Casimir Operators

Adjoint Representation

$$g\,T_a\,g^{-1} = T_b\,d_{ba}(g) \qquad d(g)\,d(g') = d(g\,g') \qquad d(g^{-1}) = d^{-1}(g)$$

Invariant tensors

$$\tau_a = d_{ab}\,\tau_b$$

Consider

$$C = \tau_a\,D(T_a)$$

$$D(g)C = \tau_a\,D(g)D(T_a) = \tau_a\,D(g\,T_a\,g^{-1})\,D(g) = T_b\,d_{ba}(g)\,\tau_a\,D(g) = C\,D(g)$$

Higher Tensors

$$\tau_{a_1 a_2 a_3} = d_{a_1 b_1}\, d_{a_2 b_2}\, d_{a_3 b_3}\, \tau_{b_1 b_2 b_3}$$

Take $\quad C = \tau_{a_1 a_2 a_3}\, D(T_{a_1})\, D(T_{a_2})\, D(T_{a_2})$

$$D(g)\, C = \tau_{a_1 a_2 a_3}\, D(g\, T_{a_1}\, g^{-1}\, g\, T_{a_2}\, g^{-1}\, g\, T_{a_2}\, g^{-1})\, D(g)$$

$$= D(T_{b_1})\, D(T_{b_2})\, D(T_{b_3})\, d_{b_1 a_1}\, d_{b_2 a_2}\, d_{b_3 a_3}\, \tau_{a_1 a_2 a_3}\; D(g)$$

$$= D(T_{b_1})\, D(T_{b_2})\, D(T_{b_3})\, \tau_{b_1 b_2 b_3}\; D(g)$$

$$= C\; D(g)$$

How to construct such tensors

$$\mathrm{Tr}(T_{a_1}\,T_{a_2}\,T_{a_3}) = \mathrm{Tr}(gT_{a_1}g^{-1}\,gT_{a_2}g^{-1}\,gT_{a_3}g^{-1}) =$$

$$= \mathrm{Tr}(T_{b_1}\,T_{b_2}\,T_{b_3})\,d_{b_1a_1}\,d_{b_2a_2}\,d_{b_3a_3}$$

Invariant but on the other side

$$\eta_{ab} = \mathrm{Tr}(T_aT_b) = \eta_{cd}\,d_{ca}\,d_{db} \qquad \eta = d^T\,\eta\,d \qquad \eta^{-1} = d^{-1}\,\eta^{-1}\,(d^T)^{-1}$$

Take $\qquad \tau_{a_1a_2a_3} = \mathrm{Tr}(T_{b_1}T_{b_2}T_{b_3})\,\eta^{-1}_{b_1a_1}\,\eta^{-1}_{b_2a_2}\,\eta^{-1}_{b_3a_3}$

$$\tau_{a_1a_2a_3} \to \mathrm{Tr}(T_{b_1}T_{b_2}T_{b_3})\,(\eta^{-1}(d^T)^{-1})_{b_1a_1}(\eta^{-1}(d^T)^{-1})_{b_2a_2}\,(\eta^{-1}(d^T)^{-1})_{b_3a_3}$$

$$= d^{-1}_{a_1b_1}\,d^{-1}_{a_2b_2}\,d^{-1}_{a_3b_3}\,\tau_{b_1b_2b_3}$$

Note $$\mathrm{Tr}(T_{b_1}T_{b_2}T_{b_3}) = \mathrm{Tr}((\frac{1}{2}\,[T_{b_1}\,,\,T_{b_2}] + \frac{1}{2}\,\{T_{b_1}\,,\,T_{b_2}\}T_{b_3})$$
$$= \frac{1}{2}\,f^{c}_{b_1 b_2}\,\mathrm{Tr}(T_c T_{b_3}) + \frac{1}{2}\,\mathrm{Tr}(\{T_{b_1}\,,\,T_{b_2}\}T_{b_3})$$

need to symmetrise it

Casimir Operators

$$C^{(n)} = \mathrm{Tr}_S(T_{b_1}T_{b_2}\ldots T_{b_n})\,\eta^{-1}_{b_1 a_1}\eta^{-1}_{b_2 a_2}\,\ldots\eta^{-1}_{b_n a_n}\,D(T_{a_1})\,D(T_{a_2})\,\ldots D(T_{a_n})$$

$$C^{(n)}\,D(g) = D(g)\,C^{(n)}$$

Schur's Lemma: in an irred. rep. $C^{(n)} \sim 1$

Number of Casimir Operators = rank of the algebra

3.6 Casimir operators

Let $\Gamma^{s_1 s_2 \ldots s_n}$ be a tensor invariant under the adjoint representation of a Lie group G. By that we mean

$$\Gamma^{s_1 s_2 \ldots s_n} = d^{s_1}_{s'_1}(g)\, d^{s_2}_{s'_2}(g) \ldots d^{s_n}_{s'_n}(g)\, \Gamma^{s'_1 s'_2 \ldots s'_n} \tag{3.57}$$

for any $g \in G$, and where $d^{s_j}_{s'_j}(g)$ is the matrix representing g in the adjoint representation, i.e. $g T_s g^{-1} = T_{s'} d^{s'}_{s}(g)$ (see (2.31)).

Consider now a representation D of G and construct the operator

$$C_n^{(D)} \equiv \Gamma^{s_1 s_2 \ldots s_n}\, D(T_{s_1})\, D(T_{s_2}) \ldots D(T_{s_n}) \tag{3.58}$$

Notice that such operator can only be defined on a given representation since it involves the product of operators and not Lie brackets of the generators.

We then have

$$\begin{aligned} D(g)\, C_n^{(D)} &= \Gamma^{s_1 s_2 \ldots s_n}\, D\left(g T_{s_1} g^{-1}\right) D\left(g T_{s_2} g^{-1}\right) \ldots D\left(g T_{s_n} g^{-1}\right) D(g) \\ &= d^{s'_1}_{s_1}(g) \ldots d^{s'_n}_{s_n}(g)\, \Gamma^{s_1 \ldots s_n} D\left(T_{s'_1}\right) \ldots D\left(T_{s'_n}\right) D(g) \\ &= \Gamma^{s'_1 \ldots s'_n} D\left(T_{s'_1}\right) \ldots D\left(T_{s'_n}\right) D(g) \\ &= C_n^{(D)} D(g) \end{aligned} \tag{3.59}$$

So, we have shown that $C_n^{(D)}$ commutes with any matrix of the representation

$$\left[C_n^{(D)} ,\, D(g) \right] = 0 \tag{3.60}$$

We are interested in operators that can not be reduced to lower orders. That implies that the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ has to be totally symmetric. Indeed, suppose that $\Gamma^{s_1 s_2 \ldots s_n}$ has an antisymmetric part in the indices s_j and s_{j+1}. Then we write

$$\begin{aligned} D\left(T_{s_j}\right) D\left(T_{s_{j+1}}\right) &= \frac{1}{2}\{D\left(T_{s_j}\right) , D\left(T_{s_{j+1}}\right)\} + \frac{1}{2}\left[D\left(T_{s_j}\right), D\left(T_{s_{j+1}}\right)\right] \\ &= \frac{1}{2}\{D\left(T_{s_j}\right) , D\left(T_{s_{j+1}}\right)\} + f^{t}_{s_j s_{j+1}} D\left(T_t\right) \end{aligned} \tag{3.61}$$

and so, $C_n^{(D)}$ will have terms involving the product of $(n-1)$ operators. Therefore, by totally symmetrizing the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ we get operators $C_n^{(D)}$ which are monomials of order n in $D\left(T_s\right)$'s. Such operators are called *Casimir operators*, and n is called their *order*. They play an important role in representation theory. From Schur's lemma 1.1 it follows that in an irreducible representation the Casimir operators have to be proportional to the identity.

One way of constructing tensors which are invariant under the adjoint representation, is by considering traces of products of generators in a given representation D', since

$$\mathrm{Tr}\left(D'\left(T_{s_1} T_{s_2} \ldots T_{s_n}\right)\right) = \mathrm{Tr}\left(D'\left(g T_{s_1} g^{-1} g T_{s_2} g^{-1} \ldots g T_{s_n} g^{-1}\right)\right) \tag{3.62}$$

Then taking

$$\Gamma_{s_1 s_2 \ldots s_n} \equiv \frac{1}{n!} \sum_{\text{permutations}} \mathrm{Tr}\left(D'\left(T_{s_1} T_{s_2} \ldots T_{s_n}\right)\right) \tag{3.63}$$

we get Casimir operators. However, one finds that after the symetrization procedure very few tensors of the form above survive. It follows that a semisimple Lie algebra of rank r possesses r invariant Casimir operators functionally independent. Their orders, for the simple Lie algebras, are given in table 3.1.

A_r	$SU(r+1)$	2, 3, 4, $\ldots$ $r+1$
B_r	$SO(2r+1)$	2, 4, 6, $\ldots$ $2r$
C_r	$Sp(r)$	2, 4, 6 $\ldots$ $2r$
D_r	$SO(2r)$	2, 4, 6 $\ldots$ $2r-2$, r
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6

3.6.1 The Quadratic Casimir operator

Notice from table 3.1 that all simple Lie groups have a quadratic Casimir operator. That is because all such groups have an invariant symmetric tensor of order two which is the Killing form (see section 2.4)

$$\eta_{st} = \mathrm{Tr}\left(d\left(T_s\right) d\left(T_t\right)\right) \tag{3.64}$$

and

$$C_2^{(D)} \equiv \eta^{st} D\left(T_s\right) D\left(T_t\right) \tag{3.65}$$

where η^{st} is the inverse of η_{st}.

Using the normalization (2.134) of the Killing form, we have that the Casimir operator in the Cartan-Weyl basis is given by

$$C_2^{(D)} = \sum_{i=1}^{r} D\left(H_i\right) D\left(H_i\right) + \sum_{\alpha>0} \frac{\alpha^2}{2}\left(D\left(E_\alpha\right) D\left(E_{-\alpha}\right) + D\left(E_{-\alpha}\right) D\left(E_\alpha\right)\right) \tag{3.66}$$

Since the Casimir operator commutes with all generators, we have from the Schur's lemma 1.1 that in an irreducible representation it must be proportional to the unit matrix. Denoting by λ the highest weight of the irreducible representation D we have

$$\begin{aligned}
C_2^{(D)} \mid \lambda\rangle &= \left(\sum_{i=1}^{r} \lambda_i^2 + \sum_{\alpha>0} \frac{\alpha^2}{2}\left[\, D\left(E_\alpha\right),\, D\left(E_{-\alpha}\right)\right]\right) \mid \lambda\rangle \\
&= \left(\lambda^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} H_\alpha^2\right) \mid \lambda\rangle \\
&= \left(\lambda^2 + \sum_{\alpha>0} \alpha \cdot \lambda\right) \mid \lambda\rangle
\end{aligned} \tag{3.67}$$

where we have used (3.28) and (2.125). So, if D, with highest weight λ, is irreducible, we can write using (3.43) that

$$C_2^{(D)} = \lambda \cdot \left(\lambda + 2\delta\right) \mathbb{1} = \left(\left(\lambda + \delta\right)^2 - \delta^2\right) \mathbb{1} \tag{3.68}$$

where $\mathbb{1}$ is the unit matrix in the representation D under consideration.

Example 3.7 *In the case of $SU(2)$ the quadratic operator is J^2 , i.e., the square of the angular momentum. Indeed, from example 3.1 we have that $\alpha = 1$, and then $\delta = 1/2$ and therefore $C_2^{(D)} = \lambda\left(\lambda + 1\right)$. Since λ is a positive integer or half integer we see that these are really the eigenvalues of J^2.*

3.7 Characters

In definition 1.13 we defined the *character* of an element g of a group G in a given finite dimensional representation of G, with highest weight λ, as being the trace of the matrix that represents that element, i.e.

$$\chi^{\lambda}(g) \equiv \mathrm{Tr}(D(g)) \tag{3.69}$$

Obviously equivalent representations (see section 1.5) have the same characters. Analogously, two conjugate elements, $g_1 = g_3 g_2 g_3^{-1}$, have the same character in all representations. Therefore the conjugacy classes can be labelled by the characters.

Example 3.8 *Using (2.27) and the commutation relations (2.58) for the algebra of* $so(3)$ *(or* $su(2)$*) one gets that*

$$e^{i\frac{\pi}{2}T_2}\, T_3\, e^{-i\frac{\pi}{2}T_2} = T_1 \tag{3.70}$$

and consequently

$$e^{i\frac{\pi}{2}T_2}\, e^{i\theta T_3}\, e^{-i\frac{\pi}{2}T_2} = e^{i\theta T_1} \tag{3.71}$$

An analogous result is obtained if we interchange the roles of the generators T_1 *,* T_2 *and* T_3*. Therefore the rotations by a given angle* θ*, no matter the axis, are conjugate. The conjugacy classes of* $SO(3)$ *are defined by the angle of rotation, and the characters in a representation of spin* j *are given by*

$$\chi^{j}(\theta) = \chi^{j}\left(e^{i\theta T_3}\right) = \sum_{m=-j}^{j} e^{im\theta} \tag{3.72}$$

where m *are the eigenvalues of* T_3 *(see section 2.5). We have a geometric progression and therefore*

$$\chi^{j}(\theta) = \frac{e^{i\left(j+\frac{1}{2}\right)\theta} - e^{-i\left(j+\frac{1}{2}\right)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \tag{3.73}$$

Notice that rotations by θ *and* $-\theta$ *have the same character.*

The relation (3.71) can be generalized for any compact Lie group. Any element of a compact group is conjugate to an element of the abelian subgroup which is the exponentiation of the Cartan subalgebra, i.e.

$$g = g' e^{i\theta \cdot H} g'^{-1} \tag{3.74}$$

Therefore the conjugacy classes, and consequently the characters, can be labelled by r parameters or "angles" (r = rank).

However, the elements of the abelian group parametrized by θ and $\sigma_\alpha(\theta)$ have the same character, since from (2.155) we have

$$S_\alpha e^{i\theta \cdot H} S_\alpha^{-1} = e^{i\sigma_\alpha(\theta) \cdot H} \tag{3.75}$$

Thus the parameter θ and its Weyl reflections parametrize the same conjugacy class.

The generalization of (3.73) to any compact group was done by H. Weyl in 1935. In a representation with highest weight the elements of the conjugacy class labelled by have a character given by

$$\chi^\lambda(\theta) = \frac{\sum_{\sigma \in W} (\text{sign}\sigma)\, e^{i\sigma(\lambda+\delta)\cdot\theta}}{e^{i\delta\cdot\theta} \prod_{\alpha>0} \left(1 - e^{-i\alpha\cdot\theta}\right)} \tag{3.76}$$

where the summation is over the elements σ of the Weyl group W, and where sign is 1 (-1) if the element σ of the Weyl group is formed by an even (odd) number of reflections. δ is the same as the one defined in (3.43). This relation is called the *Weyl character formula*.

The character can also be calculated once one knows the multiplicities of the weights of the representation. From (3.69) and (3.74) we have that

$$\chi^{\lambda}(\theta) = \mathrm{Tr}\, D^{\lambda}\left(e^{i\theta\cdot H}\right) = \sum_{\mu} m(\mu)\, e^{i\theta\cdot\mu} \tag{3.77}$$

where the summation is over the weights of the representation and $m(\mu)$ are their multiplicities. These can be obtained from Freudenthal's formula (3.42).

In the scalar representation the elements of the group are represented by the unity and the highest weight is zero. So setting $\lambda = 0$ in (3.76) we obtain what is called the *Weyl denominator formula*

$$\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\delta)\cdot\theta} = e^{i\delta\cdot\theta} \prod_{\alpha>0} \left(1 - e^{-i\alpha\cdot\theta}\right) \tag{3.78}$$

In general, such formula provides a nontrivial relation between a product and a sum. Substituting (3.78)in (3.76) we can write the Weyl character formula as the ratio of two sums:

$$\chi^{\lambda}(\theta) = \frac{\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\lambda+\delta)\cdot\theta}}{\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\delta)\cdot\theta}} \tag{3.79}$$

The dimension of the representation can be obtained from the Weyl character formula (3.76) noticing that

$$\dim D^{\lambda} = \mathrm{Tr}\left(\mathbb{1}\right) = \chi^{\lambda}\left(0\right) \tag{3.80}$$

we then obtain the so called *Weyl dimensionality formula*

$$\dim D^{\lambda} = \frac{\prod_{\alpha>0}\left(\lambda+\delta\right)\cdot\alpha}{\prod_{\alpha>0}\delta\cdot\alpha} \tag{3.81}$$

Example 3.9 *In the case of $SO(3)$ (or $SU(2)$) we have that $\alpha = 1$, $\delta = 1/2$ and consequently we have from (3.81) that*

$$\dim\, D^{j} = 2j+1 \tag{3.82}$$

This result can also be obtained from (3.73) by taking the limit $\theta \to 0$ and using L'Hospital's rule

(m_1, m_2)	dimension
$(1,0)$	(triplet) 3
$(0,1)$	(anti-triplet) 3
$(2,0)$	6
$(0,2)$	6
$(1,1)$	(adjoint) 8
$(3,0)$	10
$(0,3)$	10
$(2,1)$	15
$(1,2)$	15

Table 3.2: The dimensions of the smallest irreps. of $SU(3)$

Example 3.10 *Consider an irrep. of $SU(3)$ with highest weight λ. We can write $\lambda = m_1\lambda_1 + m_2\lambda_2$ where λ_1 and λ_2 are the fundamental weights and m_1 and m_2 are non-negative integers. From (3.56) we have that $(\delta + \lambda)^2 = (m_1 + 1)\,\lambda_1 + (m_2 + 1)\,\lambda_2$. Normalizing the roots of $SU(3)$ as $\alpha^2 = 2$ we have (from (3.4)) that $\lambda_a \cdot \alpha_b = \delta_{ab}$ $(a, b = 1, 2)$, where α_1 and α_2 are the simple roots and therefore ($\alpha_3 = \alpha_1 + \alpha_2$)*

$$\begin{aligned}
(\delta + \lambda) \cdot \alpha_1 &= m_1 + 1\,; \quad (\delta + \lambda) \cdot \alpha_2 = m_2 + 1\,; \quad (\delta + \lambda) \cdot \alpha_3 = m_1 m_2 + 2 \\
\delta \cdot \alpha_1 &= \delta \cdot \alpha_2 = 1\,; \qquad \delta \cdot \alpha_3 = 2
\end{aligned} \tag{3.83}$$

So, from (3.81) the dimension of the irrep. of $SU(3)$ with highest weight λ is

$$\dim D^{\lambda} = \dim D^{\lambda} = \frac{1}{2}\,(m_1 + 1)\,(m_2 + 1)\,(m_1 + m_2 + 2) \tag{3.84}$$

In table 3.2 we give the dimensions of the smallest irreps. of $SU(3)$.

Example 3.11 *Similarly let us consider the irreps. of $SO(5)$ (or $Sp(2)$) with highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2$. From example 2.14 we have that the positive roots of $SO(5)$ are α_1, α_2, $\alpha_3 \equiv \alpha_1 + \alpha_2$, and $\alpha_4 \equiv 2\alpha_1 + \alpha_2$, and so using (3.4) and (3.56) we get (setting $\alpha_1^2 = 1$, $\alpha_2^2 = 2$)*

$$\begin{aligned}
\frac{2\delta\cdot\alpha_1}{\alpha_1^2} = \frac{2\delta\cdot\alpha_2}{\alpha_2^2} &= 1\,; \quad \frac{2\delta\cdot\alpha_3}{\alpha_3^2} = \frac{3}{2}1\,; \quad \frac{2\delta\cdot\alpha_4}{\alpha_4^2} = 2 \\
\frac{2\,(\delta+\lambda)\cdot\alpha_1}{\alpha_1^2} &= m_1 + 1\,; \quad \frac{2\,(\delta+\lambda)\cdot\alpha_2}{\alpha_2^2} = m_2 + 1 \\
\frac{2\,(\delta+\lambda)\cdot\alpha_3}{\alpha_3^2} &= \frac{1}{2}\,(m_1 + 2m_2 + 3)\;; \quad \frac{2\,(\delta+\lambda)\cdot\alpha_4}{\alpha_4^2} = \frac{1}{2}\,(m_1 + m_2 + 2)
\end{aligned} \tag{3.85}$$

Therefore from (3.81)

$$\dim D^{(m_1,m_2)} = \frac{1}{6}\,(m_1+1)\,(m_2+1)\,(m_1+m_2+2)\,(m_1+2m_2+3) \tag{3.86}$$

The smallest irreps. of $SO(5)$ (or $Sp(2)$) are shown in table 3.3.

(m_1, m_2)	dimension
$(1,0)$	(spinor) 4
$(0,1)$	(vector) 5
$(2,0)$	(adjoint) 10
$(0,2)$	14
$(1,1)$	16
$(3,0)$	20
$(0,3)$	30
$(2,1)$	35
$(1,2)$	40

Table 3.3: The dimensions of the smallest irreps. of $SO(5)$ (or $Sp(2)$)

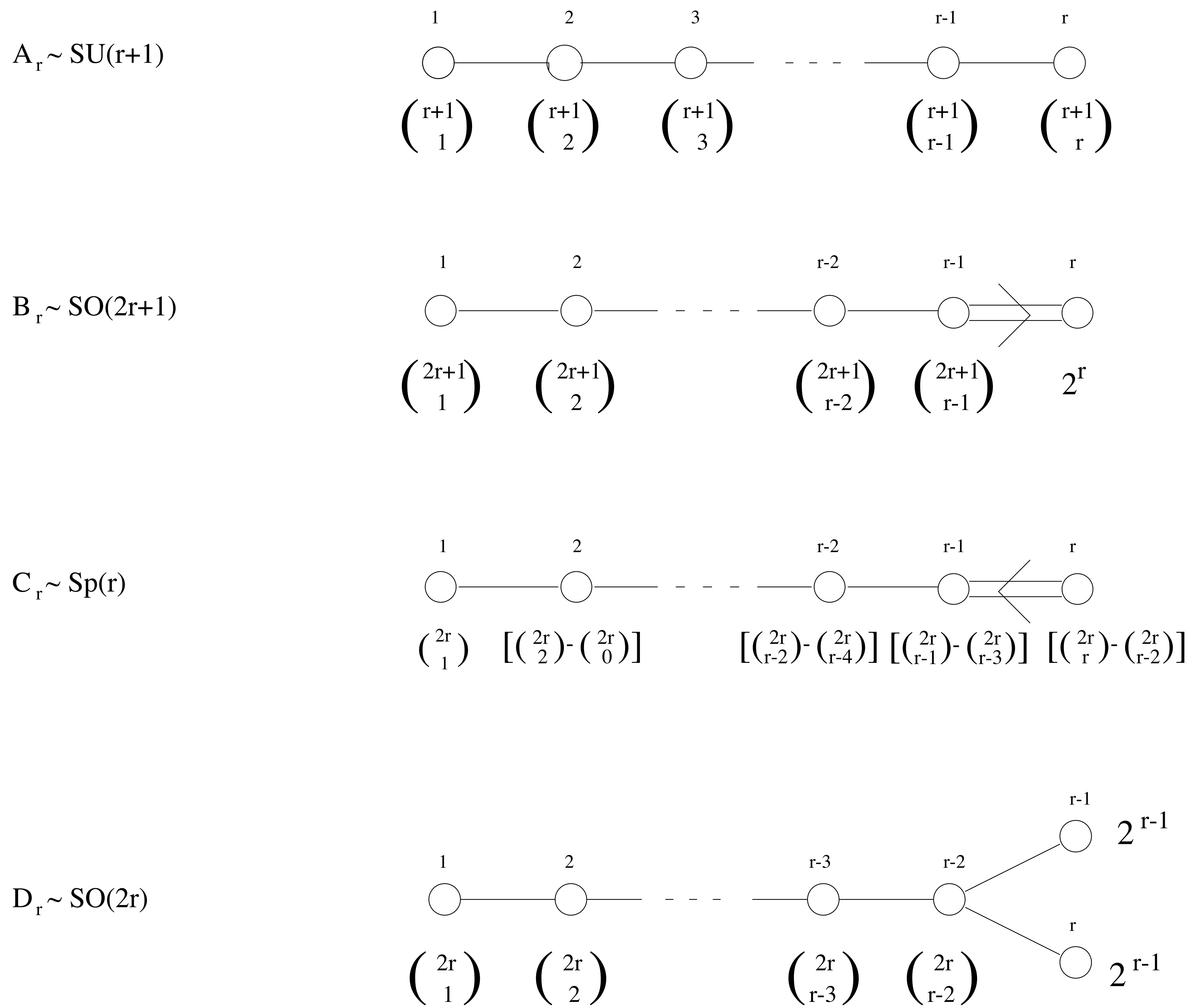

Figure 3.4: The dimensions of the fundamental representations of the classical Lie groups.

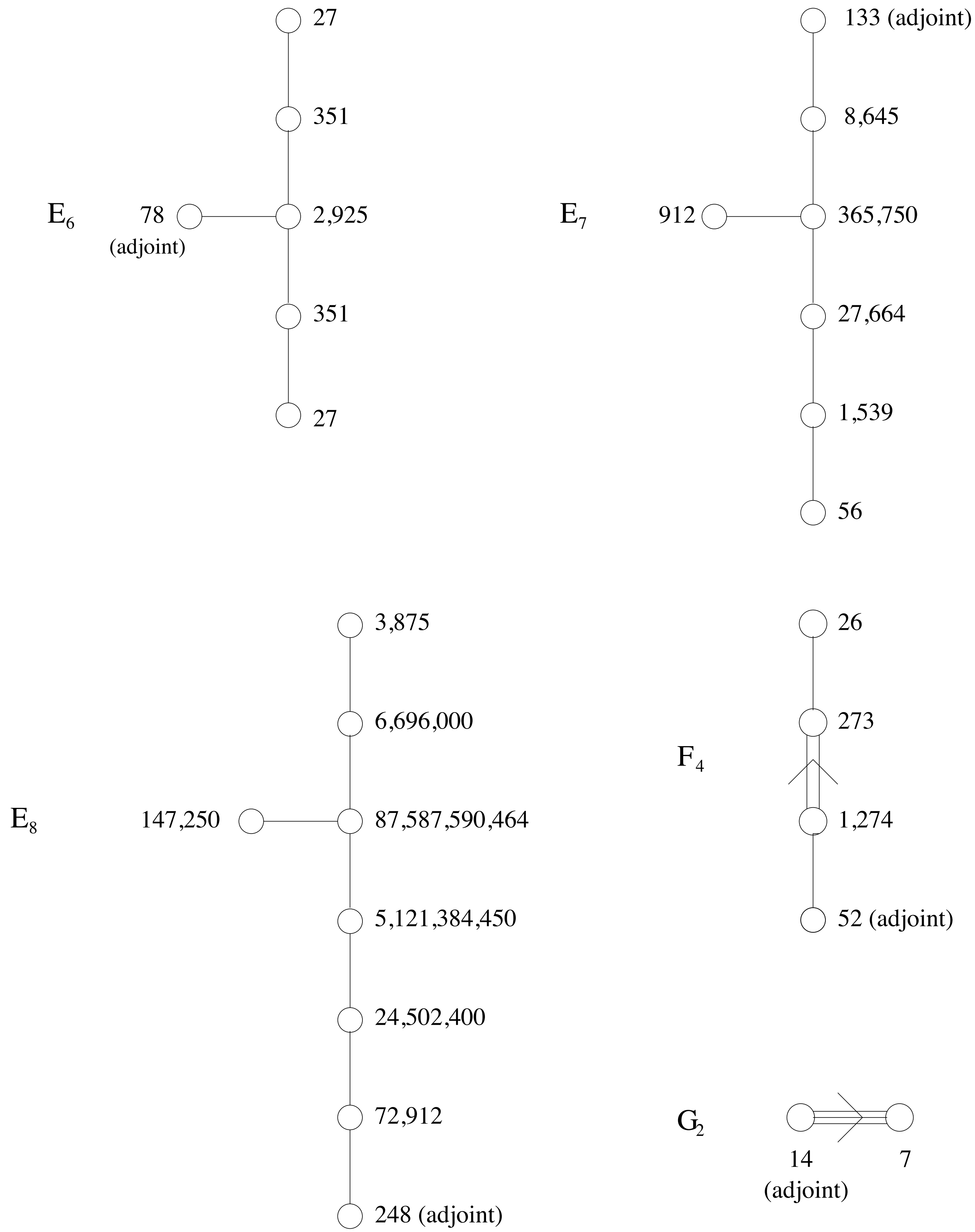

Figure 3.5: The dimensions of of the fundamental representations of the exceptional Lie groups.

3.8 Construction of matrix representations

We have seen that finite dimensional representations of compact Lie groups are equivalent to unitary ones (see theorem 3.1). In such representations the Cartan subalgebra generators and step operators can be chosen to satisfy[1]

$$H_i^\dagger = H_i \; ; \qquad E_\alpha^\dagger = E_{-\alpha} \tag{3.87}$$

We have chosen the basis of the representation to be formed by the eigenstates of the Cartan subalgebra generators. Using (3.1) and (3.87) we have

$$\langle \mu' \mid H_i \mid \mu \rangle = \mu_i \langle \mu' \mid \mu \rangle = \mu'_i \langle \mu' \mid \mu \rangle \tag{3.88}$$

and so

$$(\mu' - \mu) \, \langle \mu' \mid \mu \rangle = 0 \tag{3.89}$$

and consequently states with different weights are orthogonal. In the case a weight is degenerate, it is possible to find an orthogonal basis for the subspace generated by the states corresponding to that degenerate weight. We then shall denote the base states of the representation by $\mid \mu, k\rangle$ where μ is the corresponding weight and k is an integer number that runs from 1 to $m(\mu)$, the multiplicity of μ. We can always normalize these states such that

$$\langle \mu', k' \mid \mu, k \rangle = \delta_{\mu,\mu'} \, \delta_{kk'} \tag{3.90}$$

If T denotes an operator of the representation of the algebra then the matrix

$$D(T)_{(\mu',k')\,(\mu,k)} \equiv \langle \mu', k' \mid T \mid \mu, k\rangle \tag{3.91}$$

form a matrix representation since they reproduce the commutation relations of the algebra. Indeed

$$\begin{aligned}
[\, D(T)\, ,\, D(T')\,]_{(\mu',k')\,(\mu,k)} &= \sum_{\mu'',k''} \langle \mu', k' \mid T \mid \mu'', k''\rangle\langle \mu'', k'' \mid T' \mid \mu', k'\rangle \\
&- \sum_{\mu'',k''} \langle \mu', k' \mid T' \mid \mu'', k''\rangle\langle \mu'', k'' \mid T \mid \mu', k'\rangle \\
&= \langle \mu', k' \mid [\, T\, ,\, T'\,] \mid \mu', k'\rangle \\
&= D([\, T\, ,\, T'\,])_{(\mu',k')\,(\mu,k)}
\end{aligned} \tag{3.92}$$

where we have used the fact that

$$\mathbb{1} = \sum_{\mu,k} \mid \mu, k\rangle\langle \mu, k \mid \tag{3.93}$$

is the identity operator.

When a step operator E_α acts on a state of weight μ, it either annihilates it or produces a state of weight $\mu + \alpha$. Therefore, using (3.93) and (3.90) one gets

$$\begin{aligned}
E_\alpha \mid \mu, k\rangle &= \sum_{\mu',k'} \mid \mu', k'\rangle\langle \mu', k' \mid E_\alpha \mid \mu, k\rangle \\
&= \sum_{l=1}^{m(\mu+\alpha)} \mid \mu+\alpha, l\rangle\langle \mu+\alpha, l \mid E_\alpha \mid \mu, k\rangle
\end{aligned} \tag{3.94}$$

where the sum is over the states of weight $\mu + \alpha$. Therefore, from (3.91) one has

$$D(E_\alpha)_{(\mu',k')\,(\mu,k)} = \langle \mu+\alpha, k' \mid E_\alpha \mid \mu, k\rangle \delta_{\mu',\mu+\alpha} \tag{3.95}$$

The matrix elements of H_i are known once we have the weights of the representation, since from (3.1) and (3.90)

$$D\left(H_i\right)_{(\mu',k')\,(\mu,k)} = \langle \mu', k' \mid H_i \mid \mu, k\rangle = \mu_i\, \delta_{\mu',\mu}\, \delta_{k',k} \tag{3.96}$$

Therefore, in order to construct the matrix representation of the algebra we have to calculate the "transition amplitudes" $\langle \mu + \alpha, l \mid E_\alpha \mid \mu, k\rangle$. Notice that from (3.87)

$$\langle \mu + \alpha, l \mid E_\alpha \mid \mu, k\rangle^\dagger = \langle \mu, k \mid E_{-\alpha} \mid \mu + \alpha, l\rangle \tag{3.97}$$

Now, using the commutation relation (see (2.218))

$$[\, E_\alpha \,,\, E_{-\alpha} \,] = \frac{2\alpha \cdot H}{\alpha^2} \tag{3.98}$$

one gets

$$\begin{aligned}
\langle \mu, k \mid [\, E_\alpha \,,\, E_{-\alpha} \,] \mid \mu, k\rangle &= \langle \mu, k \mid \frac{2\alpha \cdot H}{\alpha^2} \mid \mu, k\rangle \qquad (3.99)\\
&= \frac{2\alpha \cdot \mu}{\alpha^2}\\
&= \langle \mu, k \mid E_\alpha E_{-\alpha} \mid \mu, k\rangle - \langle \mu, k \mid E_{-\alpha} E_\alpha \mid \mu, k\rangle\\
&= \sum_{l=1}^{m(\mu-\alpha)} \langle \mu, k \mid E_\alpha \mid \mu - \alpha, l\rangle \langle \mu - \alpha, l \mid E_{-\alpha} \mid \mu, k\rangle\\
&- \sum_{l=1}^{m(\mu+\alpha)} \langle \mu, k \mid E_{-\alpha} \mid \mu + \alpha, l\rangle \langle \mu + \alpha, l \mid E_\alpha \mid \mu, k\rangle
\end{aligned}$$

and so, using (3.97)

$$\sum_{l=1}^{m(\mu-\alpha)} \mid \langle \mu, k \mid E_\alpha \mid \mu - \alpha, l\rangle \mid^2 - \sum_{l=1}^{m(\mu+\alpha)} \mid \langle \mu + \alpha, l \mid E_\alpha \mid \mu, k\rangle \mid^2 = \frac{2\alpha \cdot \mu}{\alpha^2} \tag{3.100}$$

where $m\left(\mu + \alpha\right)$ and $m\left(\mu - \alpha\right)$ are the multiplicities of the weights $\mu + \alpha$ and $\mu - \alpha$ respectively.

The relation (3.100) can be used to calculate the modules of the transition amplitudes recursively. By taking α to be a positive root and μ the highest weight λ of the representation we have that the second term on the l.h.s. of (3.100) vanishes. Since, in a irrep., λ is not degenerate we can neglect the index k and write

$$\sum_{l=1}^{m(\mu-\alpha)} \mid \langle \lambda \mid E_\alpha \mid \mu - \alpha, l \rangle \mid^2 = \frac{2\alpha \cdot \lambda}{\alpha^2} = q \tag{3.101}$$

where, according to (3.41), q is the highest positive integer such that $\lambda - q\alpha$ is a weight of the representation. Taking now the second highest weight we repeat the process and so on.

The other relations that the transition amplitudes have to satisfy come from the commutation relations between step operators. If $\alpha + \beta$ is a root we have from (2.218)

$$\langle \mu + \alpha + \beta, l \mid [\, E_\alpha \,, \, E_\beta \,] \mid \mu, k \rangle = (q+1)\, \varepsilon(\alpha, \beta) \langle \mu + \alpha + \beta, l \mid E_{\alpha+\beta} \mid \mu, k \rangle \tag{3.102}$$

Then using (3.90) and (3.94) one gets

$$\begin{aligned} & \sum_{k'=1}^{m(\mu+\beta)} \langle \mu + \alpha + \beta, l \mid E_\alpha \mid \mu + \beta, k' \rangle \langle \mu + \beta, k' \mid E_\beta \mid \mu, k \rangle \\ - & \sum_{k'=1}^{m(\mu+\alpha)} \langle \mu + \alpha + \beta, l \mid E_\beta \mid \mu + \alpha, k' \rangle \langle \mu + \alpha, k' \mid E_\alpha \mid \mu, k \rangle \\ = & \; (q+1)\, \varepsilon(\alpha, \beta) \langle \mu + \alpha + \beta, l \mid E_{\alpha+\beta} \mid \mu, k \rangle \end{aligned} \tag{3.103}$$

where q is the highest positive integer such that $\beta - q\alpha$ (or equivalently $\alpha - q\beta$, since we are assuming $\alpha + \beta$ is a root) is a root, and $\varepsilon(\alpha, \beta)$ are signs determined from the Jacobi identities (see section 2.14)

We now give some examples to ilustrate how to use (3.100) and (3.103) to construct matrix representations. This method is very general and consequently difficult to use when the representation (or the algebra) is big. There are other methods which work better in specific cases.

3.8.1 The irreducible representations of $SU(2)$

In section 2.5 we have studied the representations of $SU(2)$. We have seen that the weights of $SU(2)$, denoted by m, are integers or half integers, and on a given irreducible representation with highest weight j they run from $-j$ to j in integer steps. The weights are non-degenerated and so the representations have dimensions $2j+1$. As we did in section 2.5 we shall denote the basis of the representation space as

$$\mid j, m\rangle \qquad\qquad m = -j, -j+1, \ldots, j-1, j \tag{3.104}$$

and they are orthonormal

$$\langle j, m' \mid j, m\rangle = \delta_{m,m'} \tag{3.105}$$

The Chevalley basis for $SU(2)$ satisfy the commutation relations

$$[\,H\,,\,E_{\pm}\,] = \pm E_{\pm} \qquad [\,E_{+}\,,\,E_{-}\,] = H \tag{3.106}$$

where $H = 2\alpha \cdot H/\alpha^2$, with α being the only positive root of $SU(2)$. In section 2.5 we have used the basis

$$[\,T_3\,,\,T_{\pm}\,] = \pm T_{\pm} \qquad [\,T_{+}\,,\,T_{-}\,] = 2T_3 \tag{3.107}$$

and so we have $E_{\pm} \equiv T_{\pm}$ and $H \equiv 2T_3$. Since m are eigenvalues of T_3

$$T_3 \mid j, m\rangle = m \mid j, m\rangle \tag{3.108}$$

we get from (3.91) the matrix representing T_3 as

$$D^{(j)}_{m',m}\,(T_3) = \langle j, m' \mid T_3 \mid j, m\rangle = m\,\delta_{m,m'} \tag{3.109}$$

Using the relation (3.100), which is the same as taking the expectation value on the state $\mid j, m\rangle$ of both sides of the second relation in (3.107), we get

$$\mid \langle j, m \mid T_{+} \mid j, m-1\rangle \mid^{2} - \mid \langle j, m+1 \mid T_{+} \mid j, m\rangle \mid^{2} = 2m \qquad (3.110)$$

$$c_{m-1} - c_m = 2\,m$$

where we have used the fact that $T_{+}^{\dagger} = T_{-}$ (see (3.87)). Notice that $T_{+} \mid j, j\rangle = 0$, since j is the highest weight and so

$$\mid \langle j, j \mid T_{+} \mid j, j-1\rangle \mid^{2} = 2j \qquad (3.111)$$

Clearly, such result could also be obtained directly from (3.101). The other matrix elements of T_{+} can then be obtained recursively from (3.110). Indeed, denoting $c_m \equiv \mid \langle j, m+1 \mid T_{+} \mid j, m\rangle \mid^{2}$, we get $c_{j-1} = 2j$, $c_{j-2} = 2j + 2(j-1)$, $c_{j-3} = 2j + 2(j-1) + 2(j-2)$, and so

$$c_m = \sum_{l=0}^{j-m-1} 2(j-l) = (j-m)(j+m+1) = j(j+1) - m(m+1)$$

Therefore

$$\mid \langle j, m+1 \mid T_{+} \mid j, m\rangle \mid^{2} = j(j+1) - m(m+1) \qquad (3.112)$$

and since

$$\langle j, m+1 \mid T_{+} \mid j, m\rangle^{\dagger} = \langle j, m \mid T_{-} \mid j, m+1\rangle \qquad (3.113)$$

we get

$$\mid \langle j, m-1 \mid T_{-} \mid j, m\rangle \mid^{2} = j(j+1) - m(m-1) \qquad (3.114)$$

The phases of such matrix elements can be chosen to vanish, since in $SU(2)$ we do not have a relation like (3.103) to relate them. Therefore, we get

$$T_{\pm} \mid j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \mid j, m \pm 1\rangle \qquad (3.115)$$

and so,

$$\begin{aligned} D^{(j)}_{m',m}(T_{+}) &= \langle j, m' \mid T_{+} \mid j, m\rangle \\ &= \sqrt{j(j+1) - m(m+1)}\, \delta_{m',m+1} \\ D^{(j)}_{m',m}(T_{-}) &= \langle j, m' \mid T_{-} \mid j, m\rangle \\ &= \sqrt{j(j+1) - m(m-1)}\, \delta_{m',m-1} \end{aligned} \qquad (3.116)$$

3.8.2 The triplet representation of $SU(3)$

Consider the fundamental representation of $SU(3)$ with highest weight λ_1. In example 3.10 we have seen it has dimension 3, and in fact it is the so called triplet representation of $SU(3)$. From (3.4) we have

$$\frac{2\lambda_1 \cdot \alpha_1}{\alpha_1^2} = \frac{2\lambda_1 \cdot \alpha_3}{\alpha_3^2} = 1 \tag{3.117}$$

where $\alpha_3 = \alpha - 1 + \alpha_2$, α_1 and α_2 are the the simple roots of $SU(3)$. So,from (3.41) we get that λ_1, $(\lambda_1 - \alpha_1)$ and $(\lambda_1 - \alpha_3)$ are weights of the representation. Since the representation has dimension 3 it follows that they are the only weights and they are non-degenerate. Those weights are shown in figure 3.6.

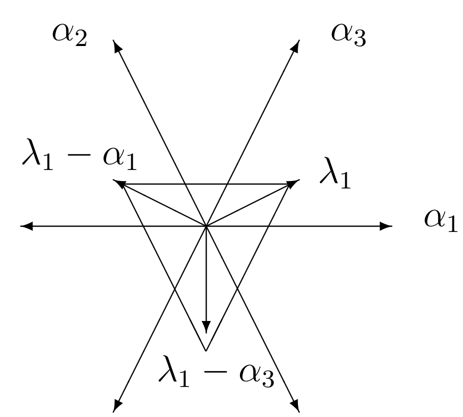

Figure 3.6: The weights of the triplet representation of $SU(3)$

Taking the Cartan subalgebra generators in the Chevalley basis we have

$$\langle \mu' \mid H_a \mid \mu \rangle = \frac{2\alpha_a \cdot \mu}{\alpha_a^2} \delta_{\mu',\mu} \qquad a = 1, 2 \tag{3.118}$$

where we have used (3.90), and where we have neglected the degeneracy index. From (3.4) and the Cartan matrix of $SU(3)$ (see example 2.13) we have

$$\begin{aligned} \frac{2\alpha_1 \cdot (\lambda_1 - \alpha_1)}{\alpha_1^2} &= -1 \qquad \frac{2\alpha_2 \cdot (\lambda_1 - \alpha_3)}{\alpha_2^2} = 1 \\ \frac{2\alpha_1 \cdot (\lambda_1 - \alpha_3)}{\alpha_1^2} &= 0 \qquad \frac{2\alpha_2 \cdot (\lambda_1 - \alpha_1)}{\alpha_2^2} = 1 \end{aligned} \tag{3.119}$$

Denoting the states as (as a matter of ordering the rows and columus of the matrices)

$$\mid 1\rangle \equiv\mid \lambda_1\rangle \; ; \qquad \mid 2\rangle \equiv\mid \lambda_1-\alpha_1\rangle \; ; \qquad \mid 3\rangle \equiv\mid \lambda_1-\alpha_3\rangle \tag{3.120}$$

we obtain from (3.117), (3.118), (3.119) and that the matrices representing the Cartan subalgebra generators are

$$D^{\lambda_1}\left(H_1\right)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D^{\lambda_1}\left(H_2\right)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.121}$$

Using (3.101) and (3.117) we have that

$$\sum_{l=1}^{m(\mu-\alpha)} \mid \langle\lambda \mid E_\alpha \mid \mu-\alpha, l\rangle \mid^2 = \frac{2\alpha\cdot\lambda}{\alpha^2}=q$$

$$\mid\langle\lambda_1 \mid E_{\alpha_1} \mid \lambda_1-\alpha_1\rangle\mid^2=\mid\langle\lambda_1 \mid E_{\alpha_3} \mid \lambda_1-\alpha_3\rangle\mid^2=1 \tag{3.122}$$

$$\frac{2\lambda_1\cdot\alpha_1}{\alpha_1^2}=\frac{2\lambda_1\cdot\alpha_3}{\alpha_3^2}=1$$

Making $\mu=\lambda_1-\alpha_1$ and $\alpha=\alpha_2$ in (3.100) and using the fact that

$$\langle\lambda_1-\alpha_1+\alpha_2 \mid E_{\alpha_2} \mid \lambda_1-\alpha_1\rangle=0 \tag{3.123}$$

since $\lambda_1-\alpha_1+\alpha_2$ is not weight, we get

$$\mid\langle\lambda_1-\alpha_1 \mid E_{\alpha_2} \mid \lambda_1-\alpha_1-\alpha_2\rangle\mid^2=1 \tag{3.124}$$

These are the only non vanishing "transition amplitudes". From (3.95) and (3.120) we see that the only non vanishing elements of the matrices representing the step operators are

$$\begin{aligned} D^{\lambda_1}\left(E_{\alpha_1}\right) &= \langle\lambda_1 \mid E_{\alpha_1} \mid \lambda_1-\alpha_1\rangle \equiv e^{i\theta} \\ D^{\lambda_1}\left(E_{\alpha_2}\right) &= \langle\lambda_1-\alpha_1 \mid E_{\alpha_2} \mid \lambda_1-\alpha_3\rangle \equiv e^{i\varphi} \\ D^{\lambda_1}\left(E_{\alpha_3}\right) &= \langle\lambda_1 \mid E_{\alpha_3} \mid \lambda_1-\alpha_3\rangle \equiv e^{i\phi} \end{aligned} \tag{3.125}$$

where, according to (3.122) and (3.124), we have introduced the angles θ, ϕ and φ. The negative step operators are obtained from these ones using (3.87).

Choosing the cocycle $\varepsilon\left(\alpha_1,\alpha_2\right)=1$ and since $\alpha_2-\alpha_1$ is not a root, we have from (3.103) that the fases have to satisfy (set $\mu=\lambda_1-\alpha_3$, $\alpha=\alpha_1$ and $\beta=\alpha_2$ in (3.103))

$$\theta+\varphi=\phi \tag{3.126}$$

There are no futher restrictions on these fases.

$$\sum_{l=1}^{m(\mu-\alpha)} \mid\langle\mu,k \mid E_\alpha \mid \mu-\alpha,l\rangle\mid^2 - \sum_{l=1}^{m(\mu+\alpha)} \mid\langle\mu+\alpha,l \mid E_\alpha \mid \mu,k\rangle\mid^2=\frac{2\alpha\cdot\mu}{\alpha^2} \tag{3.100}$$

Therefore we get that the matrices which represent the step operators in the triplet representation are

$$
\begin{aligned}
D^{\lambda_1}\left(E_{\alpha_1}\right) &= \begin{pmatrix} 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}\left(E_{-\alpha_1}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ e^{-i\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
D^{\lambda_1}\left(E_{\alpha_2}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\varphi} \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}\left(E_{-\alpha_2}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \end{pmatrix} \\
D^{\lambda_1}\left(E_{\alpha_3}\right) &= \begin{pmatrix} 0 & 0 & e^{i(\theta+\varphi)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}\left(E_{-\alpha_3}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i(\theta+\varphi)} & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{3.127}
$$

In general, the fases θ and φ are chosen to vanish. The algebra of $SU(3)$ is generated by taking real linear combination of the matrices H_a $(a = 1, 2)$, $(E_\alpha + E_{-\alpha})$ and $(E_\alpha - E_{-\alpha})$. On the other hand the algebra of $SL(3)$ is generated by the same matrices but the third one does not have the factor i. Notice that in this way the triplet representation of the group $SU(3)$ is unitary whilst the triplet of $SL(3)$ is not.

3.8.3 The anti-triplet representation of $SU(3)$

We now consider the other fundamental representation of $SU(3)$ which has highest weight λ_2 . In example 3.10 we saw it also has diemnsion 3 and it is the anti-triplet of $SU(3)$. Using (3.4) we get that the weight are λ_2, $\lambda_2 - \alpha_2$ and $\lambda_2 - \alpha_3$ and consequently they are not degenerate. They are shown in figure 3.7.

We shall denote the states as

$$| 1\rangle \equiv| \lambda_2\rangle \; ; \qquad | 2\rangle \equiv| \lambda_2 - \alpha_2\rangle \; ; \qquad | 3\rangle \equiv| \lambda_2 - \alpha_3\rangle \tag{3.128}$$

Using the Cartan matrix of $SU(3)$ (see example 2.13), (3.4) and (3.118) we get that the matrices which represent the Cartan subalgebra generators in the Chevalley basis are

$$D^{\lambda_2}(H_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad D^{\lambda_2}(H_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.129}$$

Using (3.101) we have that

$$| \langle\lambda_2 \mid E_{\alpha_2} \mid \lambda_2 - \alpha_2\rangle |^2 = | \langle\lambda_2 \mid E_{\alpha_3} \mid \lambda_2 - \alpha_3\rangle |^2 = 1 \tag{3.130}$$

and from (3.100)

$$| \langle\lambda_2 - \alpha_2 \mid E_{\alpha_1} \mid \lambda_2 - \alpha_1 - \alpha_2\rangle |^2 = 1 \tag{3.131}$$

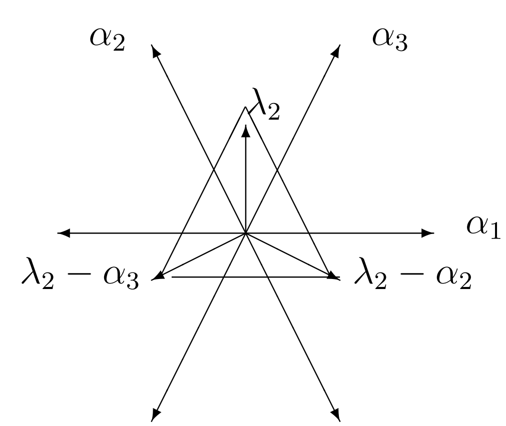

Figure 3.7: The weights of the anti-triplet representation of $SU(3)$

Using (3.95) we get that the only non vanishing matrix elements of the step operators are

$$
\begin{aligned}
D^{\lambda_2}\left(E_{\alpha_1}\right) &= \langle \lambda_2 - \alpha_2 \mid E_{\alpha_1} \mid \lambda_2 - \alpha_3 \rangle \equiv e^{i\theta} \\
D^{\lambda_2}\left(E_{\alpha_2}\right) &= \langle \lambda_2 \mid E_{\alpha_2} \mid \lambda_2 - \alpha_2 \rangle \equiv e^{i\varphi} \\
D^{\lambda_2}\left(E_{\alpha_3}\right) &= \langle \lambda_2 \mid E_{\alpha_3} \mid \lambda_2 - \alpha_3 \rangle \equiv e^{i\phi}
\end{aligned}
\tag{3.132}
$$

where, according to (3.130) and (3.131), we have introduced the fases θ, φ and ϕ. From (3.87) we obtain the matrices for the negative step operators. Using the fact that $(q+1)\,\varepsilon\left(\alpha_1, \alpha_2\right) = 1$ we get from (3.103) that these fases have to satisfy

$$
\theta + \varphi = \phi + \pi \tag{3.133}
$$

Therefore the matrices which represent the step operators in the anti-triplet representation are

$$
\begin{aligned}
D^{\lambda_2}\left(E_{\alpha_1}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_2}\left(E_{-\alpha_1}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \end{pmatrix} \\
D^{\lambda_2}\left(E_{\alpha_2}\right) &= \begin{pmatrix} 0 & e^{i\varphi} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_2}\left(E_{-\alpha_2}\right) &= \begin{pmatrix} 0 & 0 & 0 \\ e^{-i\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
D^{\lambda_2}\left(E_{\alpha_3}\right) &= -\begin{pmatrix} 0 & 0 & e^{i(\theta+\varphi)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_2}\left(E_{-\alpha_3}\right) &= -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i(\theta+\varphi)} & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{3.134}
$$

So, these matrices are obtained from those of the triplet by making the change $E_{\pm\alpha_1} \leftrightarrow E_{\pm\alpha_2}$ and $E_{\pm\alpha_3} \leftrightarrow -E_{\pm\alpha_3}$. From (3.121) and (3.129) we see the Cartan subalgebra generators are also interchanged.

3.9 Tensor product of representations

We have seen in definition 1.12 of section 1.5 the concept of tensor product of representations. The idea is quite simple. Consider two irreducible representations D^{λ} and $D^{\lambda'}$ of a Lie group G, with highest weights λ and λ' and representation spaces V^{λ} and $V^{\lambda'}$ respectively. We can construct a third representation by considering the tensor product space $V^{\lambda\otimes\lambda'} \equiv V^{\lambda} \otimes V^{\lambda'}$. The operators representing the group elements in the tensor product representation are

$$D^{\lambda\otimes\lambda'}\left(g\right) \equiv D^{\lambda}\left(g\right) \otimes D^{\lambda'}\left(g\right) \tag{3.135}$$

and they act as

$$D^{\lambda\otimes\lambda'}\left(g\right) V^{\lambda\otimes\lambda'} = D^{\lambda}\left(g\right) V^{\lambda} \otimes D^{\lambda'}\left(g\right) V^{\lambda'} \tag{3.136}$$

They form a representation since

$$\begin{aligned} D^{\lambda\otimes\lambda'}\left(g_1\right) D^{\lambda\otimes\lambda'}\left(g_2\right) &= D^{\lambda}\left(g_1\right) D^{\lambda}\left(g_2\right) \otimes D^{\lambda'}\left(g_1\right) D^{\lambda'}\left(g_2\right) \\ &= D^{\lambda}\left(g_1 g_2\right) \otimes D^{\lambda'}\left(g_1 g_2\right) \\ &= D^{\lambda\otimes\lambda'}\left(g_1 g_2\right) \end{aligned} \tag{3.137}$$

The operators representing the elements T of the Lie algebra $\mathcal{G}$ of G are given by

$$D^{\lambda\otimes\lambda'}\left(T\right) \equiv D^{\lambda}\left(T\right) \otimes \mathbb{1} + \mathbb{1} \otimes D^{\lambda'}\left(T\right) \tag{3.138}$$

Indeed

$$\begin{aligned} \left[\, D^{\lambda\otimes\lambda'}\left(T_1\right) , \, D^{\lambda\otimes\lambda'}\left(T_2\right) \right] &= \left[\, D^{\lambda}\left(T_1\right) , \, D^{\lambda}\left(T_1\right) \right] \otimes \mathbb{1} \\ &+ \mathbb{1} \otimes \left[\, D^{\lambda'}\left(T_1\right) , \, D^{\lambda'}\left(T_1\right) \right] \\ &= D^{\lambda}\left(\left[\, T_1 \, , \, T_2 \,\right]\right) \otimes \mathbb{1} + \mathbb{1} \otimes D^{\lambda'}\left(\left[\, T_1 \, , \, T_2 \,\right]\right) \\ &= D^{\lambda\otimes\lambda'}\left(\left[\, T_1 \, , \, T_2 \,\right]\right) \end{aligned} \tag{3.139}$$

Notice that if $\mid \mu, l\rangle$ and $\mid \mu', l'\rangle$ are states of the representations V^{λ} and $V^{\lambda'}$ with weights μ and μ' respectively, one gets

$$\begin{aligned} D^{\lambda\otimes\lambda'}(H_i) \mid \mu, l\rangle\otimes \mid \mu', l'\rangle &= D^{\lambda}(H_i) \mid \mu, l\rangle\otimes \mid \mu', l'\rangle \\ &+ \mid \mu, l\rangle \otimes D^{\lambda'}(H_i) \mid \mu', l'\rangle \\ &= (\mu_i + \mu'_i) \mid \mu, l\rangle\otimes \mid \mu', l'\rangle \end{aligned} \tag{3.140}$$

It then follows that the weigths of the representation $V^{\lambda\otimes\lambda'}$ are the sums of all weights of V^{λ} with all weights of $V^{\lambda'}$. If λ and λ' are the highest weights of V^{λ} and $V^{\lambda'}$ respectively, then the highest weight of $V^{\lambda\otimes\lambda'}$ is $\lambda+\lambda'$, and the corresponding state is

$$\mid \lambda + \lambda'\rangle =\mid \lambda\rangle\otimes \mid \lambda'\rangle \tag{3.141}$$

which is clearly non-degenerate.

In general, the representation $V^{\lambda\otimes\lambda'}$ is reducible and one can split it as the sum of irreducible representations of G

$$V^{\lambda\otimes\lambda'} = \oplus_{\lambda''} V^{\lambda''} \tag{3.142}$$

where $V^{\lambda''}$ are irreducible representations with highest weight λ''. The decomposition (3.142) is called the *branching* of the representation $V^{\lambda\otimes\lambda'}$.

Taking orthonormal basis $\mid \mu, l\rangle$ and $\mid \mu', l'\rangle$ for V^{λ} and $V^{\lambda'}$ respectively, we can construct an orthonormal basis for $V^{\lambda\otimes\lambda'}$ as

$$\mid \mu + \mu', k\rangle = \sum_{l=1}^{m(\mu)} \sum_{l'=1}^{m(\mu')} C^k_{l,l'} \mid \mu, l\rangle\otimes \mid \mu', l'\rangle \tag{3.143}$$

where $m(\mu)$ and $m(\mu')$ are the multiplicities of μ and μ' in V^{λ} and $V^{\lambda'}$ respectively, and $k = 1, 2, \ldots m(\mu+\mu')$, with $m(\mu+\mu')$ being the multiplicity of $\mu+\mu'$ in $V^{\lambda\otimes\lambda'}$. Clearly, $m(\mu+\mu') = m(\mu)\, m(\mu')$. The constants $C^k_{l,l'}$ are the so-called *Clebsch-Gordan coefficients*.

Example 3.12 *Let us consider the tensor product of two spinorial representations of $SU(2)$. As discussed in section 3.8.1 it is a two dimensional representation with states $\mid \frac{1}{2}, \frac{1}{2}\rangle$ and $\mid \frac{1}{2}, -\frac{1}{2}\rangle$, and satisfying*

$$T_3 \mid \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm\frac{1}{2} \mid \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \tag{3.144}$$

and (see (3.115))

$$T_{\pm} \mid j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)} \mid j, m \pm 1\rangle \tag{3.115}$$

$$\begin{aligned} T_+ \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= 0\,; & T_+ \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle &= \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ T_- \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\,; & T_- \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle &= 0 \end{aligned} \tag{3.145}$$

One can easily construct the irreducible components by taking the highest weight state $\mid \frac{1}{2}, \frac{1}{2}\rangle \otimes \mid \frac{1}{2}, \frac{1}{2}\rangle$ and act with the lowering operator. One gets

$$\begin{aligned} D^{\frac{1}{2}\otimes\frac{1}{2}}(T_-) \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= (T_- \otimes \mathbb{1} + \mathbb{1} \otimes T_-) \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ &= \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle + \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned}$$

and

$$\left(D^{\frac{1}{2}\otimes\frac{1}{2}}(T_-)\right)^2 \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle = 2 \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{3.146}$$

and

$$\left(D^{\frac{1}{2}\otimes\frac{1}{2}}(T_-)\right)^3 \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle = 0 \tag{3.147}$$

On the other hand notice that

$$D^{\frac{1}{2}\otimes\frac{1}{2}}(T_{\pm}) \left(\mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle - \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) = 0 \tag{3.148}$$

Therefore, one gets that the states

$$\begin{aligned} \mid 1, 1\rangle &\equiv \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ \mid 1, 0\rangle &\equiv \left(\mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle + \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)/\sqrt{2} \\ \mid 1, -1\rangle &\equiv \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned} \tag{3.149}$$

constitute a triplet representation (spin 1*) of* $SU(2)$*.*

The state

$$\mid 0,0\rangle \equiv \left(\mid \tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes \mid \tfrac{1}{2},\tfrac{1}{2}\rangle - \mid \tfrac{1}{2},\tfrac{1}{2}\rangle\otimes \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle\right)/\sqrt{2} \tag{3.150}$$

constitute a scalar representation (spin 0*) of* $SU(2)$*.*

The branching of the tensor product representation is usually denoted in terms of the dimensions of the irreducible representations, and in such case we have

$$\mathbf{2}\otimes\mathbf{2} = \mathbf{3}+\mathbf{1} \tag{3.151}$$

Given an irreducible representation D of a group G one observes that it is also a representation of any subgroup H of G. However, it will in general be a reducible representation of the subgroup. The decomposition of D in terms of irreducible representations of H is called the branching of D. In order to illustrate it let us discuss some examples.

Example 3.13 *The operator* T_3 *generates a subgroup* $U(1)$ *of* $SU(2)$ *(see (3.107)). From the considerations in 3.8.1 one observes that each state* $\mid j,m\rangle$ *constitutes a scalar representation of such* $U(1)$ *subgroup. Therefore, each spin* j *representation of* $SU(2)$ *decomposes into* $2j+1$ *scalars representation of* $U(1)$*.*

Example 3.14 *In example 3.6 we have seen that weights of the adjoint representation of $SU(3)$ are its roots plus the null weight which is two-fold degenerate. So, let us denote the states as*

$$\mid \pm\alpha_1\rangle \; ; \quad \mid \pm\alpha_2\rangle \; ; \quad \mid \pm\alpha_3\rangle \; ; \quad \mid 0\rangle \; ; \quad \mid 0'\rangle \tag{3.152}$$

Consider the $SU(2) \otimes U(1)$ subgroup of $SU(3)$ generated by

$$\begin{aligned} SU(2) &\equiv \{E_{\pm\alpha_1}, \frac{2\alpha_1 \cdot H}{\alpha_1^2}\} \\ U(1) &\equiv \{\frac{2\lambda_2 \cdot H}{\alpha_2^2}\} \end{aligned} \tag{3.153}$$

One can define the state $\mid 0\rangle$ as

$$\mid 0\rangle \equiv E_{-\alpha_1} \mid \alpha_1\rangle \tag{3.154}$$

and consequently the states

$$\mid \alpha_1\rangle \; ; \qquad \mid 0\rangle \; ; \qquad \mid -\alpha_1\rangle \tag{3.155}$$

constitute a triplet representation of the $SU(2)$ defined above. In addition, the states

$$\mid \alpha_2\rangle \; ; \qquad \mid \alpha_3\rangle \tag{3.156}$$

and

$$\mid -\alpha_3\rangle \; ; \qquad \mid -\alpha_2\rangle \tag{3.157}$$

constitute two dublet representations of the same $SU(2)$.

By taking $\mid 0'\rangle$ to be orthogonal to $\mid 0\rangle$ one gets that it is a singlet representation of $SU(2)$.

Clearly, each state $\mid \mu\rangle$ in (3.152) constitute a scalar representation of the $U(1)$ subgroup with eigenvalue $2\lambda_2 \cdot \mu/\alpha_2^2$. Since, $U(1)$ commutes with the $SU(2)$ it follows the states of a given irreducible representation of $SU(2)$ have to have the same eigenvalue fo the $U(1)$. Therefore, we have got the following branching of the adjoint of $SU(3)$ in terms of irreps. of $SU(2) \otimes U(1)$

$$\mathbf{8} = \mathbf{3}(0) + \mathbf{2}(1) + \mathbf{2}(-1) + \mathbf{1}(0) \tag{3.158}$$

where the numbers inside the parenteses are the $U(1)$ eigenvalues.