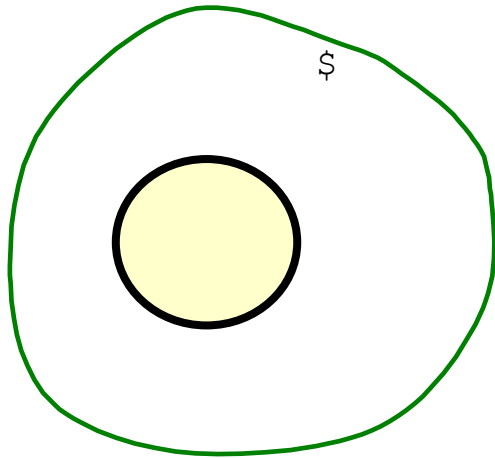
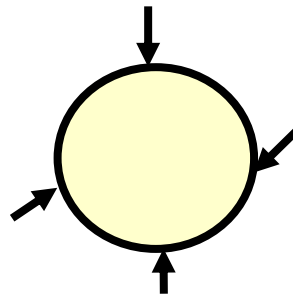
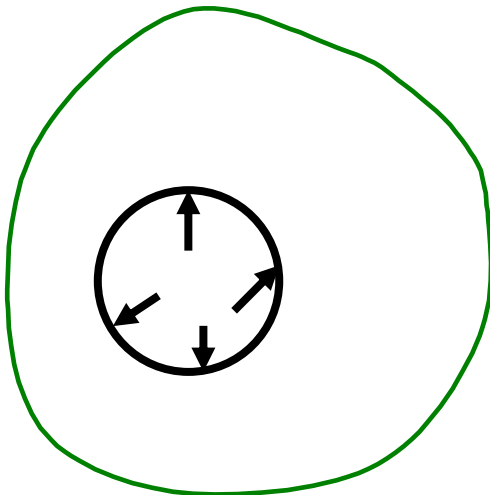


Equações Constitutivas

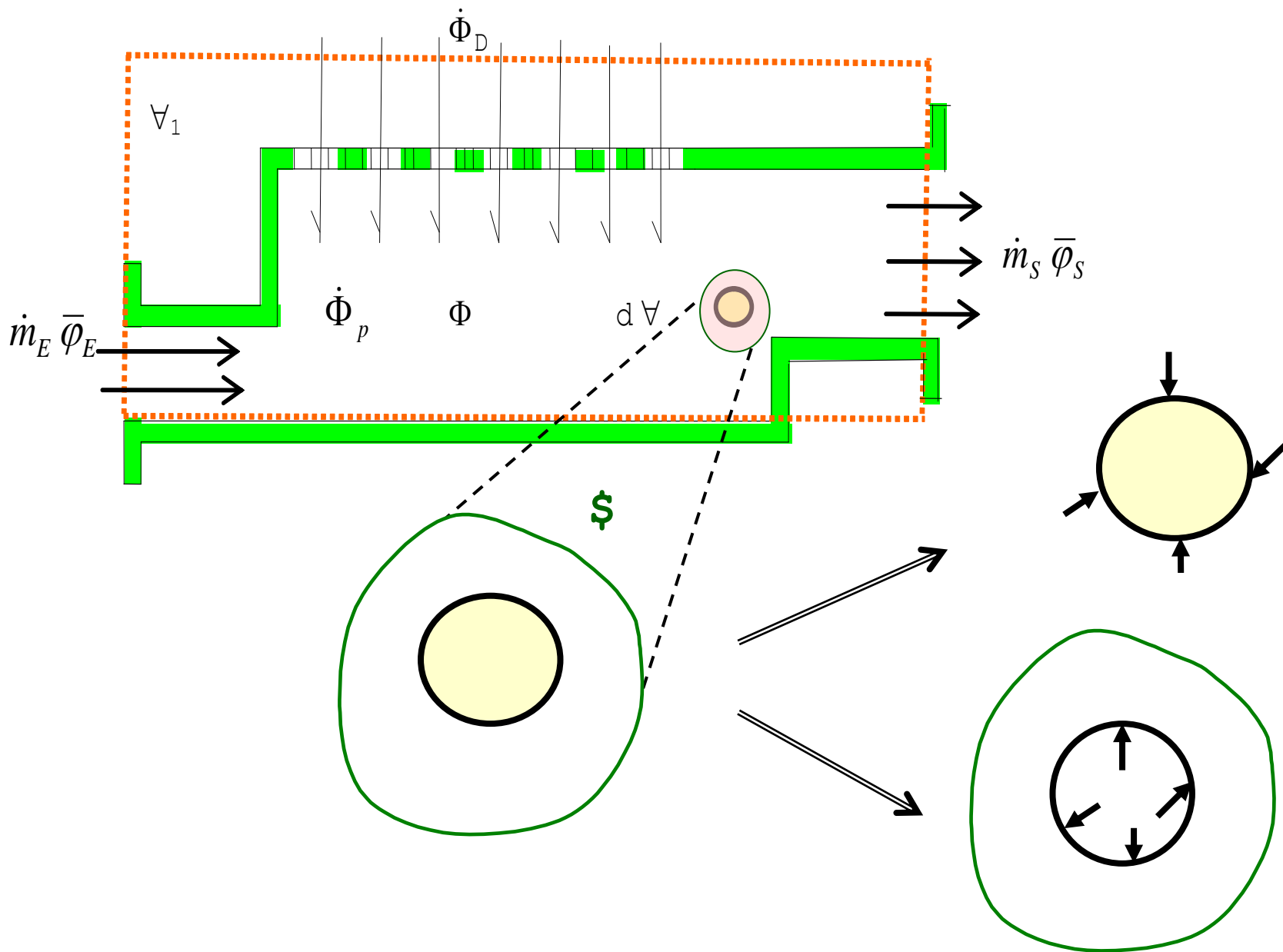
CAUCHY



VOLUME DE
FLUIDO

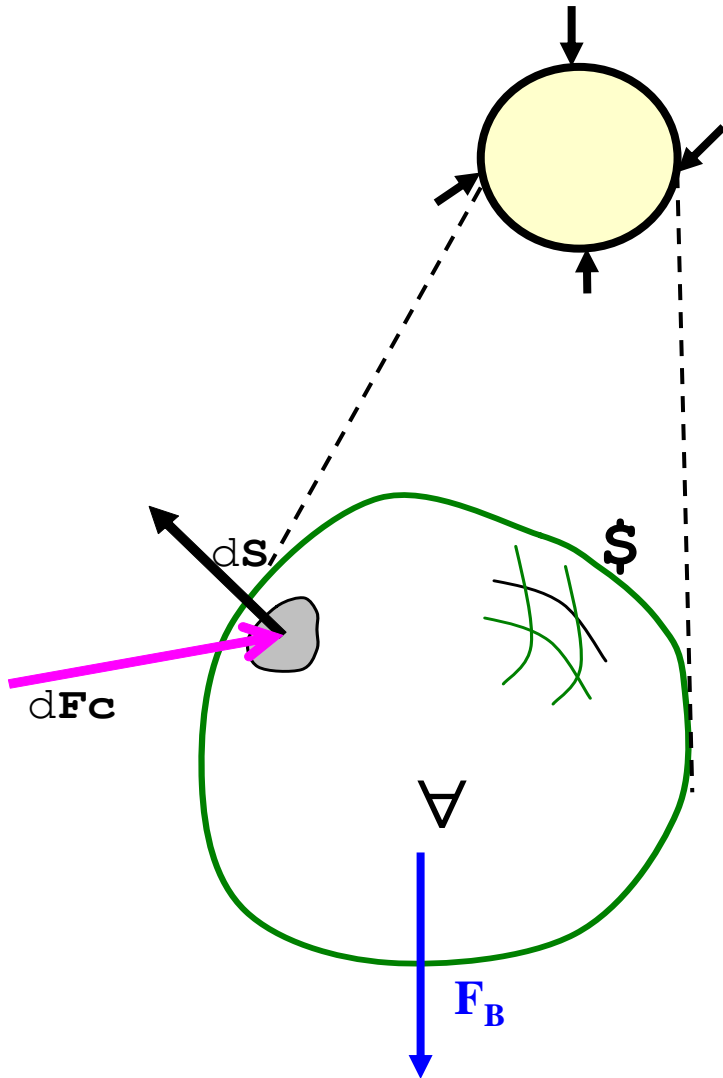


Equações Constitutivas – Escoamento - CAUCHY



Equações Constitutivas

FORÇA DE CONTATO E DE CAMPO



\mathbf{F}_B - força de campo – produção

\mathbf{F}_c - força de contato - difusão

Equações Constitutivas

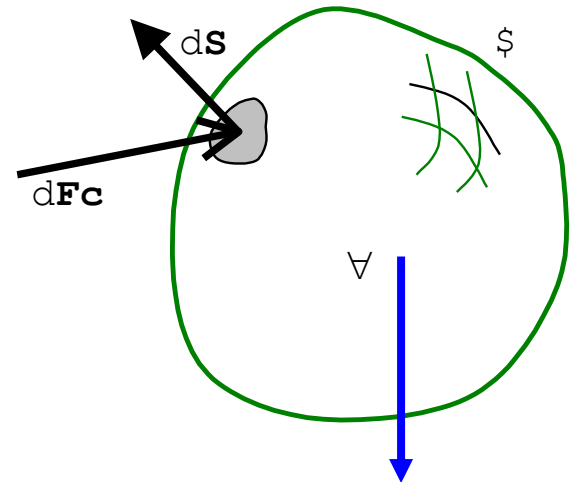
FORÇA DE CONTATO E DE CAMPO

Força de campo/volume: GRAVIDADE $\vec{F}_B = \rho \vec{g}$

O tensor das Tensões: $\vec{\sigma} = \frac{d\vec{F}_C}{d\vec{S}}$

F_B - força de campo – produção

F_C - força de contato - difusão



Equações Constitutivas

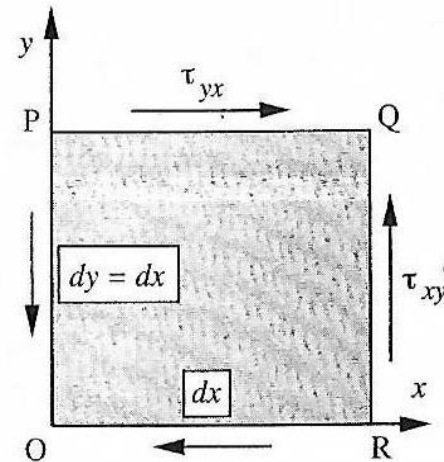
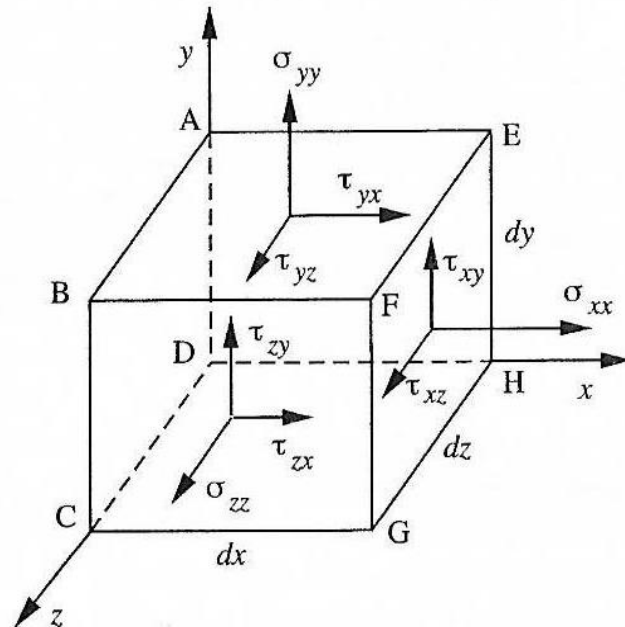
Decomposição do Tensor das TENSÕES σ

$$\vec{\sigma} = -p \vec{\delta} + \vec{\tau} \quad \left\{ \begin{array}{l} \text{tensões normais: } \sigma_{ij} = -p + \tau_{ij} \\ \text{tensões de cisalhamento: } \sigma_{ij} = \tau_{ij} \end{array} \right.$$

σ_{ij} : componente da força (de contato) na direção j por unidade de área perpendicular à direção i .

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}. \quad (5)$$

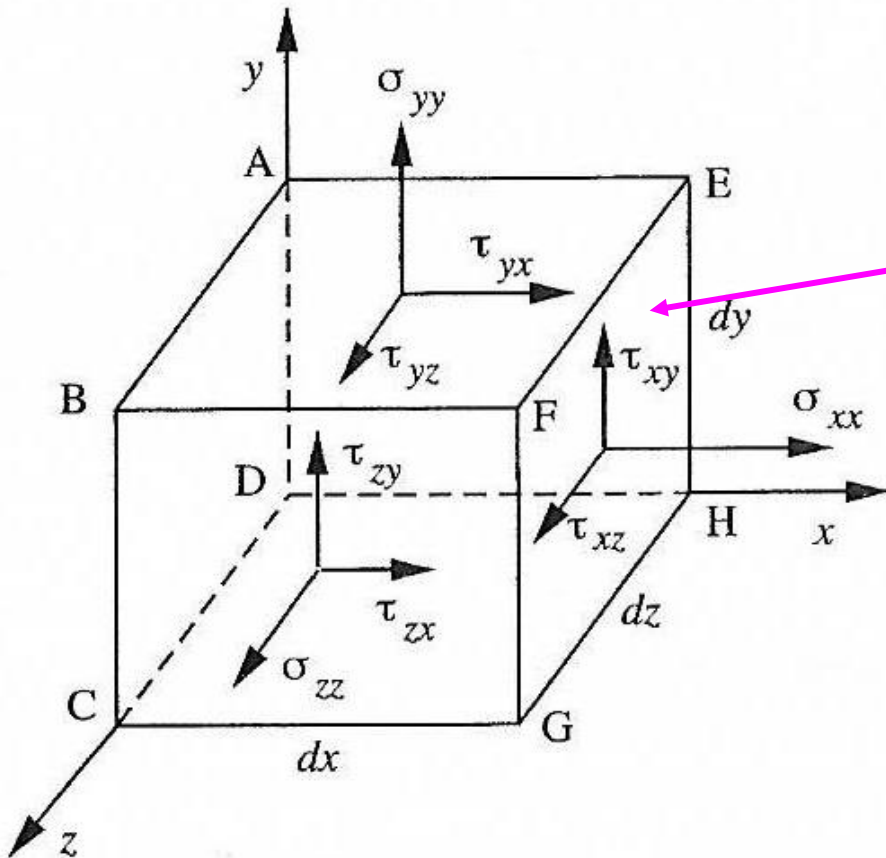
$$\sigma_{ij} = \frac{d\vec{F}_C}{d\vec{S}}$$



Força de contato

$$\sigma_{ij} = \frac{d\vec{F}_C}{d\vec{S}} \quad \left\{ \begin{array}{l} d\vec{F}_C = dF_{cx}\vec{i} + dF_{cy}\vec{j} + dF_{cz}\vec{k} \\ d\vec{A} = dA_x\vec{i} + dA_y\vec{j} + dA_z\vec{k} \end{array} \right.$$

$$\sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$



$$\sigma_{xx} = \frac{dF_{cx}}{dA_x}$$

$$\sigma_{xy} = \frac{dF_{cy}}{dA_x}$$

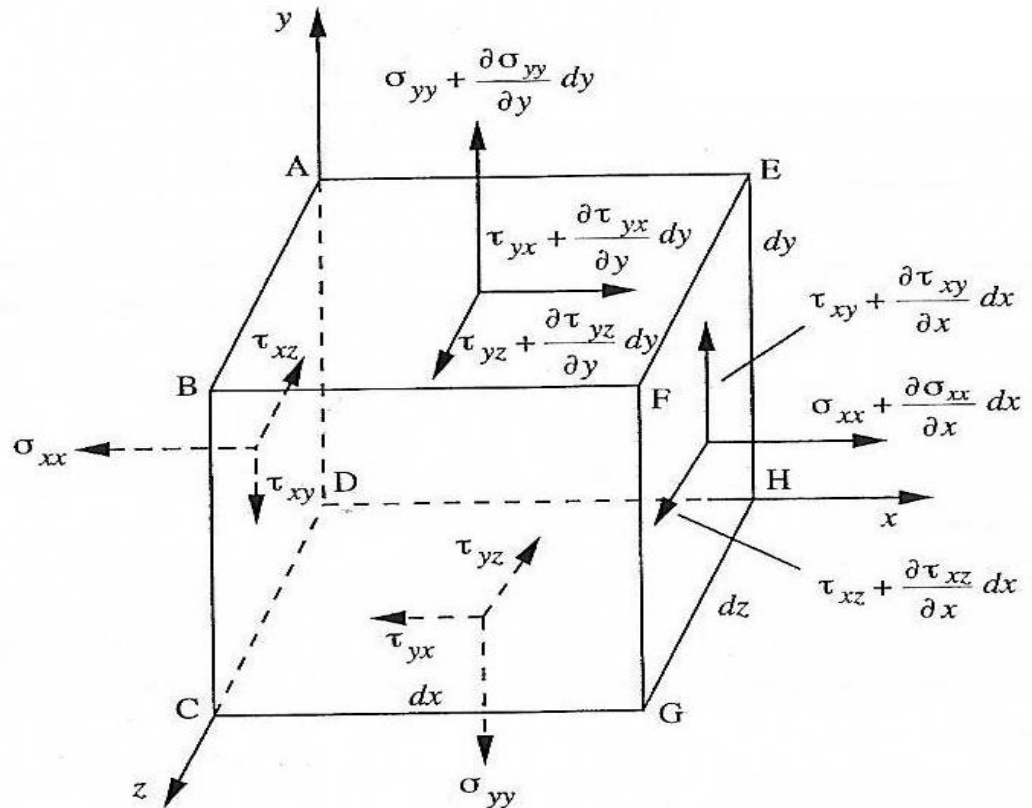
$$\sigma_{xz} = \frac{dF_{cz}}{dA_x}$$

Força de contato na direção x

$$d\vec{F}_c = dF_{cx} \vec{i} + dF_{cy} \vec{j} + dF_{cz} \vec{k}$$

$$dF_{cx} = \{[(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x] \Delta y \Delta z + [(\tau_{yx})_{y+\Delta y} - (\tau_{yx})_y] \Delta x \Delta z + [(\tau_{zx})_{z+\Delta z} - (\tau_{zx})_z] \Delta x \Delta y\} - \{[(p)_{x+\Delta x} - (p)_x] \Delta y \Delta z\}$$

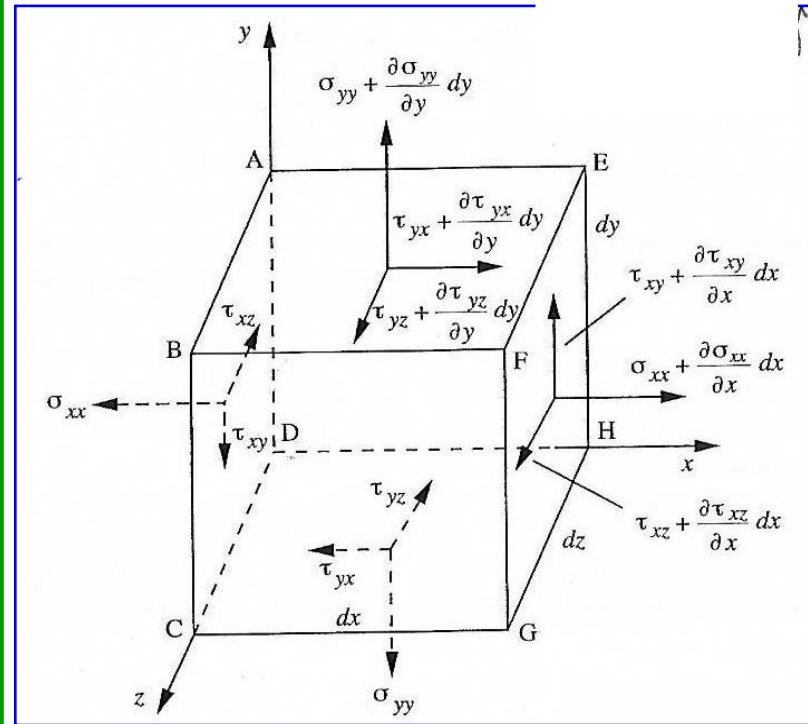
Volume de controle finito
 $\Delta V = \Delta x \Delta y \Delta z$.



Força de contato na direção x

$$dF_{cx} = \{[(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x]\Delta y\Delta z + [(\tau_{yx})_{y+\Delta y} - (\tau_{yx})_y]\Delta x\Delta z + [(\tau_{zx})_{z+\Delta z} - (\tau_{zx})_z]\Delta x\Delta y\} - \{[(p)_{x+\Delta x} - (p)_x]\Delta y\Delta z\}$$

$$\begin{aligned} dF_{cx} &= - \left[\frac{(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x}{\Delta x} \right] \Delta x \Delta y \Delta z \\ &+ \left[\frac{(\tau_{yx})_{y+\Delta y} - (\tau_{yx})_y}{\Delta y} \right] \Delta y \Delta x \Delta z \\ &+ \left[\frac{(\tau_{zx})_{z+\Delta z} - (\tau_{zx})_z}{\Delta z} \right] \Delta z \Delta x \Delta y \\ &- \left\{ \left[\frac{(p)_{x+\Delta x} - (p)_x}{\Delta x} \right] \Delta x \Delta y \Delta z \right\} \end{aligned}$$



Força de contato e campo na direção x

$$dF_{cx} = \left[\frac{(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x}{\Delta x} \right] \Delta x \Delta y \Delta z + \left[\frac{(\tau_{yx})_{y+\Delta y} - (\tau_{yx})_y}{\Delta y} \right] \Delta y \Delta x \Delta z$$
$$+ \left[\frac{(\tau_{zx})_{z+\Delta z} - (\tau_{zx})_z}{\Delta z} \right] \Delta z \Delta x \Delta y - \left\{ \left[\frac{(p)_{x+\Delta x} - (p)_x}{\Delta x} \right] \Delta x \Delta y \Delta z \right\}$$

$$\frac{dF_{cx}}{dV} = \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] - \frac{\partial p}{\partial x}$$

$$dV = \Delta z \Delta x \Delta y$$

$$dF_{bx} = \rho(\Delta x \Delta y \Delta z) g_x$$

$$\frac{dF_{bx}}{dV} = \rho g_x$$

$$(\rho dV) \frac{Dv_x}{Dt} = dF_{bx} + dF_{cx}$$

2ª lei de Newton – Forças de campo e de contato

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + \vec{v} \cdot \text{grad}\vec{v}$$

$$(\rho dV)\vec{a} = (\rho dV)\frac{D\vec{v}}{Dt} = (\rho dV)\frac{\partial\vec{v}}{\partial t} + (\rho dV)\vec{v} \cdot \text{grad}\vec{v}$$

$$(\rho dV)\frac{Dv_x}{Dt} = (\rho dV)\left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z\right]$$

$$(\rho dV)\frac{Dv_x}{Dt} = dF_{bx} + dF_{cx}$$

2ª lei de Newton – Forças de campo e de contato

$$(\rho dV) \frac{Dv_x}{Dt} = dF_{bx} + dF_{cx}$$

$$\rho \frac{Dv_x}{Dt} = \frac{dF_{bx}}{dV} + \frac{dF_{cx}}{dV}$$

$$\frac{dF_{cx}}{dV} = \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] - \frac{\partial p}{\partial x}$$

$$\frac{dF_{bx}}{dV} = \rho g_x$$

$$\begin{aligned} \rho \frac{Dv_x}{Dt} &= \rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] = \\ &= \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] - \frac{\partial p}{\partial x} + \rho g_x \end{aligned}$$

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \text{grad } p + \text{div } \vec{\tau}$$

Fluido Newtoniano: Tensão x deformação angular

$$\vec{\sigma} = -p \vec{\delta} + \vec{\tau} \left\{ \begin{array}{l} \text{tensões normais: } \sigma_{ij} = -p + \tau_{ij} \\ \text{tensões de cisalhamento: } \sigma_{ij} = \tau_{ij} \end{array} \right.$$

$\dot{\gamma}$: deformação angular

$$\dot{\gamma} = \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) & 2 \frac{\partial v_y}{\partial y} & \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) & \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) & 2 \frac{\partial v_z}{\partial z} \end{pmatrix} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T,$$

Fluido Newtoniano: Tensão x deformação angular

τ_{ij}

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu \nabla \cdot \mathbf{v},$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu \nabla \cdot \mathbf{v},$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu \nabla \cdot \mathbf{v}.$$

τ_{ij}

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right),$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right),$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right).$$

σ_{ij}

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu \nabla \cdot \mathbf{v},$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu \nabla \cdot \mathbf{v},$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu \nabla \cdot \mathbf{v}.$$

Fluido Newtoniano: Tensão x deformação angular

τ_{ij}

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} - \frac{2\mu}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]$$

$$\tau_{yy} = 2\mu \frac{\partial v_y}{\partial y} - \frac{2\mu}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]$$

$$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} - \frac{2\mu}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]$$

Caso de escoamento incompressível:

$$\text{div } \vec{v} = 0$$

τ_{ij}

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right]$$

$$\tau_{yz} = \tau_{zy} = \mu \left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right]$$

$$\tau_{xz} = \tau_{zx} = \mu \left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right]$$

$$\rho \frac{Dv_x}{Dt} = \rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] =$$

$$= \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\tau_{xx} = 2\mu \frac{\partial v_x}{\partial x} - \frac{2\mu}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]$$

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right]$$

$$\tau_{xz} = \tau_{zx} = \mu \left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right]$$

$$\rho \frac{Dv_x}{Dt} = \rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] =$$

$$\mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \frac{\mu}{3} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

incompressível: $\text{div } \vec{v} = 0$

NAVIER - STOKES

$$\rho \frac{Dv_x}{Dt} = \rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] =$$

$$\mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \frac{\mu}{3} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

$$\text{div } \vec{v} = 0$$

Escoamento
incompressível,
fluido newtoniano e
viscosidade constante

$$\rho \frac{Dv_x}{Dt} = \rho \left[\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y + \frac{\partial v_x}{\partial z} v_z \right] =$$

$$\mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \text{grad } p + \text{div } \vec{\tau}$$

NAVIER - STOKES

$$\rho \frac{D\vec{v}}{Dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \text{grad } \vec{v} = \rho \vec{g} - \text{grad } p + \mu \text{lap } \vec{v}$$

QUANTIDADE DE MOVIMENTO E NAVIER - STOKES

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x,$$
$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y,$$
$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z.$$

NAVIER - STOKES

Escoamento incompressível, newtoniano e viscosidade constante

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x,$$
$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y,$$
$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z.$$

QUANTIDADE DE MOVIMENTO – Coord. Cilíndricas

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} + \rho g_r,$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho g_\theta,$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z.$$

NAVIER – STOKES : Coord. Cilíndricas

Escoamento incompressível, newtoniano e viscosidade constante

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r,$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho g_\theta,$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z.$$