Specialized Transversality Conditions

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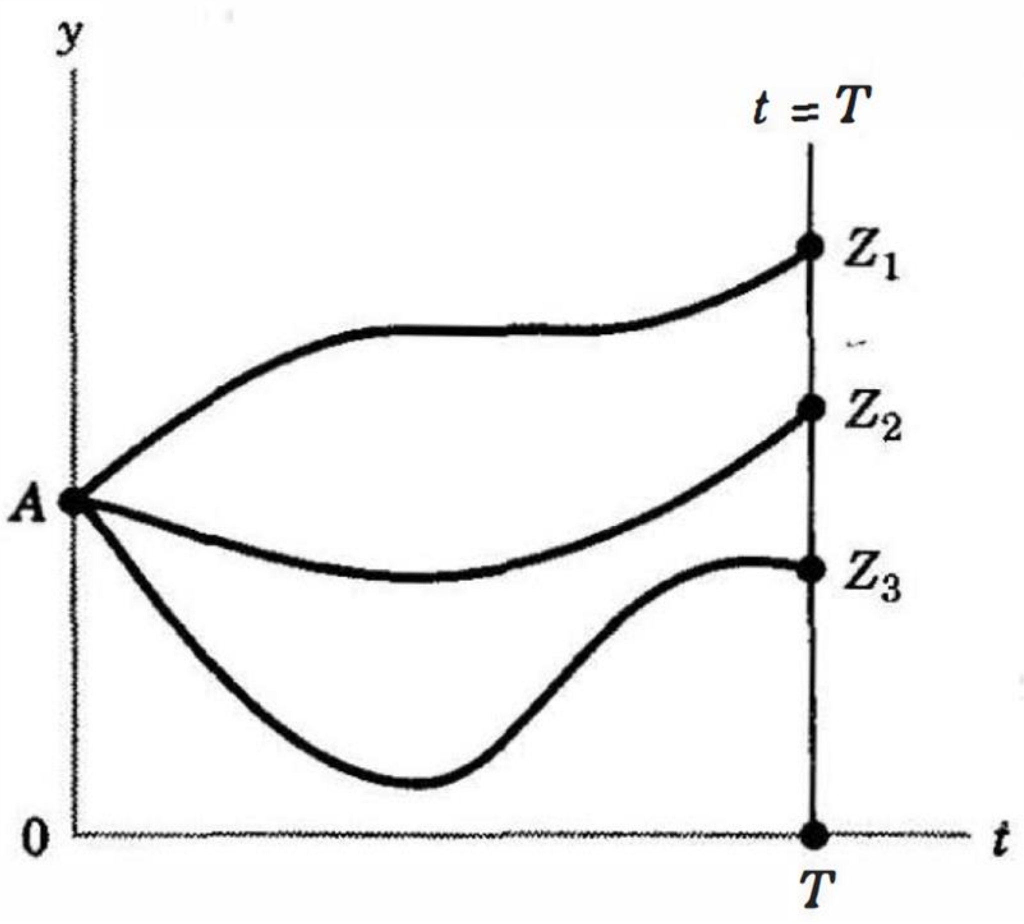
Vertical Terminal Line

(15’) $\left[F - y'F_{y'}\right]_{t=T}\Delta T + \left[F_{y'}\right]_{t=T}\Delta y_T = 0$

- This condition, unlike the Euler equation, is relevant only to one point of time, T.
- The vertical-terminal-line case involves a fixed T. Thus $\Delta T = 0$, and the first term in (15’) drops out.
- But since Δy_T is arbitrary and can take either sign, the only way to make the second term in (15’) vanish for sure is to have:

(16) $\left[F_{y'}\right]_{t=T} = 0.$

Figure 4a – Vertical terminal-line problem.

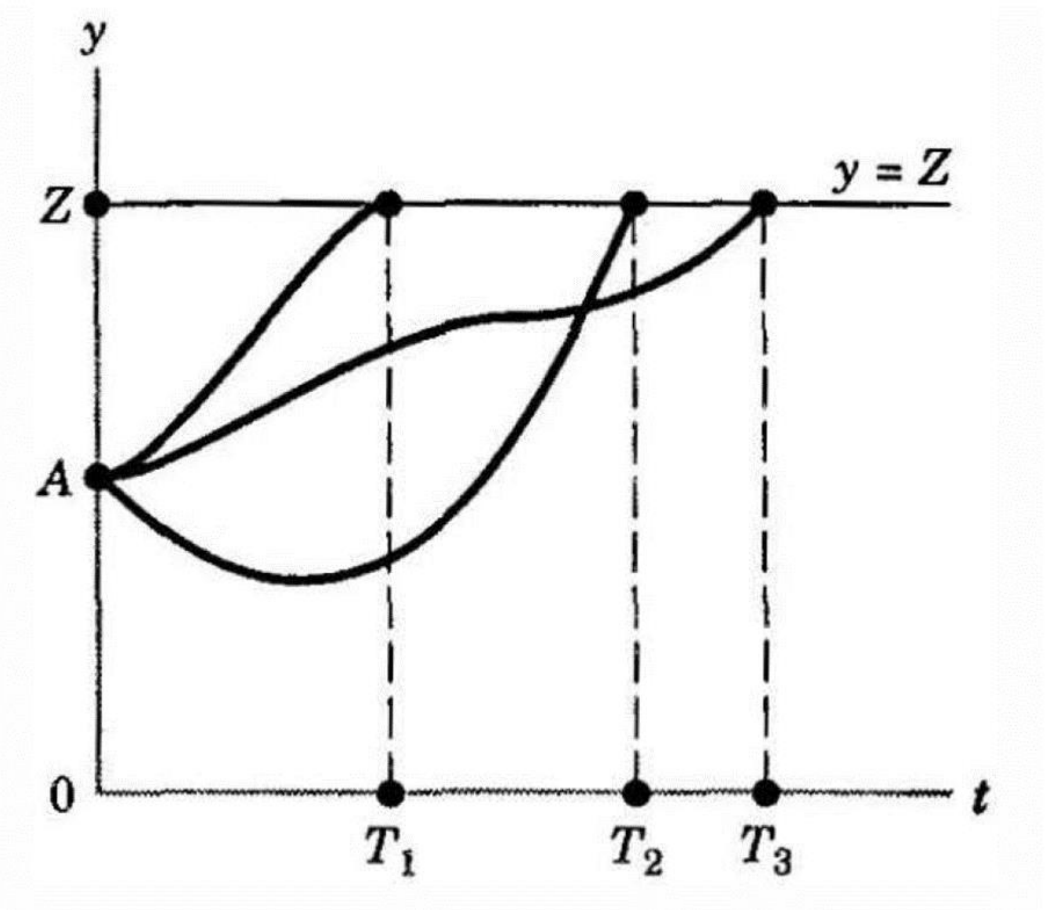


Horizontal Terminal Line

- For the horizontal-terminal-line case the situation is reversed.
- We now have $\Delta y_T = 0$ but ΔT is arbitrary. So the second term in (3.15') automatically drops out, but the first does not.
- Since ΔT is arbitrary, the only way to make the first term vanish for sure is to have the bracketed expression equal to zero.
- Thus the transversality condition is:

(17) $\left[F - y'F_{y'}\right]_{t=T} = 0$

Figure 4b – Horizontal terminal-line problem.



Specialized Transversality Conditions

- To fix ideas, let us interpret $F[t, y(t), y'(t)]$ as a profit function, where y represents capital stock, and y' represents net investment.
- Net investment entails taking resources away from the current profit-making business operation, so as to build up capital which will enhance future profit.
- Hence, there exists a tradeoff between current profit and future profit.
- At any time t, with a given capital stock y, a specific investment decision, a decision to select the investment rate y_0', will result in the current profit $F[t, y(t), y_0'(t)]$.

Specialized Transversality Conditions

- The imputed (or shadow) value to the firm of a unit of capital is measured by the derivative $F_{y'}$.
- This means that if we decide to leave (not use up) a unit of capital at the terminal time, it will entail a negative value equal to $-F_{y'}$.
- Thus, at $t = T$, the value measure of y_0' is $y_0' F_{y_0'}$.
- Accordingly, the overall profit implication of the decision to choose the investment rate y_0' is $\left[F(t, y, y_0') - y_0' F_{y_0'}\right]$.
- The general expression for this is $F - y' F_{y'}$, as in (17).

Specialized Transversality Conditions

- Now we can interpret the transversality condition (17) to mean that, in a problem with a free terminal time, the firm should select a T such that a decision to invest and accumulate capital will, at $t = T$, no longer yield any overall (current and future) profit.
- In other words, all the profit opportunities should have been fully taken advantage of by the optimally chosen terminal time.
- In addition, (16), which can equivalently be written as $\left[-F_{y'}\right]_{t=T} = 0$, instructs the firm to avoid any sacrifice of profit that will be incurred by leaving a positive terminal capital.
- In other words, in a free-terminal-state problem, in order to maximize profit in the interval $[0, T]$ but not beyond, the firm should, at time T, use up all the capital it ever accumulated.

Terminal Curve

- With a terminal curve $y_T = \phi(T)$, neither Δy_T nor ΔT is assigned a zero value, so neither term in (15’) drops out.
- However, for a small arbitrary ΔT, the terminal curve implies that $\Delta y_T = \phi' \Delta T$. So it is possible to eliminate Δy_T in (15’) and combine the two terms into the form:

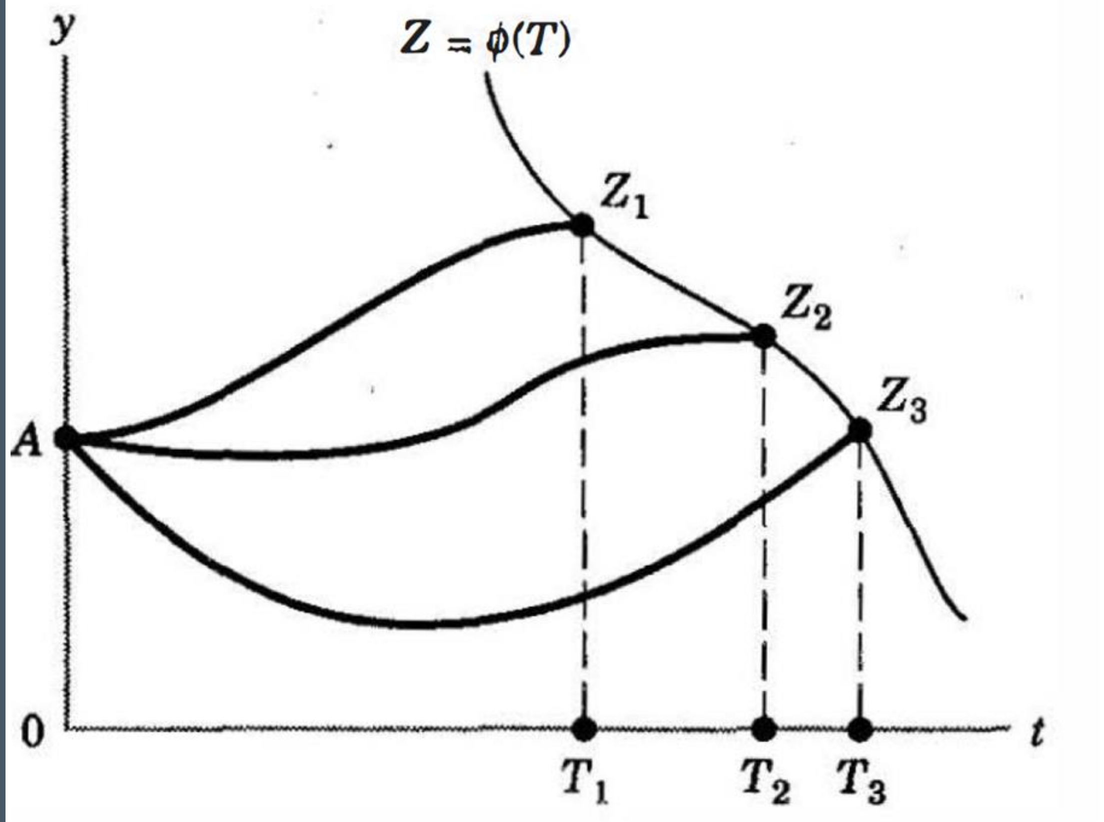
(15’) $\left[F - y'F_{y'}\right]_{t=T} \Delta T + \left[F_{y'}\right]_{t=T} \Delta y_T = 0$

(18) $\left[F - y'F_{y'} + F_{y'}\phi'\right]_{t=T} \Delta T = 0$

- Since ΔT is arbitrary, the transversality condition is:

(19) $\left[F + (\phi' - y')F_{y'}\right]_{t=T} = 0$

Figure 4c – Terminal curve problem.



Truncated Vertical Terminal Line

- The usual case of vertical terminal line, with $\Delta T = 0$, specializes (15’) to

(20) $\left[F_{y'}\right]_{t=T} \Delta y_T = 0$

- When the line is truncated, restricted by the terminal condition $y_T \geq y_{min}$, where y_{min} is a minimum permissible level of y, the optimal solution can have two possible types of outcome:

$$y_T^* > y_{min} \quad \text{or} \quad y_T^* = y_{min}$$

- **If $\boldsymbol{y_T^* > y_{min}}$**, the terminal restriction is automatically satisfied; that is, it is nonbinding. Thus, the transversality condition is in that event the same as (16):

(21) $\left[F_{y'}\right]_{t=T} = 0 \qquad \text{for} \quad y_T^* > y_{min}$

Truncated Vertical Terminal Line

- The other outcome, $y_T^* = y_{min}$, on the other hand, only admits neighboring paths with terminal values $y_T^* \geq y_{min}$.
- This means that $\Delta y_T = y_T - y_T^*$ is no longer completely arbitrary (positive or negative), but is restricted to be nonnegative.
- Assuming the perturbing curve $[y(t) = y^*(t) + \epsilon p(t)]$ to have terminal value $p(T) > 0$, $\Delta y_T \geq 0$ would mean that $\epsilon \geq 0$. The nonnegativity of ϵ means that the transversality condition (20), which has its roots in the first-order condition $dV/d\epsilon = 0$ must be changed to an inequality as in the Kuhn-Tucker conditions, and (20) should become:

  (22) $\left[F_{y'}\right]_{t=T} \Delta y_T \leq 0$

Truncated Vertical Terminal Line

- And since $\Delta y_T \geq 0$, (22) implies condition

(23) $\left[F_{y'}\right]_{t=T} \leq 0 \qquad \text{for} \quad y_T^* = y_{min}$

- Combining (21) and (23), we may write the following summary statement of the transversality condition for a maximization problem:

(24) $\left[F_{y'}\right]_{t=T} \leq 0 \qquad y_T^* \geq y_{min} \qquad (y_T^* - y_{min})\left[F_{y'}\right]_{t=T} = 0$

- [for maximization of V]
- If the problem is instead to minimize V, then the inequality sign in (22) must be reversed, and the transversality condition becomes

(25) $\left[F_{y'}\right]_{t=T} \geq 0 \qquad y_T^* \geq y_{min} \qquad (y_T^* - y_{min})\left[F_{y'}\right]_{t=T} = 0$

Truncated Horizontal Terminal Line

- The horizontal terminal line may be truncated by the restriction $T \leq T_{max}$, where T_{max} represents a maximum permissible time for completing a task deadline.
- The analysis of such a situation is very similar to the truncated vertical terminal line just discussed. By analogous reasoning, we can derive the following transversality condition for a maximization problem:

(26) $\left[F - y'F_{y'}\right]_{t=T} \geq 0 \qquad T^* \leq T_{max} \qquad (T^* - T_{max})\left[F - y'F_{y'}\right]_{t=T} = 0$

- [for maximization of V]
- If the problem is t o minimize V, the first inequality in (26) must be changed, and the transversality condition is

(27) $\left[F - y'F_{y'}\right]_{t=T} \leq 0 \qquad T^* \leq T_{max} \qquad (T^* - T_{max})\left[F - y'F_{y'}\right]_{t=T} = 0$

- [for maximization of V]

Second-Order Conditions

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Second-Order Conditions

- Our discussion has so far concentrated on the identification of the extremal(s) of a problem, without attention to whether they maximize or minimize the functional $V[y]$.
- This involves checking the second-order conditions.
- To distinguish between maximization and minimization problems, we can take the second derivative $d^2V/d\epsilon^2$, and use the following standard second-order necessary conditions in calculus:

$\frac{d^2V}{d\epsilon^2} \leq 0$ for a maximization of $V[y]$

$\frac{d^2V}{d\epsilon^2} \geq 0$ for a minimization of $V[y]$

Second-Order Conditions

- Second-order sufficient conditions:

$$\frac{d^2V}{d\epsilon^2} < 0 \qquad \text{for a maximization of } V[y]$$

$$\frac{d^2V}{d\epsilon^2} > 0 \qquad \text{for a minimization of } V[y]$$

- To find $d^2V/d\epsilon^2$, we differentiate $dV/d\epsilon$ with respect to ϵ, bearing in mind that:

1. all the partial derivatives of $F(t, y, y')$ are, like F itself, functions of t, y and y';
2. y and y' are, in turn, both functions of ϵ.

Second-Order Derivative of V

- Remember that:

$$V[\epsilon] = \int_0^T F\left[t, \underbrace{y^*(t) + \epsilon p(t)}_{y(t)}, \underbrace{y^{*\prime}(t) + \epsilon p'(t)}_{y'(t)}\right] dt$$

$$\frac{dV}{d\epsilon} = \int_0^T \frac{dF}{d\epsilon} dt = \int_0^T \left(\frac{\partial F}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial F}{\partial y'}\frac{dy'}{d\epsilon}\right) dt = \int_0^T \left[F_y p(t) + F_{y'} p'(t)\right] dt$$

- Therefore:

(1) $\frac{dy}{d\epsilon} = p(t)$ and $\frac{dy'}{d\epsilon} = p'(t)$

- Thus, we have:

(2) $\frac{d^2V}{d\epsilon^2} = \frac{d}{d\epsilon}\left(\frac{dV}{d\epsilon}\right) = \frac{d}{d\epsilon}\int_0^T \left[F_y p(t) + F_{y'} p'(t)\right] dt$

Second-Order Derivative of V

(2’) $\frac{d^2V}{d\epsilon^2} = \int_0^T \left[p(t)\frac{d}{d\epsilon}F_y + p'(t)\frac{d}{d\epsilon}F_{y'}\right]dt$

[by Leibniz 's rule]

- In view of the fact that

(3) $\frac{d}{d\epsilon}F_y = F_{yy}\frac{dy}{d\epsilon} + F_{y'y}\frac{dy'}{d\epsilon} = F_{yy}p(t) + F_{y'y}p'(t)$

(3’) $\frac{d}{d\epsilon}F_{y'} = F_{yy'}\frac{dy}{d\epsilon} + F_{y'y'}\frac{dy'}{d\epsilon} = F_{yy'}p(t) + F_{y'y'}p'(t)$

- the second derivative (2’) emerges as

(4) $\frac{d^2V}{d\epsilon^2} = \int_0^T \big[F_{yy}p(t)p(t) + F_{y'y}p(t)p'(t) + F_{yy'}p(t)p'(t) +$
$F_{y'y'}p'(t)p'(t)\big]dt$

Second-Order Derivative of V

(4') $\frac{d^2V}{d\epsilon^2} = \int_0^T \left[F_{yy}p^2(t) + 2F_{yy'}p(t)p'(t) + F_{y'y'}p'^2(t)\right]dt$

- if it can be established that the quadratic form, with F_{yy}, $F_{yy'}$ and $F_{y'y'}$ evaluated on the extremal, is negative definite for every t, then $d^2V/d\epsilon^2 < 0$, and the extremal maximizes V.
- Similarly, positive definiteness of the quadratic form for every t is sufficient for minimization of V.
- Even if we can only establish sign semidefiniteness, we can at least have the second-order necessary conditions checked.