MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

Prof. Dr. Luis Carlos de Castro Santos

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO

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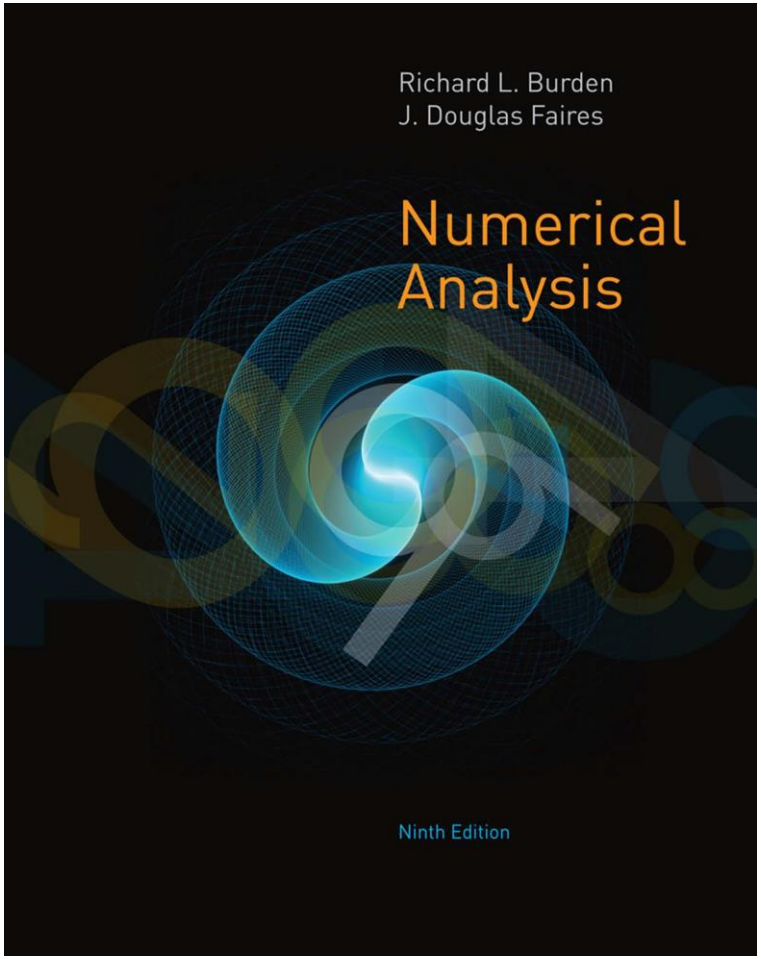
MAP2210



Numerical Analysis

NINTH EDITION

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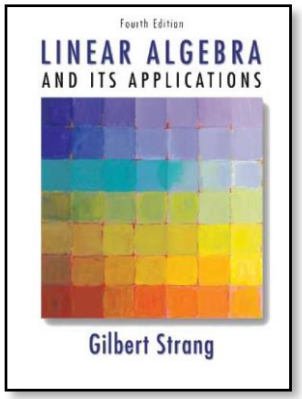

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ALGORITHM 9.1

Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ matrix A given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $||\mathbf{x}||_\infty = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 Find the smallest integer p with $1 \le p \le n$ and $|x_p| = ||\mathbf{x}||_\infty$.

Step 3 Set $\mathbf{x} = \mathbf{x}/x_p$.

Step 4 While $(k \le N)$ do Steps 5–11.

Step 5 Set $\mathbf{y} = A\mathbf{x}$.

Step 6 Set $\mu = y_p$.

Autovalor

Step 7 Find the smallest integer p with $1 \le p \le n$ and $|y_p| = \|\mathbf{y}\|_\infty$.

Step 8 If $y_p = 0$ then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has the eigenvalue 0, select a new vector $\mathbf{x}$ and restart');
STOP.

Step 9 Set $ERR = ||\mathbf{x} - (\mathbf{y}/y_p)||_\infty$;
$\mathbf{x} = \mathbf{y}/y_p$.

Autovetor

Step 10 If $ERR < TOL$ then OUTPUT $(\mu, \mathbf{x})$;
(*The procedure was successful.*)
STOP.

Step 11 Set $k = k + 1$.

Step 12 OUTPUT ('The maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

■

2.5 Accelerating Convergence

Implementing the Δ^2 procedure in Algorithm 9.1 is accomplished by modifying the algorithm as follows:

Step 1 Set $k = 1$;
$\mu_0 = 0$;
$\mu_1 = 0$.

.
.
.

Step 6 Set $\mu = y_p$;

$$\hat{\mu} = \mu_0 - \frac{(\mu_1 - \mu_0)^2}{\mu - 2\mu_1 + \mu_0}.$$

Step 10 If $ERR < TOL$ and $k \geq 4$ then OUTPUT $(\hat{\mu}, \mathbf{x})$;
STOP.

Step 11 Set $k = k + 1$;
$\mu_0 = \mu_1$;
$\mu_1 = \mu$.

ALGORITHM 9.2

Symmetric Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ symmetric matrix A, given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_2 = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$;
$\mathbf{x} = \mathbf{x}/\|\mathbf{x}\|_2$.

Step 2 While $(k \leq N)$ do Steps 3–8.

Autovalor

Step 3 Set $\mathbf{y} = A\mathbf{x}$.

Step 4 Set $\mu = \mathbf{x}^t\mathbf{y}$.

Step 5 If $\|\mathbf{y}\|_2 = 0$, then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has eigenvalue 0, select new vector $\mathbf{x}$ and restart');
STOP.

Step 6 Set $ERR = \left\| \mathbf{x} - \frac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\|_2$;
$\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_2$.

Autovetor

Step 7 If $ERR < TOL$ then OUTPUT $(\mu, \mathbf{x})$;
(*The procedure was successful.*)
STOP.

Step 8 Set $k = k + 1$.

Step 9 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

■

Example 2

Apply both the Power method and the Symmetric Power method to the matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix},$$

using Aitken's Δ^2 method to accelerate the convergence.

Table 9.3

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\|\mathbf{x}^{(m)}\|_2 = 1$
0	(1, 0, 0)			(1, 0, 0)
1	(4, −1, 1)	4	7	(0.942809, −0.235702, 0.235702)
2	(4.242641, −2.121320, 2.121320	5	6.047619	(0.816497, −0.408248, 0.408248)
3	(4.082483, −2.857738, 2.857738)	5.666667	6.002932	(0.710669, −0.497468, 0.497468)
4	(3.837613, −3.198011, 3.198011)	5.909091	6.000183	(0.646997, −0.539164, 0.539164)
5	(3.666314, −3.342816, 3.342816)	5.976744	6.000012	(0.612836, −0.558763, 0.558763)
6	(3.568871, −3.406650, 3.406650)	5.994152	6.000000	(0.595247, −0.568190, 0.568190)
7	(3.517370, −3.436200, 3.436200)	5.998536	6.000000	(0.586336, −0.572805, 0.572805)
8	(3.490952, −3.450359, 3.450359)	5.999634		(0.581852, −0.575086, 0.575086)
9	(3.477580, −3.457283, 3.457283)	5.999908		(0.579603, −0.576220, 0.576220)
10	(3.470854, −3.460706, 3.460706)	5.999977		(0.578477, −0.576786, 0.576786)

The Symmetric Power method gives considerably faster convergence for this matrix than the Power method. The eigenvector approximations in the Power method converge to $(1, -1, 1)^t$, a vector with unit l_∞-norm. In the Symmetric Power method, the convergence is to the parallel vector $(\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^t$, which has unit l_2-norm.

If λ is a real number that approximates an eigenvalue of a symmetric matrix A and $\mathbf{x}$ is an associated approximate eigenvector, then $A\mathbf{x} - \lambda\mathbf{x}$ is approximately the zero vector. The following theorem relates the norm of this vector to the accuracy of λ to the eigenvalue.

Theorem 9.19

Suppose that A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If we have $\|A\mathbf{x} - \lambda\mathbf{x}\|_2 < \varepsilon$ for some real number λ and vector $\mathbf{x}$ with $\|\mathbf{x}\|_2 = 1$, then

$$\min_{1 \le j \le n} |\lambda_j - \lambda| < \varepsilon.$$

■

Proof Suppose that $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ form an orthonormal set of eigenvectors of A associated, respectively, with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. By Theorems 9.5 and 9.3, $\mathbf{x}$ can be expressed, for some unique set of constants $\beta_1, \beta_2, \ldots, \beta_n$, as

$$\mathbf{x} = \sum_{j=1}^{n} \beta_j \mathbf{v}^{(j)}.$$

Thus

$$\|A\mathbf{x} - \lambda\mathbf{x}\|_2^2 = \left\| \sum_{j=1}^{n} \beta_j (\lambda_j - \lambda)\mathbf{v}^{(j)} \right\|_2^2 = \sum_{j=1}^{n} |\beta_j|^2 |\lambda_j - \lambda|^2 \ge \min_{1 \le j \le n} |\lambda_j - \lambda|^2 \sum_{j=1}^{n} |\beta_j|^2.$$

But

$$\sum_{j=1}^{n} |\beta_j|^2 = \|\mathbf{x}\|_2^2 = 1, \quad \text{so} \quad \varepsilon \ge \|A\mathbf{x} - \lambda\mathbf{x}\|_2 > \min_{1 \le j \le n} |\lambda_j - \lambda|.$$

■ ■ ■

Inverse Power Method

The **Inverse Power method** is a modification of the Power method that gives faster convergence. It is used to determine the eigenvalue of A that is closest to a specified number q.

Suppose the matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$. The eigenvalues of $(A - qI)^{-1}$, where $q \neq \lambda_i$, for $i = 1, 2, \ldots, n$, are

$$\frac{1}{\lambda_1 - q}, \quad \frac{1}{\lambda_2 - q}, \quad \ldots, \quad \frac{1}{\lambda_n - q},$$

with these same eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$. (See Exercise 15 of Section 7.2.)

Applying the Power method to $(A - qI)^{-1}$ gives

$$\mathbf{y}^{(m)} = (A - qI)^{-1}\mathbf{x}^{(m-1)},$$

$$\mu^{(m)} = y_{p_{m-1}}^{(m)} = \frac{y_{p_{m-1}}^{(m)}}{x_{p_{m-1}}^{(m-1)}} = \frac{\sum_{j=1}^{n} \beta_j \frac{1}{(\lambda_j - q)^m} v_{p_{m-1}}^{(j)}}{\sum_{j=1}^{n} \beta_j \frac{1}{(\lambda_j - q)^{m-1}} v_{p_{m-1}}^{(j)}}, \tag{9.4}$$

and

$$\mathbf{x}^{(m)} = \frac{\mathbf{y}^{(m)}}{y_{p_m}^{(m)}},$$

where, at each step, p_m represents the smallest integer for which $|y_{p_m}^{(m)}| = ||\mathbf{y}^{(m)}||_\infty$. The sequence $\{\mu^{(m)}\}$ in Eq. (9.4) converges to $1/(\lambda_k - q)$, where

$$\frac{1}{|\lambda_k - q|} = \max_{1 \leq i \leq n} \frac{1}{|\lambda_i - q|},$$

and $\lambda_k \approx q + 1/\mu^{(m)}$ is the eigenvalue of A closest to q.

With k known, Eq. (9.4) can be written as

$$\mu^{(m)} = \frac{1}{\lambda_k - q} \left[\frac{\beta_k v_{p_{m-1}}^{(k)} + \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_j \left[\frac{\lambda_k - q}{\lambda_j - q} \right]^m v_{p_{m-1}}^{(j)}}{\beta_k v_{p_{m-1}}^{(k)} + \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_j \left[\frac{\lambda_k - q}{\lambda_j - q} \right]^{m-1} v_{p_{m-1}}^{(j)}} \right]. \tag{9.5}$$

Thus, the choice of q determines the convergence, provided that $1/(\lambda_k - q)$ is a unique dominant eigenvalue of $(A - qI)^{-1}$ (although it may be a multiple eigenvalue). The closer q is to an eigenvalue λ_k, the faster the convergence since the convergence is of order

$$O\left(\left| \frac{(\lambda - q)^{-1}}{(\lambda_k - q)^{-1}} \right|^m \right) = O\left(\left| \frac{(\lambda_k - q)}{(\lambda - q)} \right|^m \right),$$

where λ represents the eigenvalue of A that is second closest to q.

The vector $\mathbf{y}^{(m)}$ is obtained by solving the linear system

$$(A - qI)\mathbf{y}^{(m)} = \mathbf{x}^{(m-1)}.$$

In general, Gaussian elimination with pivoting is used, but as in the case of the LU factorization, the multipliers can be saved to reduce the computation. The selection of q can be based on the Geršgorin Circle Theorem or on another means of localizing an eigenvalue.

Algorithm 9.3 computes q from an initial approximation to the eigenvector $\mathbf{x}^{(0)}$ by

$$q = \frac{\mathbf{x}^{(0)t} A \mathbf{x}^{(0)}}{\mathbf{x}^{(0)t} \mathbf{x}^{(0)}}.$$

This choice of q results from the observation that if $\mathbf{x}$ is an eigenvector of A with respect to the eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$. So $\mathbf{x}^t A\mathbf{x} = \lambda \mathbf{x}^t \mathbf{x}$ and

$$\lambda = \frac{\mathbf{x}^t A \mathbf{x}}{\mathbf{x}^t \mathbf{x}} = \frac{\mathbf{x}^t A \mathbf{x}}{\|\mathbf{x}\|_2^2}.$$

If q is close to an eigenvalue, the convergence will be quite rapid, but a pivoting technique should be used in Step 6 to avoid contamination by round-off error.

Algorithm 9.3 is often used to approximate an eigenvector when an approximate eigenvalue q is known.

ALGORITHM 9.3

Inverse Power Method

To approximate an eigenvalue and an associated eigenvector of the $n \times n$ matrix A given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_\infty = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $q = \dfrac{\mathbf{x}^t A\mathbf{x}}{\mathbf{x}^t\mathbf{x}}$.

Step 2 Set $k = 1$.

Step 3 Find the smallest integer p with $1 \le p \le n$ and $|x_p| = \|\mathbf{x}\|_\infty$.

Step 4 Set $\mathbf{x} = \mathbf{x}/x_p$.

Step 5 While $(k \le N)$ do Steps 6–12.

Step 6 Solve the linear system $(A - qI)\mathbf{y} = \mathbf{x}$.

Step 7 If the system does not have a unique solution, then
OUTPUT ('q is an eigenvalue', q);
STOP.

Step 8 Set $\mu = y_p$.

Step 9 Find the smallest integer p with $1 \le p \le n$ and $|y_p| = \|\mathbf{y}\|_\infty$.

Step 10 Set $ERR = \|\mathbf{x} - (\mathbf{y}/y_p)\|_\infty$;

$\mathbf{x} = \mathbf{y}/y_p$.

Step 11 If $ERR < TOL$ then set $\mu = (1/\mu) + q$;
OUTPUT $(\mu, \mathbf{x})$;
(*The procedure was successful.*)
STOP.

Step 12 Set $k = k + 1$.

Step 13 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

■

The convergence of the Inverse Power method is linear, so Aitken Δ^2 method can again be used to speed convergence. The following example illustrates the fast convergence of the Inverse Power method if q is close to an eigenvalue.

Example 3 Apply the Inverse Power method with $\mathbf{x}^{(0)} = (1, 1, 1)^t$ to the matrix

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{with} \quad q = \frac{\mathbf{x}^{(0)t} A \mathbf{x}^{(0)}}{\mathbf{x}^{(0)t} \mathbf{x}^{(0)}} = \frac{19}{3},$$

and use Aitken's Δ^2 method to accelerate the convergence.

Solution The Power method was applied to this matrix in Example 1 using the initial vector $\mathbf{x}^{(0)} = (1, 1, 1)^t$. It gave the approximate eigenvalue $\mu^{(12)} = 6.000837$ and eigenvector $(\mathbf{x}^{(12)})^t = (1, 0.714316, -0.249895)^t$.

For the Inverse Power method we consider

$$A - qI = \begin{bmatrix} -\frac{31}{3} & 14 & 0 \\ -5 & \frac{20}{3} & 0 \\ -1 & 0 & -\frac{13}{3} \end{bmatrix}$$

With $\mathbf{x}^{(0)} = (1, 1, 1)^t$, the method first finds $\mathbf{y}^{(1)}$ by solving $(A - qI)\mathbf{y}^{(1)} = \mathbf{x}^{(0)}$. This gives

$$\mathbf{y}^{(1)} = \left(-\frac{33}{5}, -\frac{24}{5}, \frac{84}{65}\right)^t = (-6.6, -4.8, 1.292\overline{307692})^t.$$

So

$$||\mathbf{y}^{(1)}||_\infty = 6.6, \quad \mathbf{x}^{(1)} = \frac{1}{-6.6}\mathbf{y}^{(1)} = (1, 0.7272727, -0.1958042)^t,$$

and

$$\mu^{(1)} = -\frac{1}{6.6} + \frac{19}{3} = 6.1818182.$$

Subsequent results are listed in Table 9.4, and the right column lists the results of Aitken's Δ^2 method applied to the $\mu^{(m)}$. These are clearly superior results to those obtained with the Power method. ■

Table 9.4

m	$\mathbf{x}^{(m)t}$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$
0	(1, 1, 1)		
1	(1, 0.7272727, −0.1958042)	6.1818182	6.000098
2	(1, 0.7155172, −0.2450520)	6.0172414	6.000001
3	(1, 0.7144082, −0.2495224)	6.0017153	6.000000
4	(1, 0.7142980, −0.2499534)	6.0001714	6.000000
5	(1, 0.7142869, −0.2499954)	6.0000171	
6	(1, 0.7142858, −0.2499996)	6.0000017	

If A is symmetric, then for any real number q, the matrix $(A - qI)^{-1}$ is also symmetric, so the Symmetric Power method, Algorithm 9.2, can be applied to $(A - qI)^{-1}$ to speed the convergence to

$$O\left(\left|\frac{\lambda_k - q}{\lambda - q}\right|^{2m}\right).$$

Deflation Methods

Numerous techniques are available for obtaining approximations to the other eigenvalues of a matrix once an approximation to the dominant eigenvalue has been computed. We will restrict our presentation to **deflation techniques**.

Deflation techniques involve forming a new matrix B whose eigenvalues are the same as those of A, except that the dominant eigenvalue of A is replaced by the eigenvalue 0 in B. The following result justifies the procedure. The proof of this theorem can be found in [Wil2], p. 596.

Theorem 9.20

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A with associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ and that λ_1 has multiplicity 1. Let $\mathbf{x}$ be a vector with $\mathbf{x}^t\mathbf{v}^{(1)} = 1$. Then the matrix

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t$$

has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\mathbf{v}^{(1)}$, $\mathbf{w}^{(2)}$, $\mathbf{w}^{(3)}, \ldots, \mathbf{w}^{(n)}$, where $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are related by the equation

$$\mathbf{v}^{(i)} = (\lambda_i - \lambda_1)\mathbf{w}^{(i)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(i)})\mathbf{v}^{(1)}, \tag{9.6}$$

for each $i = 2, 3, \ldots, n$. ■

There are many choices of the vector $\mathbf{x}$ that could be used in Theorem 9.20. **Wielandt deflation** proceeds from defining

$$\mathbf{x} = \frac{1}{\lambda_1 v_i^{(1)}}(a_{i1}, a_{i2}, \ldots, a_{in})^t, \tag{9.7}$$

where $v_i^{(1)}$ is a nonzero coordinate of the eigenvector $\mathbf{v}^{(1)}$, and the values $a_{i1}, a_{i2}, \ldots, a_{in}$ are the entries in the ith row of A.

With this definition,

$$\mathbf{x}^t\mathbf{v}^{(1)} = \frac{1}{\lambda_1 v_i^{(1)}}[a_{i1}, a_{i2}, \ldots, a_{in}](v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)})^t = \frac{1}{\lambda_1 v_i^{(1)}}\sum_{j=1}^{n} a_{ij}v_j^{(1)},$$

where the sum is the ith coordinate of the product $A\mathbf{v}^{(1)}$. Since $A\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)}$, we have

$$\sum_{j=1}^{n} a_{ij}v_j^{(1)} = \lambda_1 v_i^{(1)},$$

which implies that

$$\mathbf{x}^t\mathbf{v}^{(1)} = \frac{1}{\lambda_1 v_i^{(1)}}(\lambda_1 v_i^{(1)}) = 1.$$

So $\mathbf{x}$ satisfies the hypotheses of Theorem 9.20. Moreover (see Exercise 20), the ith row of $B = A - \lambda_1\mathbf{v}^{(1)}\mathbf{x}^t$ consists entirely of zero entries.

If $\lambda \neq 0$ is an eigenvalue with associated eigenvector $\mathbf{w}$, the relation $B\mathbf{w} = \lambda\mathbf{w}$ implies that the ith coordinate of $\mathbf{w}$ must also be zero. Consequently the ith column of the matrix B makes no contribution to the product $B\mathbf{w} = \lambda\mathbf{w}$. Thus, the matrix B can be replaced by an $(n-1) \times (n-1)$ matrix B' obtained by deleting the ith row and column from B. The matrix B' has eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$.

If $|\lambda_2| > |\lambda_3|$, the Power method is reapplied to the matrix B' to determine this new dominant eigenvalue and an eigenvector, $\mathbf{w}^{(2)'}$, associated with λ_2, with respect to the matrix B'. To find the associated eigenvector $\mathbf{w}^{(2)}$ for the matrix B, insert a zero coordinate between the coordinates $w_{i-1}^{(2)'}$ and $w_i^{(2)'}$ of the $(n-1)$-dimensional vector $\mathbf{w}^{(2)'}$ and then calculate $\mathbf{v}^{(2)}$ by the use of Eq. (9.6).

Example 4 The matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

has the dominant eigenvalue $\lambda_1 = 6$ with associated unit eigenvector $\mathbf{v}^{(1)} = (1, -1, 1)^t$. Assume that this dominant eigenvalue is known and apply deflation to approximate the other eigenvalues and eigenvectors.

Solution The procedure for obtaining a second eigenvalue λ_2 proceeds as follows:

$$\mathbf{x} = \frac{1}{6}\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)^t,$$

$$\mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{2}{3}, & -\frac{1}{6}, & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix},$$

and

$$B = A - \lambda_1 \mathbf{v}^{(1)}\mathbf{x}^t = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} - 6\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ -\frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

Deleting the first row and column gives

$$B' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_2 = 3$ and $\lambda_3 = 1$. For $\lambda_2 = 3$, the eigenvector $\mathbf{w}^{(2)'}$ can be obtained by solving the linear system

$$(B' - 3I)\mathbf{w}^{(2)'} = \mathbf{0}, \quad \text{resulting in} \quad \mathbf{w}^{(2)'} = (1, -1)^t.$$

Adding a zero for the first component gives $\mathbf{w}^{(2)} = (0, 1, -1)^t$ and, from Eq. (9.6), we have the eigenvector $\mathbf{v}^{(2)}$ of A corresponding to $x_2 = 3$:

$$\begin{aligned} \mathbf{v}^{(2)} &= (\lambda_2 - \lambda_1)\mathbf{w}^{(2)} + \lambda_1(\mathbf{x}^t\mathbf{w}^{(2)})\mathbf{v}^{(1)} \\ &= (3-6)(0,1,-1)^t + 6\left[\left(\frac{2}{3}, -\frac{1}{6}, \frac{1}{6}\right)(0,1,-1)^t\right](1,-1,1)^t = (-2,-1,1)^t. \end{aligned}$$

■

Although this deflation process can be used to find approximations to all of the eigenvalues and eigenvectors of a matrix, the process is susceptible to round-off error. After deflation is used to approximate an eigenvalue of a matrix, the approximation should be used as a starting value for the Inverse Power method applied to the original matrix. This will ensure convergence to an eigenvalue of the original matrix, not to one of the reduced matrix, which likely contains errors. When all the eigenvalues of a matrix are required, techniques considered in Section 9.5, based on similarity transformations, should be used.

We close this section with Algorithm 9.4, which calculates the second most dominant eigenvalue and associated eigenvector for a matrix, once the dominant eigenvalue and associated eigenvector have been determined.

ALGORITHM
9.4

Wielandt Deflation

To approximate the second most dominant eigenvalue and an associated eigenvector of the $n \times n$ matrix A given an approximation λ to the dominant eigenvalue, an approximation $\mathbf{v}$ to a corresponding eigenvector, and a vector $\mathbf{x} \in \mathbb{R}^{n-1}$:

INPUT dimension n; matrix A; approximate eigenvalue λ with eigenvector $\mathbf{v} \in \mathbb{R}^n$; vector $\mathbf{x} \in \mathbb{R}^{n-1}$, tolerance TOL, maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{u}$ or a message that the method fails.

Step 1 Let i be the smallest integer with $1 \le i \le n$ and $|v_i| = \max_{1 \le j \le n} |v_j|$.

Step 2 If $i \neq 1$ then
for $k = 1, \ldots, i-1$
for $j = 1, \ldots, i-1$

$$\text{set } b_{kj} = a_{kj} - \frac{v_k}{v_i} a_{ij}.$$

Step 3 If $i \neq 1$ and $i \neq n$ then
for $k = i, \ldots, n-1$
for $j = 1, \ldots, i-1$

$$\text{set } b_{kj} = a_{k+1,j} - \frac{v_{k+1}}{v_i} a_{ij};$$

$$b_{jk} = a_{j,k+1} - \frac{v_j}{v_i} a_{i,k+1}.$$

Step 4 If $i \neq n$ then
for $k = i, \ldots, n-1$
for $j = i, \ldots, n-1$

$$\text{set } b_{kj} = a_{k+1,j+1} - \frac{v_{k+1}}{v_i} a_{i,j+1}.$$

Step 5 Perform the power method on the $(n-1) \times (n-1)$ matrix $B' = (b_{kj})$ with $\mathbf{x}$ as initial approximation.

Step 6 If the method fails, then OUTPUT ('Method fails');
STOP
else let μ be the approximate eigenvalue and
$\mathbf{w}' = (w'_1, \ldots, w'_{n-1})^t$ the approximate eigenvector.

Step 7 If $i \neq 1$ then for $k = 1, \ldots, i-1$ set $w_k = w'_k$.

Step 8 Set $w_i = 0$.

Step 9 If $i \neq n$ then for $k = i+1, \ldots, n$ set $w_k = w'_{k-1}$.

Step 10 For $k = 1, \ldots, n$

$$\text{set } u_k = (\mu - \lambda) w_k + \left(\sum_{j=1}^{n} a_{ij} w_j \right) \frac{v_k}{v_i}.$$

(*Compute the eigenvector using Eq.* (9.6).)

Step 11 OUTPUT $(\mu, \mathbf{u})$; (*The procedure was successful.*)
STOP.

■

EXERCISE SET 9.3

1. Find the first three iterations obtained by the Power method applied to the following matrices.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (1, -1, 2)^t$.

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 0, 1)^t$.

c. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 2, 1)^t$.

d. $\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (1, -2, 0, 3)^t$.

3. Repeat Exercise 1 using the Inverse Power method.

13. Use Wielandt deflation and the results of Exercise 7 to approximate the second most dominant eigenvalue of the matrices in Exercise 1. Iterate until a tolerance of 10^{-4} is achieved or until the number of iterations exceeds 25.

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