MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO

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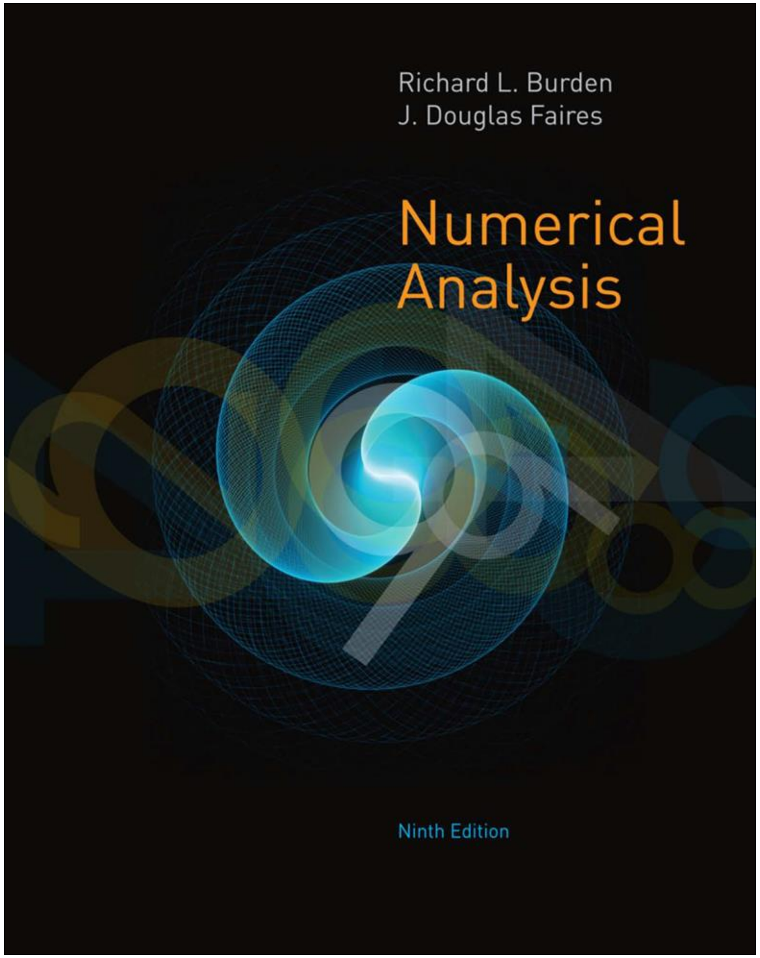

Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

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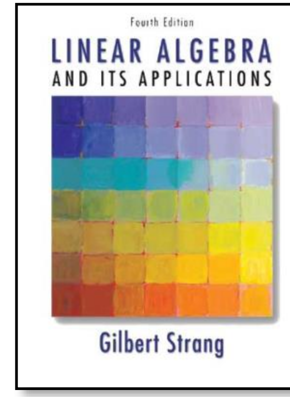

Fourth Edition
LINEAR ALGEBRA
AND ITS APPLICATIONS
Gilbert Strang

+

9.3 The Power Method

The **Power method** is an iterative technique used to determine the dominant eigenvalue of a matrix—that is, the eigenvalue with the largest magnitude. By modifying the method slightly, it can also used to determine other eigenvalues. One useful feature of the Power method is that it produces not only an eigenvalue, but also an associated eigenvector. In fact, the Power method is often applied to find an eigenvector for an eigenvalue that is determined by some other means.

To apply the Power method, we assume that the $n \times n$ matrix A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with an associated collection of linearly independent eigenvectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n)}\}$. Moreover, we assume that A has precisely one eigenvalue, λ_1, that is largest in magnitude, so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0.$$

Example 4 of Section 9.1 illustrates that an $n \times n$ matrix need not have n linearly independent eigenvectors. When it does not the Power method may still be successful, but it is not guaranteed to be.

If $\mathbf{x}$ is any vector in $\mathbb{R}^n$, the fact that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n)}\}$ is linearly independent implies that constants $\beta_1, \beta_2, \ldots, \beta_n$ exist with

$$\mathbf{x} = \sum_{j=1}^{n} \beta_j \mathbf{v}^{(j)}.$$

Multiplying both sides of this equation by $A, A^2, \ldots, A^k, \ldots$ gives

$$A\mathbf{x} = \sum_{j=1}^{n} \beta_j A\mathbf{v}^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j \mathbf{v}^{(j)}, \quad A^2\mathbf{x} = \sum_{j=1}^{n} \beta_j \lambda_j A\mathbf{v}^{(j)} = \sum_{j=1}^{n} \beta_j \lambda_j^2 \mathbf{v}^{(j)},$$

and generally, $A^k\mathbf{x} = \sum_{j=1}^{n} \beta_j \lambda_j^k \mathbf{v}^{(j)}$.

If λ_1^k is factored from each term on the right side of the last equation, then

$$A^k\mathbf{x} = \lambda_1^k \sum_{j=1}^{n} \beta_j \left(\frac{\lambda_j}{\lambda_1}\right)^k \mathbf{v}^{(j)}.$$

Since $|\lambda_1| > |\lambda_j|$, for all $j = 2, 3, \ldots, n$, we have $\lim_{k\to\infty}(\lambda_j/\lambda_1)^k = 0$, and

$$\lim_{k\to\infty} A^k\mathbf{x} = \lim_{k\to\infty} \lambda_1^k \beta_1 \mathbf{v}^{(1)}. \tag{9.2}$$

The sequence in Eq. (9.2) converges to $\mathbf{0}$ if $|\lambda_1| < 1$ and diverges if $|\lambda_1| > 1$, provided, of course, that $\beta_1 \neq 0$. As a consequence, the entries in the $A^k\mathbf{x}$ will grow with k if $|\lambda_1| > 1$ and will go to 0 if $|\lambda_1| < 1$, perhaps resulting in overflow or underflow. To take care of that possibility, we scale the powers of $A^k\mathbf{x}$ in an appropriate manner to ensure that the limit in Eq. (9.2) is finite and nonzero. The scaling begins by choosing $\mathbf{x}$ to be a unit vector $\mathbf{x}^{(0)}$ relative to $\|\cdot\|_\infty$ and choosing a component $x_{p_0}^{(0)}$ of $\mathbf{x}^{(0)}$ with

$$x_{p0}^{(0)} = 1 = \|\mathbf{x}^{(0)}\|_\infty.$$

Componente de y que norma do máximo de x (p_0)

Let $\mathbf{y}^{(1)} = A\mathbf{x}^{(0)}$, and define $\mu^{(1)} = y^{(1)}_{p_0}$. Then

$$\mu^{(1)} = y^{(1)}_{p_0} = \frac{y^{(1)}_{p_0}}{x^{(0)}_{p_0}} = \frac{\beta_1 \lambda_1 v^{(1)}_{p_0} + \sum_{j=2}^{n} \beta_j \lambda_j v^{(j)}_{p_0}}{\beta_1 v^{(1)}_{p_0} + \sum_{j=2}^{n} \beta_j v^{(j)}_{p_0}} = \lambda_1 \left[\frac{\beta_1 v^{(1)}_{p_0} + \sum_{j=2}^{n} \beta_j (\lambda_j/\lambda_1) v^{(j)}_{p_0}}{\beta_1 v^{(1)}_{p_0} + \sum_{j=2}^{n} \beta_j v^{(j)}_{p_0}} \right].$$

Let p_1 be the least integer such that

$$|y^{(1)}_{p_1}| = \|\mathbf{y}^{(1)}\|_\infty,$$

p_1 é o índice da maior componente de y (correspondendo a norma do máximo)

and define $\mathbf{x}^{(1)}$ by

$$\mathbf{x}^{(1)} = \frac{1}{y^{(1)}_{p_1}} \mathbf{y}^{(1)} = \frac{1}{y^{(1)}_{p_1}} A\mathbf{x}^{(0)}.$$

Then

$$x^{(1)}_{p_1} = 1 = \|\mathbf{x}^{(1)}\|_\infty.$$

Now define

$$\mathbf{y}^{(2)} = A\mathbf{x}^{(1)} = \frac{1}{y^{(1)}_{p_1}} A^2 \mathbf{x}^{(0)}$$

and

$$\mu^{(2)} = y_{p_1}^{(2)} = \frac{y_{p_1}^{(2)}}{x_{p_1}^{(1)}} = \frac{\left[\beta_1 \lambda_1^2 v_{p_1}^{(1)} + \sum_{j=2}^{n} \beta_j \lambda_j^2 v_{p_1}^{(j)}\right] \Big/ y_{p_1}^{(1)}}{\left[\beta_1 \lambda_1 v_{p_1}^{(1)} + \sum_{j=2}^{n} \beta_j \lambda_j v_{p_1}^{(j)}\right] \Big/ y_{p_1}^{(1)}}$$

$$= \lambda_1 \left[\frac{\beta_1 v_{p_1}^{(1)} + \sum_{j=2}^{n} \beta_j (\lambda_j/\lambda_1)^2 v_{p_1}^{(j)}}{\beta_1 v_{p_1}^{(1)} + \sum_{j=2}^{n} \beta_j (\lambda_j/\lambda_1) v_{p_1}^{(j)}} \right].$$

Let p_2 be the smallest integer with

$$|y_{p_2}^{(2)}| = \|\mathbf{y}^{(2)}\|_\infty,$$

and define

$$\mathbf{x}^{(2)} = \frac{1}{y_{p_2}^{(2)}} \mathbf{y}^{(2)} = \frac{1}{y_{p_2}^{(2)}} A\mathbf{x}^{(1)} = \frac{1}{y_{p_2}^{(2)} y_{p_1}^{(1)}} A^2 \mathbf{x}^{(0)}.$$

In a similar manner, define sequences of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ and $\{\mathbf{y}^{(m)}\}_{m=1}^{\infty}$, and a sequence of scalars $\{\mu^{(m)}\}_{m=1}^{\infty}$ inductively by

$$\mathbf{y}^{(m)} = A\mathbf{x}^{(m-1)},$$

$$\mu^{(m)} = y_{p_{m-1}}^{(m)} = \lambda_1 \left[\frac{\beta_1 v_{p_{m-1}}^{(1)} + \sum_{j=2}^{n} (\lambda_j/\lambda_1)^m \beta_j v_{p_{m-1}}^{(j)}}{\beta_1 v_{p_{m-1}}^{(1)} + \sum_{j=2}^{n} (\lambda_j/\lambda_1)^{m-1} \beta_j v_{p_{m-1}}^{(j)}} \right], \tag{9.3}$$

and

$$\mathbf{x}^{(m)} = \frac{\mathbf{y}^{(m)}}{y_{p_m}^{(m)}} = \frac{A^m \mathbf{x}^{(0)}}{\prod_{k=1}^{m} y_{p_k}^{(k)}},$$

where at each step, p_m is used to represent the smallest integer for which

$$|y_{p_m}^{(m)}| = \|\mathbf{y}^{(m)}\|_{\infty}.$$

By examining Eq. (9.3), we see that since $|\lambda_j/\lambda_1| < 1$, for each $j = 2, 3, \ldots, n$, $\lim_{m\to\infty} \mu^{(m)} = \lambda_1$, provided that $\mathbf{x}^{(0)}$ is chosen so that $\beta_1 \neq 0$. Moreover, the sequence of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ converges to an eigenvector associated with λ_1 that has l_∞ norm equal to one.

Illustration

The matrix

$$A = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$$

Has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = (1, -2)^t$ and $\mathbf{v}_2 = (1, -1)^t$. If we start with the arbitrary vector $\mathbf{x}_0 = (1, 1)^t$ and multiply by the matrix A we obtain

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} -5 \\ 13 \end{bmatrix}, \qquad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} -29 \\ 61 \end{bmatrix}, \qquad \mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} -125 \\ 253 \end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} -509 \\ 1021 \end{bmatrix}, \qquad \mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} -2045 \\ 4093 \end{bmatrix}, \qquad \mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} -8189 \\ 16381 \end{bmatrix}.$$

As a consequence, approximations to the dominant eigenvalue $\lambda_1 = 4$ are

$$\lambda_1^{(1)} = \frac{61}{13} = 4.6923, \qquad \lambda_1^{(2)} = \frac{253}{61} = 4.14754, \qquad \lambda_1^{(3)} = \frac{1021}{253} = 4.03557,$$

$$\lambda_1^{(4)} = \frac{4093}{1021} = 4.00881, \qquad \lambda_1^{(5)} = \frac{16381}{4093} = 4.00200.$$

An approximate eigenvector corresponding to $\lambda_1^{(5)} = \dfrac{16381}{4093} = 4.00200$ is

$$\mathbf{x}_6 = \begin{bmatrix} -8189 \\ 16381 \end{bmatrix}, \quad \text{which, divided by 16381, normalizes to} \quad \begin{bmatrix} -0.49908 \\ 1 \end{bmatrix} \approx \mathbf{v}_1.$$

□

Illustration

The matrix

$$A = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$$

Has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{v}_1 = (1, -2)^t$ and $\mathbf{v}_2 = (1, -1)^t$. If we start with the arbitrary vector $\mathbf{x}_0 = (1, 1)^t$ and multiply by the matrix A we obtain

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□

ALGORITHM 9.1

Power Method

MAP2210

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ matrix A given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $||\mathbf{x}||_\infty = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 Find the smallest integer p with $1 \le p \le n$ and $|x_p| = ||\mathbf{x}||_\infty$.

Step 3 Set $\mathbf{x} = \mathbf{x}/x_p$.

Step 4 While ($k \le N$) do Steps 5–11.

Multiplicação matriz vetor pode ser customizada de acordo com a estrutura da matriz

Step 5 Set $\mathbf{y} = A\mathbf{x}$.

Step 6 Set $\mu = y_p$. Autovalor

Step 7 Find the smallest integer p with $1 \le p \le n$ and $|y_p| = \|\mathbf{y}\|_\infty$.

Step 8 If $y_p = 0$ then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has the eigenvalue 0, select a new vector $\mathbf{x}$ and restart');
STOP.

Step 9 Set $ERR = ||\mathbf{x} - (\mathbf{y}/y_p)||_\infty$;

$\mathbf{x} = \mathbf{y}/y_p$. Autovetor

Step 10 If $ERR < TOL$ then OUTPUT ($\mu, \mathbf{x}$);
(*The procedure was successful.*)
STOP.

Step 11 Set $k = k + 1$.

Step 12 OUTPUT ('The maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

The Power method has the disadvantage that it is unknown at the outset whether or not the matrix has a single dominant eigenvalue. Nor is it known how $\mathbf{x}^{(0)}$ should be chosen so as to ensure that its representation in terms of the eigenvectors of the matrix will contain a nonzero contribution from the eigenvector associated with the dominant eigenvalue, should it exist.

Accelerating Convergence

Choosing, in Step 7, the smallest integer p_m for which $|y_{p_m}^{(m)}| = \|\mathbf{y}^{(m)}\|_\infty$ will generally ensure that this index eventually becomes invariant. The rate at which $\{\mu^{(m)}\}_{m=1}^{\infty}$ converges to λ_1 is determined by the ratios $|\lambda_j/\lambda_1|^m$, for $j = 2, 3, \ldots, n$, and in particular by $|\lambda_2/\lambda_1|^m$. The rate of convergence is $O(|\lambda_2/\lambda_1|^m)$ (see [IK, p. 148]), so there is a constant k such that for large m,

$$|\mu^{(m)} - \lambda_1| \approx k \left|\frac{\lambda_2}{\lambda_1}\right|^m,$$

which implies that

$$\lim_{m\to\infty} \frac{|\mu^{(m+1)} - \lambda_1|}{|\mu^{(m)} - \lambda_1|} \approx \left|\frac{\lambda_2}{\lambda_1}\right| < 1.$$

The sequence $\{\mu^{(m)}\}$ converges linearly to λ_1, so Aitken's Δ^2 procedure discussed in Section 2.5 can be used to speed the convergence.

2.5 Accelerating Convergence

Aitken's Δ^2 Method

Suppose $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p. To motivate the construction of a sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ that converges more rapidly to p than does $\{p_n\}_{n=0}^{\infty}$, let us first assume that the signs of $p_n - p$, $p_{n+1} - p$, and $p_{n+2} - p$ agree and that n is sufficiently large that

$$\frac{p_{n+1} - p}{p_n - p} \approx \frac{p_{n+2} - p}{p_{n+1} - p}.$$

Then

$$(p_{n+1} - p)^2 \approx (p_{n+2} - p)(p_n - p),$$

so

$$p_{n+1}^2 - 2p_{n+1}p + p^2 \approx p_{n+2}p_n - (p_n + p_{n+2})p + p^2$$

and

$$(p_{n+2} + p_n - 2p_{n+1})p \approx p_{n+2}p_n - p_{n+1}^2.$$

Solving for p gives

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Adding and subtracting the terms p_n^2 and $2p_np_{n+1}$ in the numerator and grouping terms appropriately gives

$$p \approx \frac{p_np_{n+2} - 2p_np_{n+1} + p_n^2 - p_{n+1}^2 + 2p_np_{n+1} - p_n^2}{p_{n+2} - 2p_{n+1} + p_n}$$

$$= \frac{p_n(p_{n+2} - 2p_{n+1} + p_n) - (p_{n+1}^2 - 2p_np_{n+1} + p_n^2)}{p_{n+2} - 2p_{n+1} + p_n}$$

$$= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}.$$

Aitken's Δ^2 method is based on the assumption that the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$, defined by

$$\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}, \qquad (2.14)$$

converges more rapidly to p than does the original sequence $\{p_n\}_{n=0}^{\infty}$.

Implementing the Δ^2 procedure in Algorithm 9.1 is accomplished by modifying the algorithm as follows:

Step 1 Set $k = 1$;
$\mu_0 = 0$;
$\mu_1 = 0$.

$$\vdots$$

Step 6 Set $\mu = y_p$;

$$\hat{\mu} = \mu_0 - \frac{(\mu_1 - \mu_0)^2}{\mu - 2\mu_1 + \mu_0}.$$

Step 10 If $ERR < TOL$ and $k \geq 4$ then OUTPUT $(\hat{\mu}, \mathbf{x})$;
STOP.

Step 11 Set $k = k + 1$;
$\mu_0 = \mu_1$;
$\mu_1 = \mu$.

In actuality, it is not necessary for the matrix to have distinct eigenvalues for the Power method to converge. If the matrix has a unique dominant eigenvalue, λ_1, with multiplicity r greater than 1 and $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(r)}$ are linearly independent eigenvectors associated with λ_1, the procedure will still converge to λ_1. The sequence of vectors $\{\mathbf{x}^{(m)}\}_{m=0}^{\infty}$ will, in this case, converge to an eigenvector of λ_1 of l_∞ norm equal to one that depends on the choice of the initial vector $\mathbf{x}^{(0)}$ and is a linear combination of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(r)}$. (See [Wil2], page 570.)

Example 1

Use the Power method to approximate the dominant eigenvalue of the matrix

$$A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix},$$

and then apply Aitken's Δ^2 method to the approximations to the eigenvalue of the matrix to accelerate the convergence.

Solution This matrix has eigenvalues $\lambda_1 = 6, \lambda_2 = 3$, and $\lambda_3 = 2$, so the Power method described in Algorithm 9.1 will converge. Let $\mathbf{x}^{(0)} = (1, 1, 1)^t$, then

$$\mathbf{y}^{(1)} = A\mathbf{x}^{(0)} = (10, 8, 1)^t,$$

so

$$||\mathbf{y}^{(1)}||_\infty = 10, \quad \mu^{(1)} = y_1^{(1)} = 10, \quad \text{and} \quad \mathbf{x}^{(1)} = \frac{\mathbf{y}^{(1)}}{10} = (1, 0.8, 0.1)^t.$$

Continuing in this manner leads to the values in Table 9.1, where $\hat{\mu}^{(m)}$ represents the sequence generated by the Aitken's Δ^2 procedure. An approximation to the dominant eigenvalue, 6, at this stage is $\hat{\mu}^{(10)} = 6.000000$. The approximate l_∞-unit eigenvector for the eigenvalue 6 is $(\mathbf{x}^{(12)})^t = (1, 0.714316, -0.249895)^t$.

Although the approximation to the eigenvalue is correct to the places listed, the eigenvector approximation is considerably less accurate to the true eigenvector, $(1, 5/7, -1/4)^t \approx (1, 0.714286, -0.25)^t$. ■

Table 9.1

m	$(\mathbf{x}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$
0	(1, 1, 1)		
1	(1, 0.8, 0.1)	10	6.266667
2	(1, 0.75, −0.111)	7.2	6.062473
3	(1, 0.730769, −0.188803)	6.5	6.015054
4	(1, 0.722200, −0.220850)	6.230769	6.004202
5	(1, 0.718182, −0.235915)	6.111000	6.000855
6	(1, 0.716216, −0.243095)	6.054546	6.000240
7	(1, 0.715247, −0.246588)	6.027027	6.000058
8	(1, 0.714765, −0.248306)	6.013453	6.000017
9	(1, 0.714525, −0.249157)	6.006711	6.000003
10	(1, 0.714405, −0.249579)	6.003352	6.000000
11	(1, 0.714346, −0.249790)	6.001675	
12	(1, 0.714316, −0.249895)	6.000837	

Symmetric Matrices

When A is symmetric, a variation in the choice of the vectors $\mathbf{x}^{(m)}$ and $\mathbf{y}^{(m)}$ and the scalars $\mu^{(m)}$ can be made to significantly improve the rate of convergence of the sequence $\{\mu^{(m)}\}_{m=1}^{\infty}$ to the dominant eigenvalue λ_1. In fact, although the rate of convergence of the general Power method is $O(|\lambda_2/\lambda_1|^m)$, the rate of convergence of the modified procedure given in Algorithm 9.2 for symmetric matrices is $O(|\lambda_2/\lambda_1|^{2m})$. (See [IK, pp. 149 ff].) Because the sequence $\{\mu^{(m)}\}$ is still linearly convergent, Aitken's Δ^2 procedure can also be applied.

ALGORITHM 9.2

Symmetric Power Method

To approximate the dominant eigenvalue and an associated eigenvector of the $n \times n$ symmetric matrix A, given a nonzero vector $\mathbf{x}$:

INPUT dimension n; matrix A; vector $\mathbf{x}$; tolerance TOL; maximum number of iterations N.

OUTPUT approximate eigenvalue μ; approximate eigenvector $\mathbf{x}$ (with $\|\mathbf{x}\|_2 = 1$) or a message that the maximum number of iterations was exceeded.

Step 1 Set $k = 1$;
$\mathbf{x} = \mathbf{x}/\|\mathbf{x}\|_2$.

Step 2 While ($k \leq N$) do Steps 3–8.

Step 3 Set $\mathbf{y} = A\mathbf{x}$.

Multiplicação matriz vetor pode ser customizada para aproveitar a simetria e estrutura

Step 4 Set $\mu = \mathbf{x}^t\mathbf{y}$.

Autovalor

Step 5 If $\|\mathbf{y}\|_2 = 0$, then OUTPUT ('Eigenvector', $\mathbf{x}$);
OUTPUT ('A has eigenvalue 0, select new vector $\mathbf{x}$ and restart');
STOP.

Step 6 Set $ERR = \left\| \mathbf{x} - \dfrac{\mathbf{y}}{\|\mathbf{y}\|_2} \right\|_2$;
$\mathbf{x} = \mathbf{y}/\|\mathbf{y}\|_2$.

Autovetor

Step 7 If $ERR < TOL$ then OUTPUT ($\mu, \mathbf{x}$);
(*The procedure was successful.*)
STOP.

Step 8 Set $k = k + 1$.

Step 9 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

Example 2

Apply both the Power method and the Symmetric Power method to the matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix},$$

using Aitken's Δ^2 method to accelerate the convergence.

Solution This matrix has eigenvalues $\lambda_1 = 6$, $\lambda_2 = 3$, and $\lambda_3 = 1$. An eigenvector for the eigenvalue 6 is $(1, -1, 1)^t$. Applying the Power method to this matrix with initial vector $(1, 0, 0)^t$ gives the values in Table 9.2. ■

Table 9.2

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\Vert\mathbf{x}^{(m)}\Vert_\infty = 1$
0				$(1, 0, 0)$
1	$(4, -1, 1)$	4		$(1, -0.25, 0.25)$
2	$(4.5, -2.25, 2.25)$	4.5	7	$(1, -0.5, 0.5)$
3	$(5, -3.5, 3.5)$	5	6.2	$(1, -0.7, 0.7)$
4	$(5.4, -4.5, 4.5)$	5.4	6.047617	$(1, -0.833\overline{3}, 0.833\overline{3})$
5	$(5.66\overline{6}, -5.166\overline{6}, 5.166\overline{6})$	$5.66\overline{6}$	6.011767	$(1, -0.911765, 0.911765)$
6	$(5.823529, -5.558824, 5.558824)$	5.823529	6.002931	$(1, -0.954545, 0.954545)$
7	$(5.909091, -5.772727, 5.772727)$	5.909091	6.000733	$(1, -0.976923, 0.976923)$
8	$(5.953846, -5.884615, 5.884615)$	5.953846	6.000184	$(1, -0.988372, 0.988372)$
9	$(5.976744, -5.941861, 5.941861)$	5.976744		$(1, -0.994163, 0.994163)$
10	$(5.988327, -5.970817, 5.970817)$	5.988327		$(1, -0.997076, 0.997076)$

We will now apply the Symmetric Power method to this matrix with the same initial vector $(1, 0, 0)^t$. The first steps are

$$\mathbf{x}^{(0)} = (1, 0, 0)^t, \quad A\mathbf{x}^{(0)} = (4, -1, 1)^t, \mu^{(1)} = 4,$$

and

$$\mathbf{x}^{(1)} = \frac{1}{||A\mathbf{x}^{(0)}||_2} \cdot A\mathbf{x}^{(0)} = (0.942809, -0.235702, 0.235702)^t.$$

The remaining entries are shown in Table 9.3.

Table 9.3

m	$(\mathbf{y}^{(m)})^t$	$\mu^{(m)}$	$\hat{\mu}^{(m)}$	$(\mathbf{x}^{(m)})^t$ with $\\|\mathbf{x}^{(m)}\\|_2 = 1$
0	(1, 0, 0)			(1, 0, 0)
1	(4, −1, 1)	4	7	(0.942809, −0.235702, 0.235702)
2	(4.242641, −2.121320, 2.121320	5	6.047619	(0.816497, −0.408248, 0.408248)
3	(4.082483, −2.857738, 2.857738)	5.666667	6.002932	(0.710669, −0.497468, 0.497468)
4	(3.837613, −3.198011, 3.198011)	5.909091	6.000183	(0.646997, −0.539164, 0.539164)
5	(3.666314, −3.342816, 3.342816)	5.976744	6.000012	(0.612836, −0.558763, 0.558763)
6	(3.568871, −3.406650, 3.406650)	5.994152	6.000000	(0.595247, −0.568190, 0.568190)
7	(3.517370, −3.436200, 3.436200)	5.998536	6.000000	(0.586336, −0.572805, 0.572805)
8	(3.490952, −3.450359, 3.450359)	5.999634		(0.581852, −0.575086, 0.575086)
9	(3.477580, −3.457283, 3.457283)	5.999908		(0.579603, −0.576220, 0.576220)
10	(3.470854, −3.460706, 3.460706)	5.999977		(0.578477, −0.576786, 0.576786)

The Symmetric Power method gives considerably faster convergence for this matrix than the Power method. The eigenvector approximations in the Power method converge to $(1, -1, 1)^t$, a vector with unit l_∞-norm. In the Symmetric Power method, the convergence is to the parallel vector $(\sqrt{3}/3, -\sqrt{3}/3, \sqrt{3}/3)^t$, which has unit l_2-norm.

EXERCISE SET 9.3

1. Find the first three iterations obtained by the Power method applied to the following matrices.

a. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (1, -1, 2)^t$.

b. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 0, 1)^t$.

c. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (-1, 2, 1)^t$.

d. $\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$;
Use $\mathbf{x}^{(0)} = (1, -2, 0, 3)^t$.

5. Find the first three iterations obtained by the Symmetric Power method

Considerem já implementar em Python

Fim...

AULA 16