# The Concavity/Convexity Sufficient Condition

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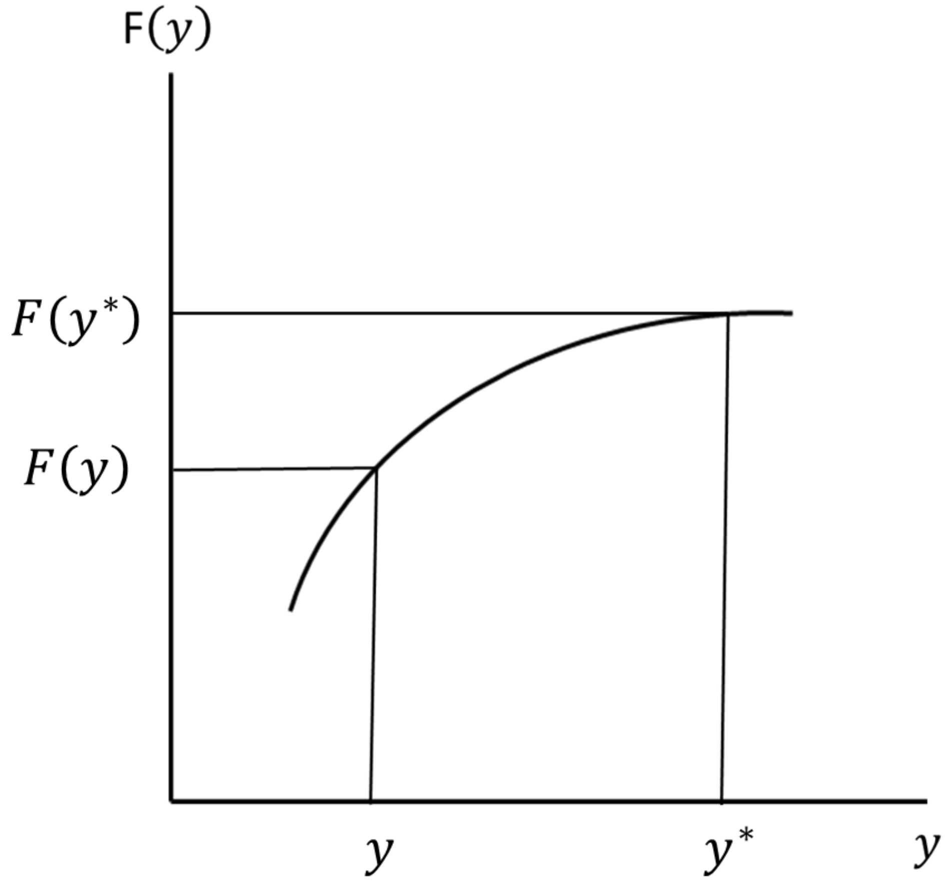
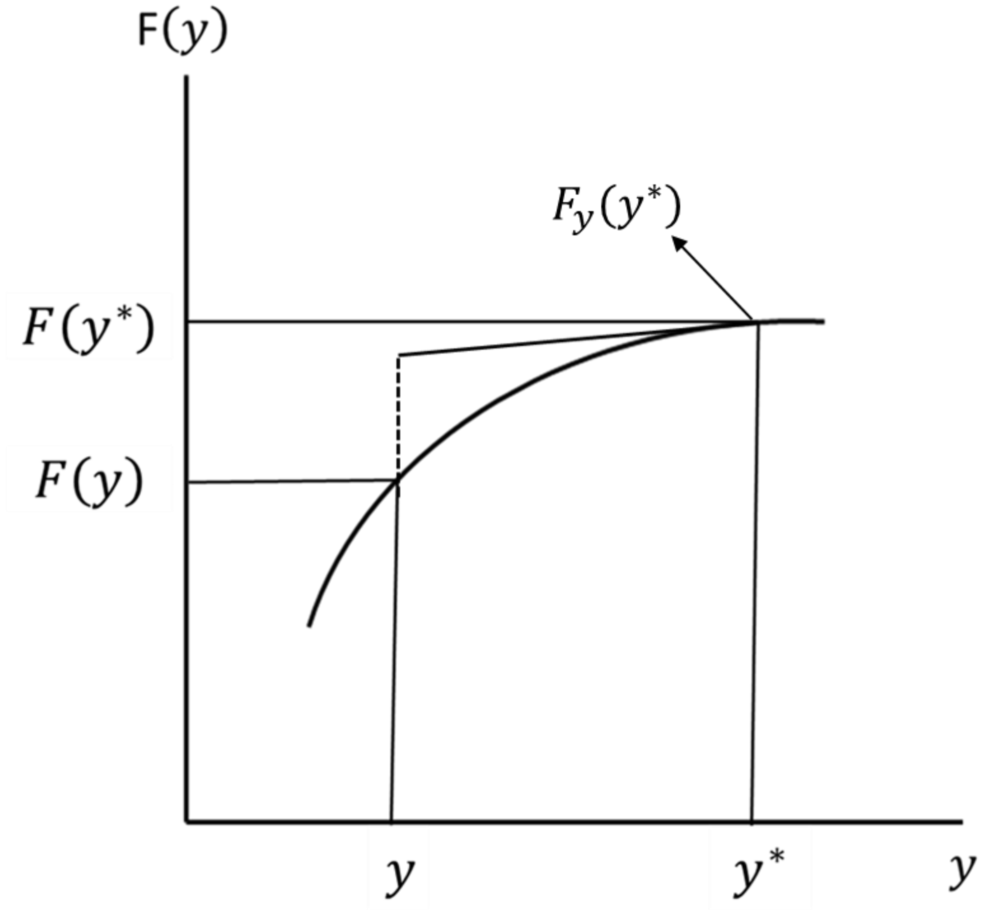
# The Concavity/Convexity Sufficient Condition

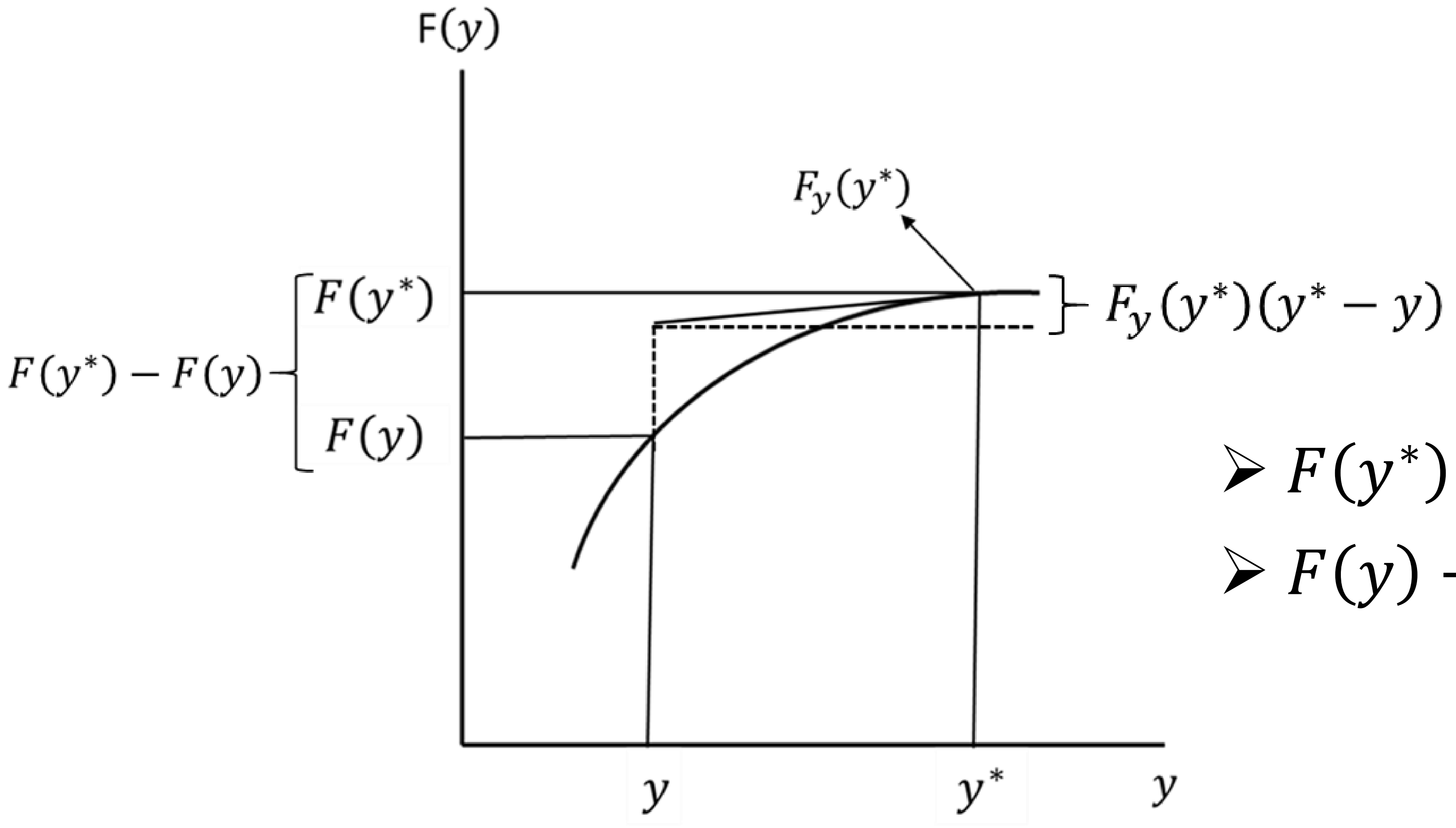
- Just as a concave (convex) objective function in a static optimization problem is sufficient to identify an extremum as an absolute maximum (minimum), a similar sufficiency theorem holds in the calculus of variations.
- For the fixed-endpoint problem Maximize ou minimize $\mathrm{V}[y] = \int_0^T F[t, y(t), y'(t)]dt$, if the integrand function $F(t, y, y')$ is concave in $(y, y')$, then the Euler equation is sufficient for an absolute maximum of $\mathrm{V}[y]$.
- Similarly, if $F(t, y, y')$ is convex in $(y, y')$, then the Euler equation is sufficient for an absolute minimum of $\mathrm{V}[y]$.

# Concavity



# Concavity




- $F(y^*) - F(y) \geq F_y(y^*)[y^* - y]$
- $F(y) - F(y^*) \leq F_y(y^*)[y - y^*]$

# The Concavity/Convexity Sufficient Condition

- It should be pointed out that concavity/convexity in $(y, y')$ means concavity/convexity in the two variables $y(t)$ and $y'(t)$ jointly, not in each variable separately.
- The function $F(t, y, y')$ is concave in $(y, y')$ if, and only if, for any pair of distinct points in the domain, $\left(t, y^*, y^{*'}\right)$ and $(t, y, y')$, we have

(5) $F(t, y, y') - F\left(t, y^*, y^{*'}\right) \leq F_y\left(t, y^*, y^{*'}\right)(y - y^*) + F_{y'}\left(t, y^*, y^{*'}\right)\left(y' - y^{*'}\right)$

- Since $y(t) = y^*(t) + \epsilon p(t)$, and $y'(t) = y^{*\prime}(t) + \epsilon p'(t)$, equation (5) may be expressed as:

(5’)

$$F(t, y, y') - F\left(t, y^*, y^{*'}\right) \leq F_y\left(t, y^*, y^{*'}\right)\epsilon p(t) + F_{y'}\left(t, y^*, y^{*'}\right)\epsilon p'(t)$$

# The Concavity/Convexity Sufficient Condition

- Here $y^*(t)$ denotes the optimal path, and $y(t)$ denotes any other path. By integrating both sides of (5') with respect to $t$ over the interval $[0, T]$, we obtain:

$$\int_0^T \left[F(t,y,y') - F\left(t,y^*,{y^*}'\right)\right] dt \leq \int_0^T \left[ F_y\left(t,y^*,{y^*}'\right)\epsilon p(t) + F_{y'}\left(t,y^*,{y^*}'\right)\epsilon p'(t)\right] dt$$

(6) $\quad \mathrm{V}[y] - \mathrm{V}[y^*] \leq \epsilon \int_0^T \left[ F_y\left(t,y^*,{y^*}'\right)p(t) + F_{y'}\left(t,y^*,{y^*}'\right)p'(t)\right] dt$

- We have already seen that

$$\int_0^T F_{y'} p'(t) dt = \left[F_{y'} p(t)\right]_0^T - \int_0^T p(t) \frac{d}{dt} F_{y'} dt \quad \text{(integration by parts).}$$

- Therefore, $\int_0^T \left[F_{y'} p'(t)\right] dt = -\int_0^T \left[p(t) \frac{d}{dt} F_{y'}\right] dt$ when $p(0) = p(T) = 0$, and

(6') $\quad \mathrm{V}[y] - \mathrm{V}[y^*] \leq \epsilon \int_0^T p(t) \left[ \underbrace{F_y\left(t,y^*,{y^*}'\right) - \frac{d}{dt} F_{y'}\left(t,y^*,{y^*}'\right)}_{Euler\ Equation} \right] dt = 0$

# The Concavity/Convexity Sufficient Condition

- Equation (6’) is equal to zero $(= 0)$ since $y^*(t)$ satisfies the Euler equation $\left(F_y - \frac{d}{dt}F_{y'} = 0\right)$.
- In other words, $\mathbf{V}[\boldsymbol{y}] \leq \mathbf{V}[\boldsymbol{y}^*]$, where $y(t)$ can refer to any other path.
- We have thus identified $y^*(t)$ as a $\mathrm{V}$-maximizing path, and at the same time demonstrated that the **Euler equation is a sufficient condition, given the assumption of a concave $\boldsymbol{F}$ function**.
- The opposite case of a convex $F$ function for minimizing V can be proved analogously.
- If the $\boldsymbol{F}$ function is **strictly concave** in $(y, y')$, then the weak inequality $(\leq)$ will become **the strict inequality** $(<)$.

# Generalization to Variable Terminal Point

- The proof above is based on the assumption of **fixed endpoints**. But it can easily be generalized to problems with a vertical terminal line or truncated vertical terminal line.
- Recall that the integration-by-parts process:

$$\int_0^T F_{y'} p'(t)dt = \left[F_{y'} p(t)\right]_0^T - \int_0^T p(t) \frac{d}{dt} F_{y'} dt$$

- originally produced an **extra term** $\left[\boldsymbol{F_{y'} p(t)}\right]_{\boldsymbol{0}}^{\boldsymbol{T}}$ which later **drops out** because it reduces to zero because $\boldsymbol{p(0) = p(T) = 0}$.
- When we switch to the problem with a **variable terminal point**, with $T$ fixed but $y(T)$ free, $\boldsymbol{p(T)}$ **is no longer required to be zero**.

# Generalization to Variable Terminal Point

- Multiplying the above equation by $\epsilon$ [see equation (6)]:
- $\epsilon \int_0^T F_{y'} p'(t) dt = \epsilon \left[ F_{y'} p(t) \right]_0^T - \epsilon \int_0^T p(t) \frac{d}{dt} F_{y'} dt$
- For this reason, we must admit an extra term [recall that $y(t) = y^*(t) + \epsilon p(t)$]:

  (7) $\epsilon \left[ F_{y'} p(t) \right]_0^T = \epsilon \left[ F_{y'} p(t) \right]_{t=T} = \left[ F_{y'} (y - y^*) \right]_{t=T}$

  on the right-hand side of the second and the third lines of (6’).
- Therefore:

  (8) $\epsilon \int_0^T F_{y'} p'(t) dt = \left[ F_{y'} (y - y^*) \right]_{t=T} - \epsilon \int_0^T p(t) \frac{d}{dt} F_{y'} dt$

# Generalization to Variable Terminal Point

- Using (8) in (6):

(9) $$\mathrm{V}[y]-\mathrm{V}[y^*] \leq \epsilon \int_0^T p(t) \underbrace{\left[ F_y\left(t, y^*, {y^*}'\right) - \frac{d}{dt} F_{y'} \right]}_{Euler\ Equation} dt + \left[F_{y'}(y-y^*)\right]_{t=T}$$

- Considering Euler Equation is valid, (9) now becomes

(10) $$\mathrm{V}[y]-\mathrm{V}[y^*] \leq \left[F_{y'}(y-y^*)\right]_{t=T}$$

- where $F_{y'}$ is to be evaluated along the optimal path, and $(y-y^*)$ represents the deviation of any admissible neighboring path $y(t)$ from the optimal path $y^*(t)$.

# Generalization to Variable Terminal Point

- If the last term in the last inequality is zero, then obviously the original conclusion-that $\mathrm{V}[y^*]$ is an absolute maximum-still stands.
- It is only when $\left[F_{y'}(y - y^*)\right]_{t=T}$ is positive that we are thrown into doubt.
- In short, the concavity condition on $F(t, y, y^*)$ only needs to be supplemented in the present case by a no positivity condition on the expression $\left[F_{y'}(y - y^*)\right]_{t=T}$.
- But this supplementary condition is automatically met when the transversality condition is satisfied for the vertical-terminal-line problem: $\left[F_{y'}\right]_{t=T} = 0.$

# Generalization to Variable Terminal Point

- As for the truncated case, the transversality condition calls for either $\left[F_{y'}\right]_{t=T} = 0$ (when $y_{min}$ is nonbinding), or $y^* = y_{min}$ (when that terminal value is binding, thereby in effect turning the problem into one with a fixed terminal point).
- Either way, the supplementary condition is met.
- Thus, if the integrand function $F$ is concave (convex) in the variables $(y, y')$ in a problem with a vertical terminal line or a truncated vertical terminal line, then the Euler equation plus the transversality condition are sufficient for an absolute maximum (minimum) of $V[y]$.