha \underbrace{- \frac{1}{\pi}\int_0^\pi \eta'_c(\theta) d\theta}_{+\alpha_{L0}} \Rightarrow \alpha + \alpha_{L0}$$

$$A_n = \frac{2}{\pi}\int_0^\pi \eta'_c(\theta)\cos(n\theta) d\theta$$

$$C_l = 2\pi(\alpha - \alpha_{L0})$$

$$C_{mac} = \frac{\pi}{4}(A_2 - A_1)$$

$$\tilde{x}_{ac} = 1/4$$

$$\eta'_c = \begin{cases} 0.25(0.8 - 2\tilde{x}) \Rightarrow 0.25(\cos\theta - 0.2) & 0 \le \theta \le \theta_s \\ 0.111(0.8 - 2\tilde{x}) \Rightarrow 0.111(\cos\theta - 0.2) & \theta_s \le \theta \le \end{cases}$$

$$\cos(\theta_s) = 0.2 \Rightarrow \theta_s \cong 1.3694 \text{ rd}$$

$$I_0 = \int_a^b k'_c(\theta)d\theta = \int[\cos\theta - 0.2]d\theta = [\sin\theta]_c^b - [\theta]_c^b$$

$$I_1 = \int_a^b k'_c(\theta)\cos\theta d\theta = \int_a^b(\cos^2\theta - 0.2\cos\theta)d\theta =$$

$$= \frac{1}{2}[\theta]_a^b + \frac{1}{4}[\sin(2\theta)]_a^b - \frac{2}{10}[\sin\theta]_a^b$$

$$I_2 = \int_a^b k'_c(\theta)\cos(2\theta)d\theta = \int_a^b(\cos\theta - 0.2)\cos(2\theta)d\theta =$$

$$= \frac{1}{6}[\sin(3\theta)]_a^b - 0.1[\sin(2\theta)]_a^b + \frac{1}{2}[\sin(\theta)]_a^b$$

$$\alpha_{L0} = -\frac{1}{\pi}\left\{0.25[I_1 - I_0]_0^{\theta_s} + 0.111[I_1 - I_0]_{\theta_s}^{\pi}\right\}$$

$$\alpha_{L0} \cong -0.0724274 \text{ rd} \cong -4.15^\circ$$

$$C_l = 2\pi(\alpha - \alpha_{L0}) \cong 2\pi(\alpha + 0.0724)$$

$$C_l\Big|_{\alpha=3^\circ} = 0.784061$$

$$A_1 = \frac{2}{\pi}\left\{0.25[I_1]_0^{\theta_s} + 0.111[I_1]_{\theta_s}^{\pi}\right\} \cong 0.162921$$

$$A_2 = \frac{2}{\pi}\left\{0.25[I_2]_0^{\theta_s} + 0.111[I_2]_{\theta_s}^{\pi}\right\} \cong 0.0277447$$

$$C_{mac} \cong -0.106167$$

Circulation:

$$\Gamma \equiv \int_{\mathcal{C}} \vec{u}\cdot d\vec{l} = \oiint_{S_{\mathcal{C}}} (\nabla\times\vec{u})\cdot\hat{n}\, ds = \oiint_{S_{\mathcal{C}}} \vec{\omega}\cdot\hat{n}\, ds$$

↳ Stokes Theorem

$$\frac{D\vec{\omega}}{Dt} = \underbrace{\vec{\omega}\cdot\nabla\vec{u} - \vec{\omega}\,\nabla\cdot\vec{u}}_{\text{Vorticity distribution remains}} + \nabla T\times\nabla S - \frac{\nabla p}{\rho^2}\times\nabla\cdot\tau + \frac{\nabla\times(\nabla\cdot\tau)}{\rho}$$

↳ sources have been removed.

Kelving-Helmholtz's Theorem

$$\frac{D\Gamma}{Dt} = \oint_{\mathcal{C}} \frac{D\vec{u}}{Dt}\cdot d\vec{l} = \oint_{\mathcal{C}} \vec{a}\cdot d\vec{l} \Rightarrow \frac{D\Gamma}{Dt} = \underbrace{\oint_{\mathcal{C}} \rho\,\vec{g}\cdot dl - \int_{\mathcal{C}} \frac{\nabla p}{\rho}\cdot d\vec{l}}_{} =$$

There fore, we get:

$$\frac{D\Gamma}{Dt} = 0$$

Stokes Theorem implies these two integrals vanish. their kernels are irrotational.

Reference:

Kuethe A.M. and Chow, C.Y. "Foundations of Aerodynamics Design". NY, J. Wiley, 1998 5th ed.

Chapter 6

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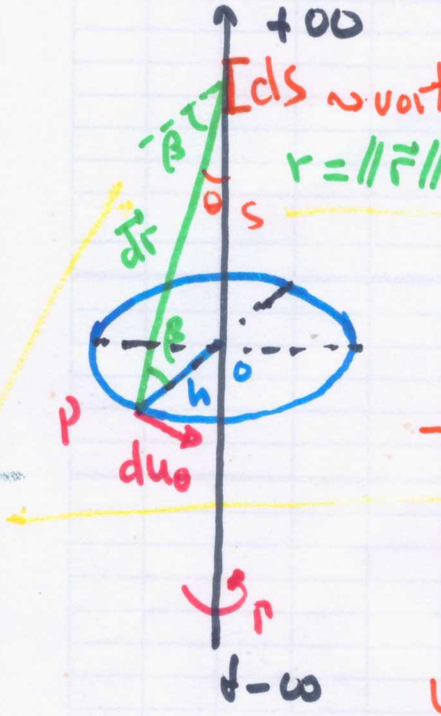
straight vortex



$$\frac{h}{s} = \tan\theta \Rightarrow ds = \frac{-h\,d\theta}{\sin^2\theta}$$

$$\sin\theta = \frac{h}{r} \Rightarrow r = \frac{h}{\sin\theta}$$

$$\beta + \theta = \frac{\pi}{2} \Rightarrow \cos\beta = \sin\theta$$

$$du_\theta = -\frac{\Gamma}{4\pi}\,\frac{\cos\beta\,ds}{r^2}$$

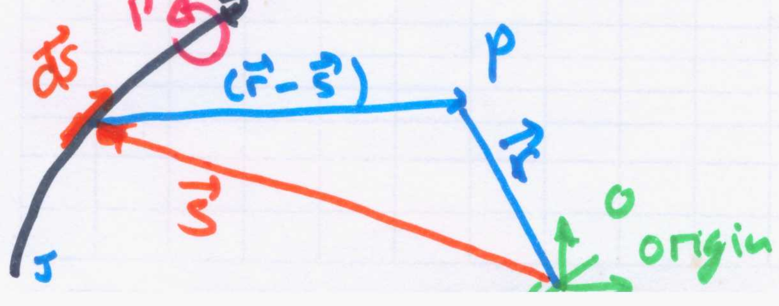
$$-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$$

Plane $\perp$ vortex

$$u_\theta = -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\cos\beta\,ds}{r^2} = -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta}{r^2}\,ds$$

$$u_\theta = \frac{\Gamma}{4\pi}\int_0^{\pi}\frac{h\sin^3(\theta)}{h^2\sin^2(\theta)}\,d\theta = \frac{\Gamma}{4\pi h}\int_0^{\pi}\sin\theta\,d\theta \Rightarrow \boxed{U_\theta = \frac{\Gamma}{2\pi h}}$$

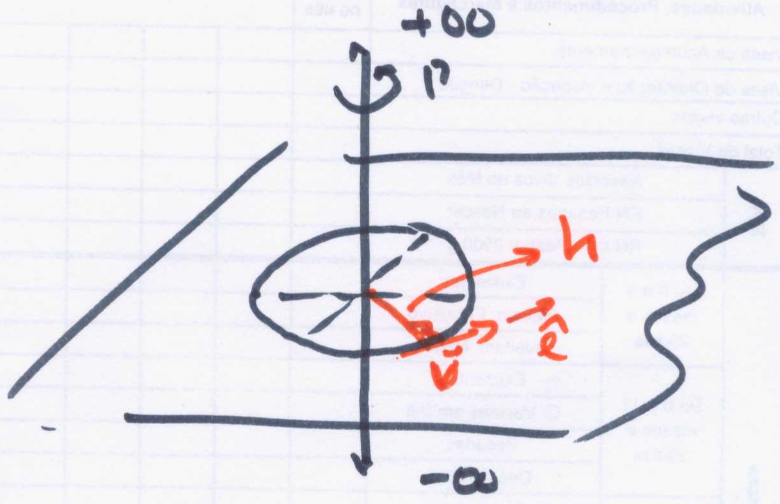
Biot-Savart Law, General form:



$$\vec{V}(\vec{r}) = \frac{\Gamma}{4\pi}\oint_J \frac{d\vec{s}\times(\vec{r}-\vec{s})}{\|\vec{r}-\vec{s}\|^3}$$

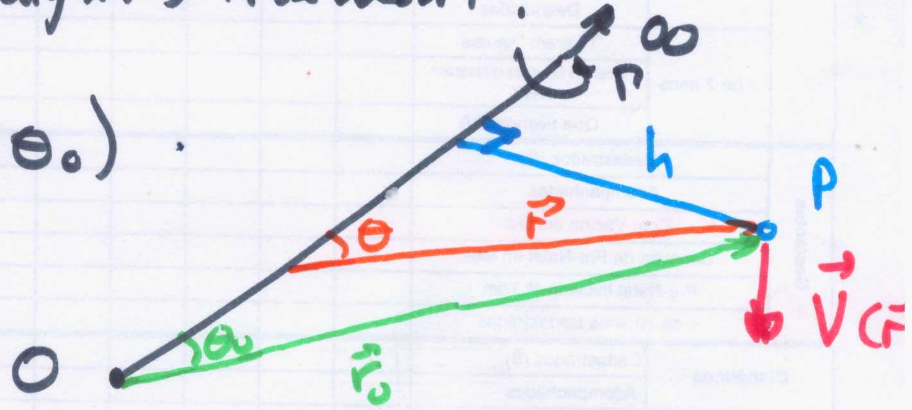
infinite vortex filament: (straight)

$\vec{V}(\vec{r}) = \frac{\Gamma}{2\pi h}\hat{l}$ where the axis $\hat{l}$



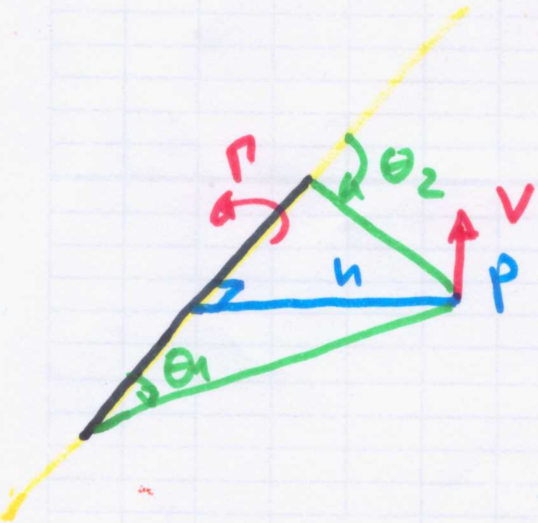
semi-infinite (straight) filament:

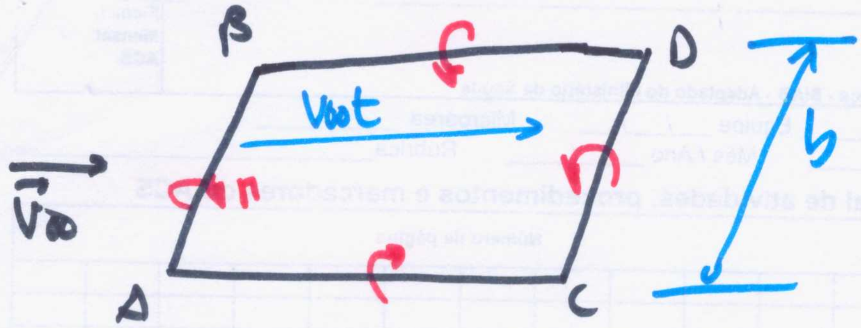
$\vec{V}(\vec{r}) = \frac{\Gamma}{4\pi h}(1+\cos\theta_0)$.

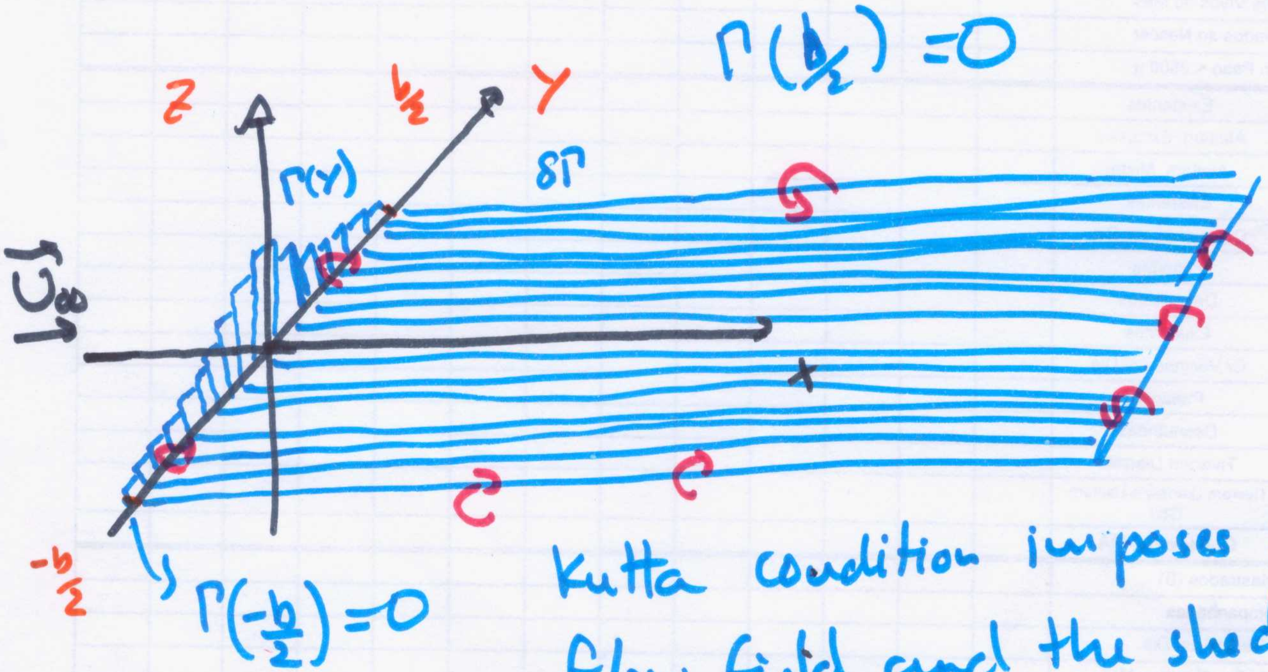


finite (straight) vortex filament:

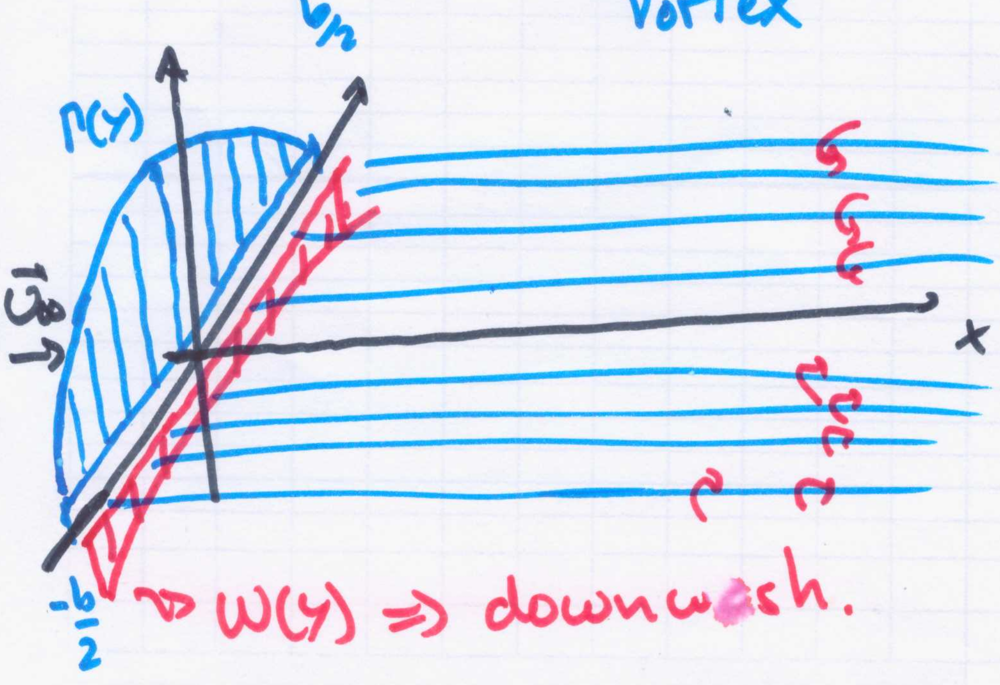
$\vec{V}(\vec{r}) = \frac{\Gamma}{4\pi h}(\cos\theta_1 - \cos\theta_2)$





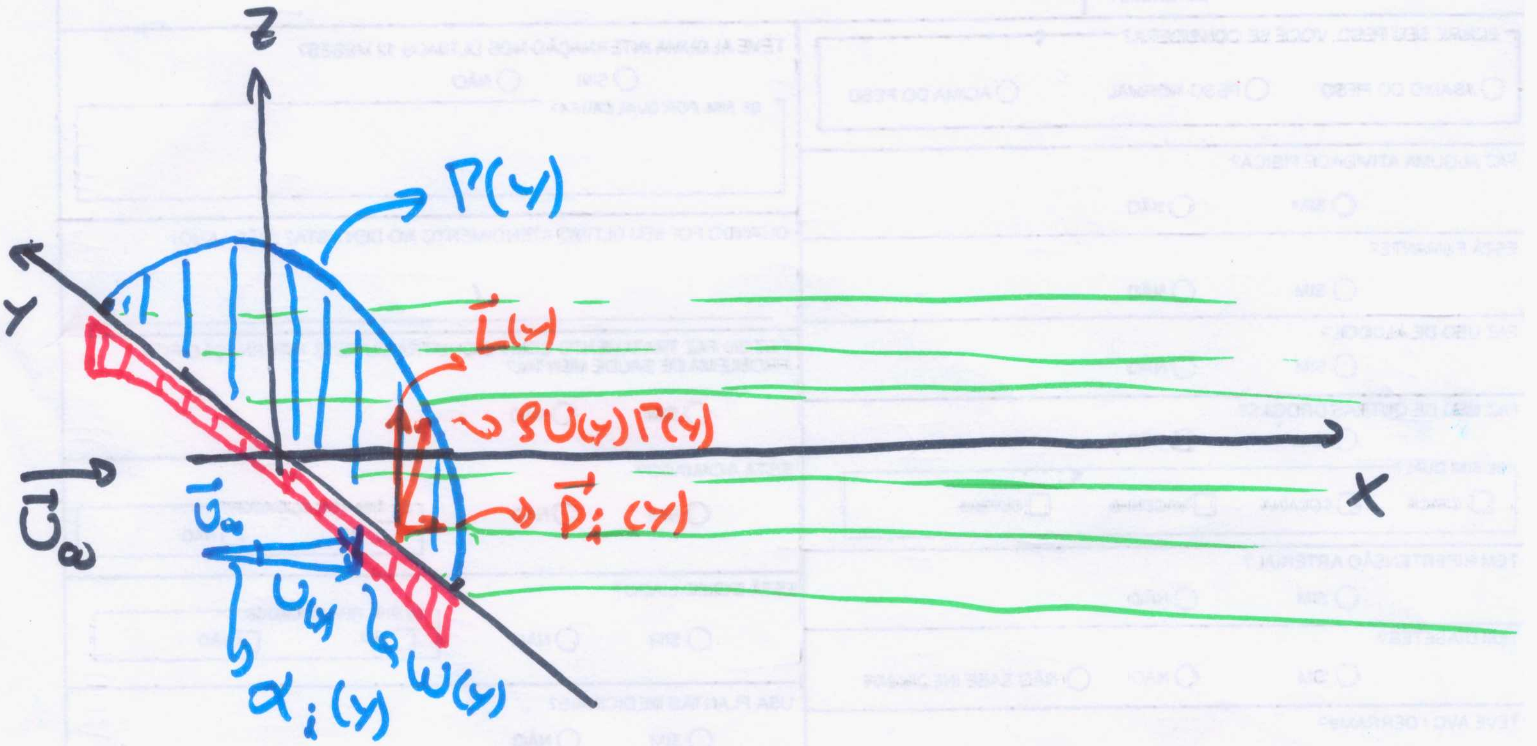


Kutta condition imposes smooth flow field and the shedding of the vortex sheet at the trailing edge. This sheet extends dow to the starting vortex



$\Rightarrow W(y) \Rightarrow$ downwash.

18/06/2020 ①

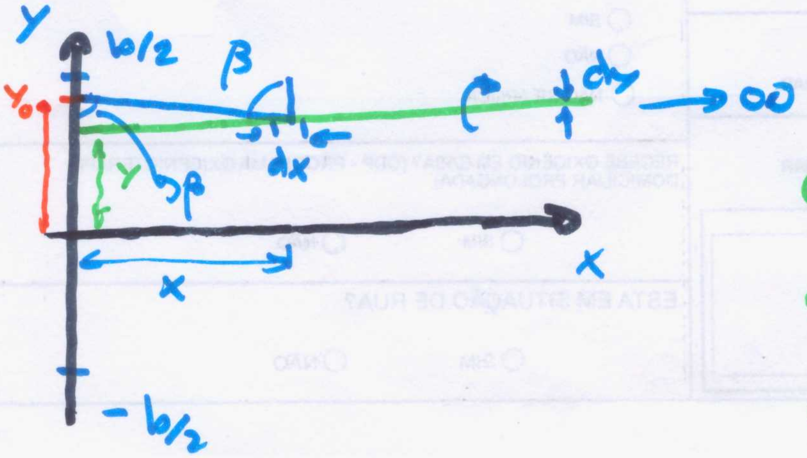


$$d\Gamma = \left(\frac{d\Gamma}{dy}\right)_{wing} dy \qquad \alpha_i \cong \frac{w}{U_\infty}$$

induced angle of attack ⇒ $\alpha_i = \tan^{-1}\left(\frac{w}{U_\infty}\right)$

$$L'(y) = \rho U(y)\Gamma\cos(\alpha_i) \cong \rho U(y)\Gamma(y) \cong \rho U_\infty\Gamma(y)$$

$$D'_i(y) = -\rho U(y)\Gamma\sin(\alpha_i) \cong -L'\alpha_i = -\rho w\Gamma(y)$$



$$dw_{y_0,y} = -\frac{d\Gamma}{4\pi}\frac{\cos\beta\, dx}{r^2}$$

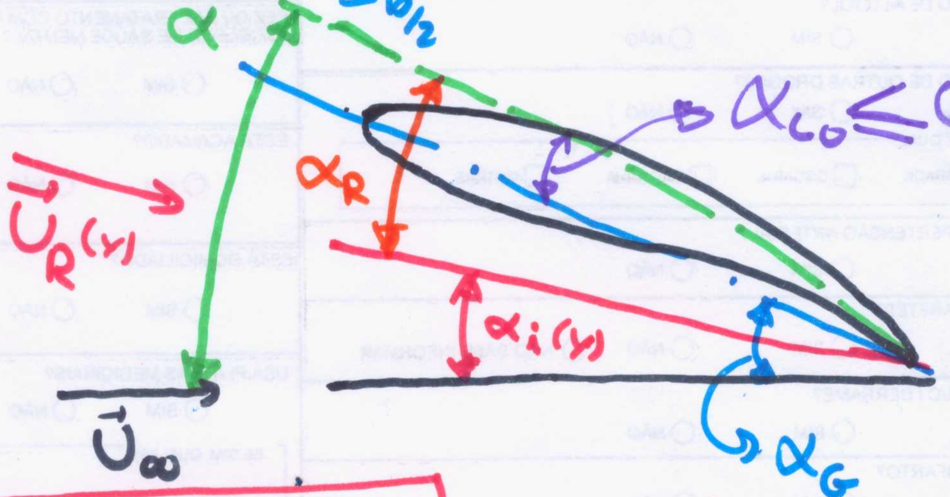
$$dw_{y_0,y} = -\frac{d\Gamma}{4\pi h}(1+\cos\theta_0)$$

$$\theta_0 \frac{\pi}{2}, h = (y_0 - y)$$

$$dw_{y_0 y} = -\frac{d\Gamma}{4\pi(y_0-y)}$$

$$dw_{y_0 y} = -\frac{d\Gamma}{4\pi}\int_0^\infty \frac{\cos\beta\, dx}{r^2} = -\frac{d\Gamma}{4\pi}\frac{1}{(y_0-y)}$$

$$\alpha_i(y_0) = \frac{w_i(y_0)}{U_\infty} = -\frac{1}{4\pi U_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)_{wing}\, dy}{(y_0-y)}$$



2-D a.o.a:

$\alpha = \alpha_G - \alpha_{L0}$

3D a.o.a

$\alpha_R = \alpha(y) - \alpha_i(y)$

$\alpha_R = \alpha_R(y)$

$$\boxed{\alpha_R(y) = (\alpha_G - \alpha_{L0}(y)) - \alpha_i(y)}$$

$\alpha_i(y) \geq 0$

From the thin airfoil theory, we have:

$\Gamma = \frac{1}{2}\frac{dc_l}{d\alpha}\alpha_R U_R c$ $\qquad \frac{dc_l}{d\alpha} = a_0 = m_0 = 2\pi$

↳ chord length

$$\Gamma(y) = \frac{a_0}{2} c(y) U_\infty \left[\alpha(y) - \alpha_i(y)\right]$$

$$\boxed{\Gamma(y) = \frac{a_0}{2} c(y)\left[U_\infty \alpha(y) - \frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{d\Gamma}{dy}(\eta)\frac{d\eta}{(y-\eta)}\right]}$$

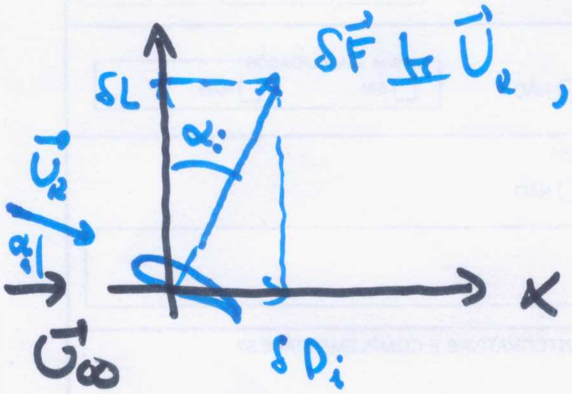
Fundamental equation of Prandtl's Lifting Line Theory

③

$$\Gamma(y) = \frac{a_0}{2} c(y) \left[\underbrace{U_\infty \alpha(y)}_{2.D\,a.o.a.} - \underbrace{\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{d\Gamma}{dy}(k) \frac{dk}{(y-k)}}_{U_\infty \alpha_i(y)} \right]$$

wing tips: $\Gamma\left(-\frac{b}{2}\right) = \Gamma\left(\frac{b}{2}\right) = 0$

$$a_0 = C_{l\alpha} = \frac{dC_l}{d\alpha} = 2\pi$$



$\delta \vec{F} \perp \vec{U}_k$, $\vec{\delta L} \perp \vec{U}_\infty$, $\vec{\delta D_i} \parallel \vec{U}_\infty$

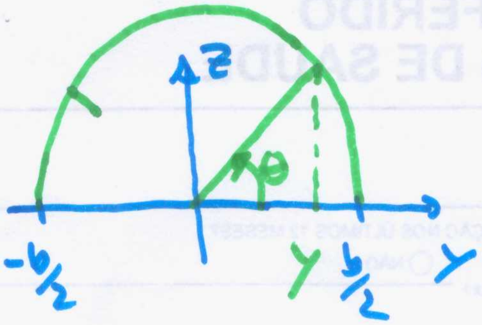
$\Gamma' = \frac{d\Gamma(y)}{dy}$, $\boxed{\alpha_i << 1}$

$$\delta L = \rho U_\infty \Gamma'(y) dy \Rightarrow \delta F(y) \cos\alpha_i \Rightarrow \boxed{\delta L(y) \cong \delta F(y)}$$

$$\delta D_i(y) = \delta F(y) \sin\alpha_i \Rightarrow \delta D_i(y) = \delta L(y) \alpha_i(y)$$

$$\begin{cases} \delta L \cong \rho U_\infty \Gamma'(y) dy \\ \delta D_i \cong \rho U_\infty \Gamma'(y) \alpha_i(y) dy \end{cases}$$

$$w_i(y) = U_\infty \alpha_i(y)$$



$$y = \frac{b}{2}\cos\theta \; ; \; 0 \le \theta \le \pi$$

$$\int_0^\pi \frac{\cos(n\theta)d\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin(n\theta_0)}{\sin(\theta_0)}$$

$$\Gamma(\theta) = 2bU_\infty \sum_{n=1}^{\infty} A_n \sin(n\theta)$$

$$W(\theta) = U_\infty \sum_{n=1}^{\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)} \; ; \; A_n = \frac{a_n}{bU_\infty \left(b/4\right)^n}$$

$$\mu(\theta) \equiv \frac{\overset{a_\alpha = 2\pi}{\cancel{a_0}}(\theta)\, C(\theta)}{4b}$$

Aspect Ratio:

$$AR = \frac{b^2}{S_w}$$

b^2 → wing span

S_w → wing plan Area

$$\boxed{\sum_{n=1}^{\infty} A_n \sin(n\theta)\left[n\mu(\theta) + \sin(\theta)\right] = \mu(\theta)\alpha(\theta)\sin(\theta)}$$

$$C_L = \frac{L}{\frac{\rho U_\infty^2}{2} S_w} = \pi AR\, A_1$$

$$C_{D_i} = \frac{D_i}{\frac{\rho U_\infty^2}{2} S_w} = \frac{C_L^2}{\pi AR}(1+\delta) \; ; \; \delta = \sum_{n=2}^{\infty} n \left(\frac{A_n}{A_1}\right)^2$$

$\delta \geq 0$

$$C_{m_R} = \frac{M_R}{\frac{\rho U_\infty^2}{2} S_w \bar{c}} = -\frac{\pi}{4}(AR^2) A_2 \quad (\text{Roll})$$

$$C_{m_y} = \frac{M_y}{\rho U_\infty^2 S_w \bar{c}} = \frac{\pi}{4}(AR^2) \sum_{n=1}^{\infty} (2n+1) A_n A_{n+1} \quad (\text{Yaw})$$

Let's assume a situation where only $A_1 \neq 0$ and $A_k = 0 \;\; \forall k > 1$

$C_L = \pi AR A_1$, $C_{Di} = \frac{C_L^2}{\pi AR}$ $(\delta = 0)$, $C_{m_R} = C_{m_y} = 0$

For an ellipitc loaded wing, we have

$\Gamma(\theta) = \Gamma_0 \sin\theta$, $\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$A_1 = \frac{\Gamma_0}{2bU_\infty}$