The nature of dynamic optimization

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Dynamic optimization problem

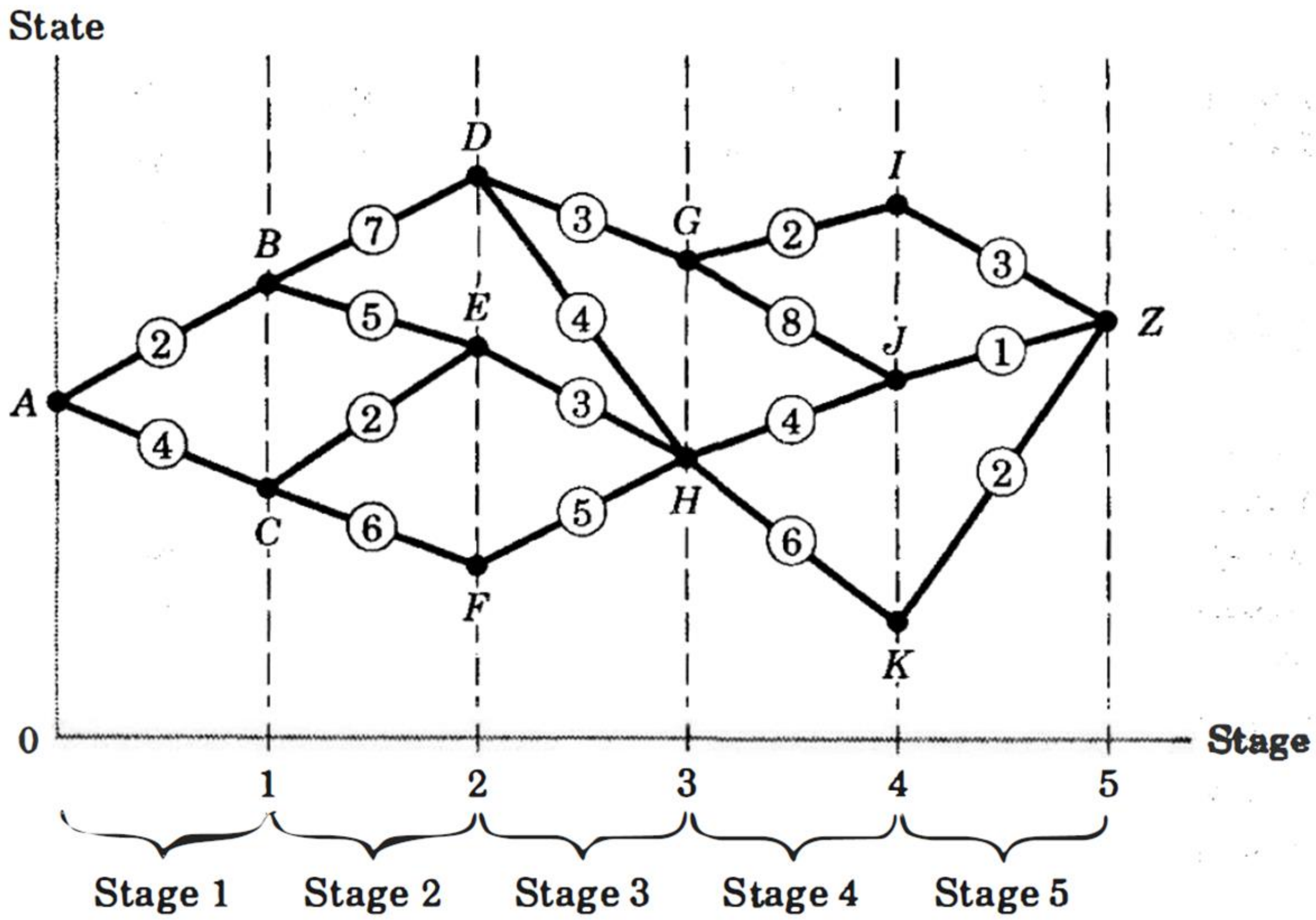
- Optimization is a predominant theme in economic analysis.
- Useful as they are, such tools are applicable only to static optimization problems
- The solution sought in such problems usually consists of a single optimal magnitude for every choice variable.
- In contrast, a dynamic optimization problem poses the question of what is the optimal magnitude of a choice variable in each period of time within the planning period (discrete-time case) or at each point of time in a given time interval, say [0, T] (continuous-time case).

Dynamic optimization problem

- Although dynamic optimization is mostly couched in terms of a sequence of time, it is also possible to envisage the planning horizon as a sequence of stages in an economic process.
- In that case, dynamic optimization can be viewed as a problem of multistage decision making. The distinguishing feature, however, remains the fact that the optimal solution would involve more than one single value for the choice variable.
- Suppose that a firm engages in transforming a certain substance from an initial state A (raw material state) into a terminal state Z (finished product state) through a five-stage production process.
- In every stage, the firm faces the problem of choosing among several possible alternative subprocesses, each entailing a specific cost. The question is: How should the firm select the sequence of subprocesses through the five stages in order to minimize the total cost?

Figure 1 - Cost minimization

In Figure 1, we illustrate such a problem by plotting the stages horizontally and the states vertically.
The initial state A is shown by the leftmost point (at the beginning of state 1); the terminal state Z is shown by the rightmost point (at the end of stage 5).
The other points B, C, . . . , K show the various intermediate states into which the substance may be transformed during the process.



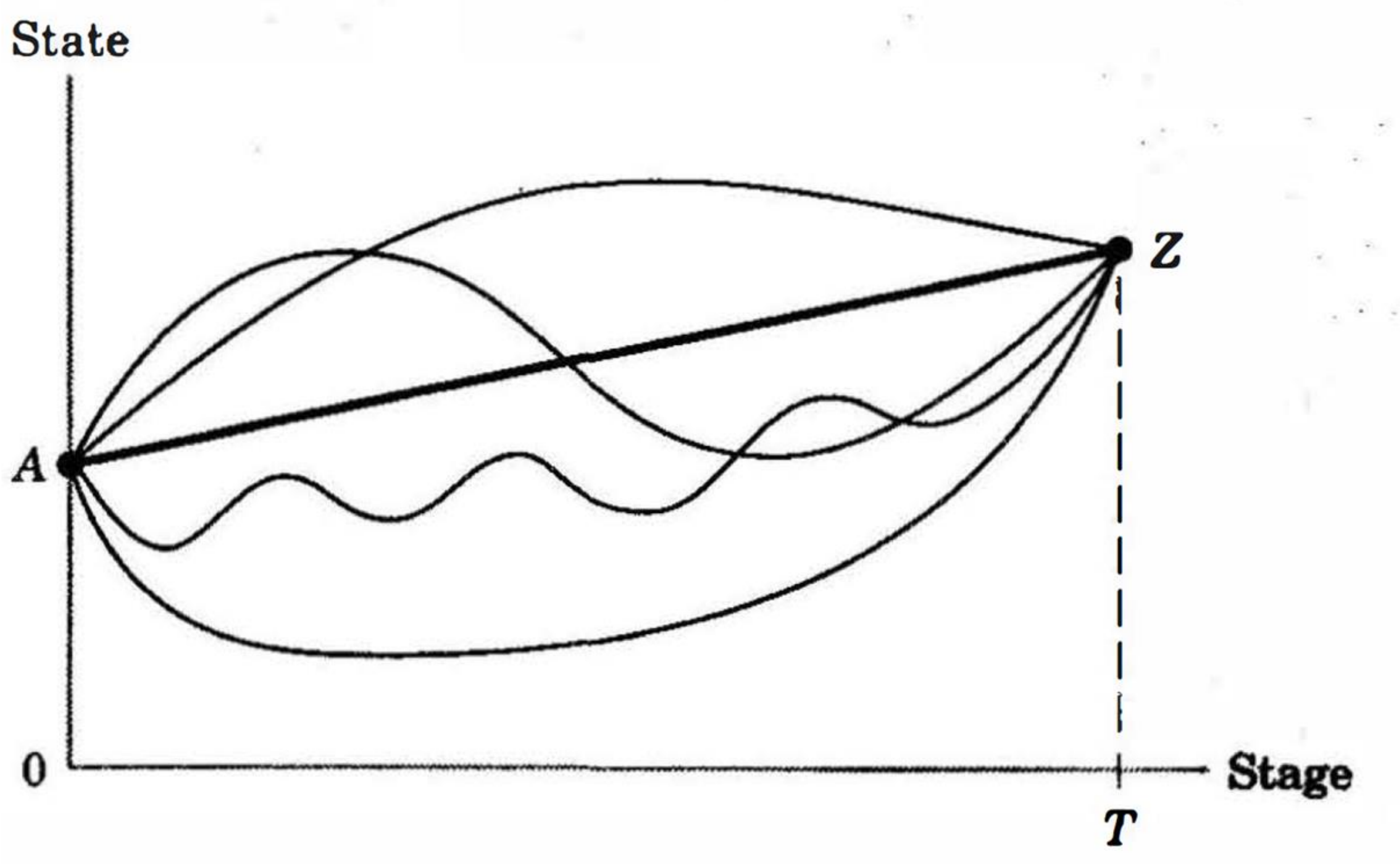
Cost minimization problem

- Our problem is to choose a connected sequence of arcs going from left to right, starting at A and terminating at Z, such that the sum of the values of the component arcs is minimized. Such a sequence of arcs will constitute an optimal path.
- A myopic, one-stage-at-a-time optimization procedure will not in general yield the optimal path. For example, a myopic decision maker would have chosen arc AB over arc AC in the first stage, because the former involves only half the cost of the latter; yet, over the span of five stages, the more costly first-stage arc AC should be selected instead.
- The example in Fig. 1 is simple enough so that a solution may be found by enumerating all the admissible paths from A to Z and picking the one with the least total arc values. For more complicated problems, however, a systematic method of attack is needed.

Figure 2 – The continous variable version

In Figure 2, each possible path is now seen to travel through an infinite number of stages in the interval [0, T]. There is also an infinite number of states on each path, each state being the result of a particular choice made in a specific stage.

For concreteness, let us visualize Fig. 1.2 to be a map of an open terrain, with the stage variable representing the longitude, and the state variable representing the latitude. Our assigned task is to transport a load of cargo from location A to location Z at minimum cost by selecting an appropriate travel path.



The continous variable version

- For most of the problems discussed in the following, the stage variable will represent time; then the curves in Fig. 2 will depict time paths.
- As a concrete example, consider a firm with an initial capital stock equal to A at time 0, and a predetermined target capital stock equal to Z at time T.
- Each investment plan implies a specific capital path and entails a specific potential profit for the firm. In this case, we can interpret the curves in Fig. 2 as possible capital paths and their path values as the corresponding profits.
- The problem of the firm is to identify the investment plan, hence the capital path-that yields the maximum potential profit.

The basic elements

- a simple type of dynamic optimization problem would contain the following basic ingredients:

1. a given initial point and a given terminal point;
2. a set of admissible paths from the initial point to the terminal point;
3. a set of path values serving as performance indices (cost, profit, etc.) associated with the various paths; and
4. a specified objective-either to maximize or to minimize the path value or performance index by choosing the optimal path.

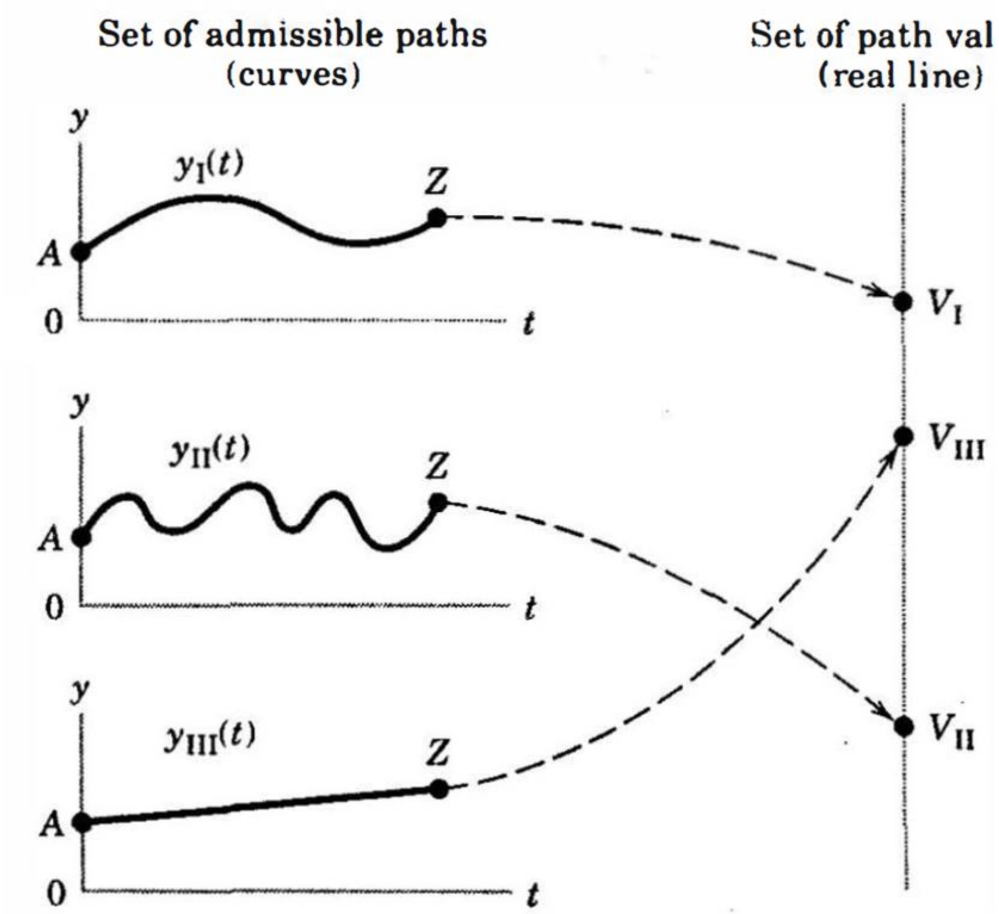
The concept of a functional

- A functional is a mapping from paths (curves) to real numbers (performance indices). Let us think of the paths in question as time paths, and denote them by $y_I(t)$, $y_{II}(t)$, and so on.
- Then the mapping is as shown in Fig. 3, where V_I, V_{II} represent the associated path values. The general notation for the mapping should therefore be $V[y(t)]$. In the symbol $V[y(t)]$ the $y(t)$ component comes as an integral unit - to indicate time paths - and therefore we should not take V to be a function of t. Instead, V should be understood to be a function of "$y(t)$".
- This type of mapping is given a distinct name: functional.
- To further avoid confusion, many writers omit the "(t)“ part of the symbol and write the functional as $V[y]$.

Example of a functional

- When the symbol y is used to indicate a certain state, it is suffixed, and appears as, say, $y(0)$ for the initial state or $y(T)$ for the terminal state.
- In contrast, in the path connotation, the t in $y(t)$ is not assigned a specific value.
- When we want to stress the specific time interval involved in a path or a segment thereof, we shall use the notation $y[0, T]$ or $y[0, \tau)$.
- The optimal time path is then denoted by $y^*(t)$, or the y^* path.

Figure 3 – Examples of functional



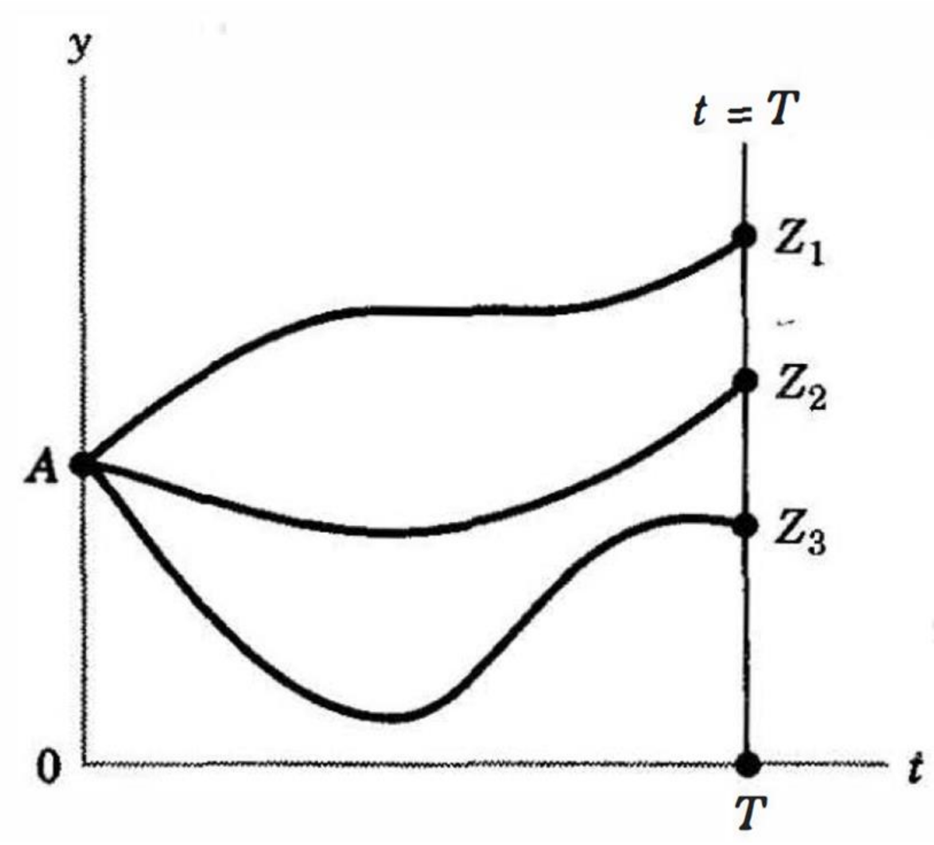
VARIABLE ENDPOINTS AND TRANSVERSALITY CONDITIONS

- In the usual problem, the optimizing plan must start from some specific initial position, say, the current position.
- The terminal position, on the other hand, may very well turn out to be a flexible matter, with no inherent need for it to be predetermined.
- We may, for instance, face only a fixed terminal time, but have complete freedom to choose the terminal state.
- On the other hand, we may also be assigned a rigidly specified terminal state, but are free to select the terminal time.
- In such a case, the terminal point becomes a part of the optimal choice.
- We shall take the stage variable to be continuous time.
- We shall also retain the symbols 0 and T for the initial time and terminal time.
- The symbols A and Z for the initial and terminal states.

Vertical terminal-line problem

- As the first type of variable terminal point, we may be given a fixed terminal time T, but a free terminal state.
- In Fig. 4a, while the planning horizon is fixed at time T, any point on the vertical line $\mathrm{t} = T$ is acceptable as a terminal point, such as Z_1, Z_2, and Z_3.
- This type of problem is commonly referred to as a fixed-time-horizon problem, or fixed-time problem.
- Alternatively, we may refer to the fixed-time problem as the vertical-terminal-line problem.

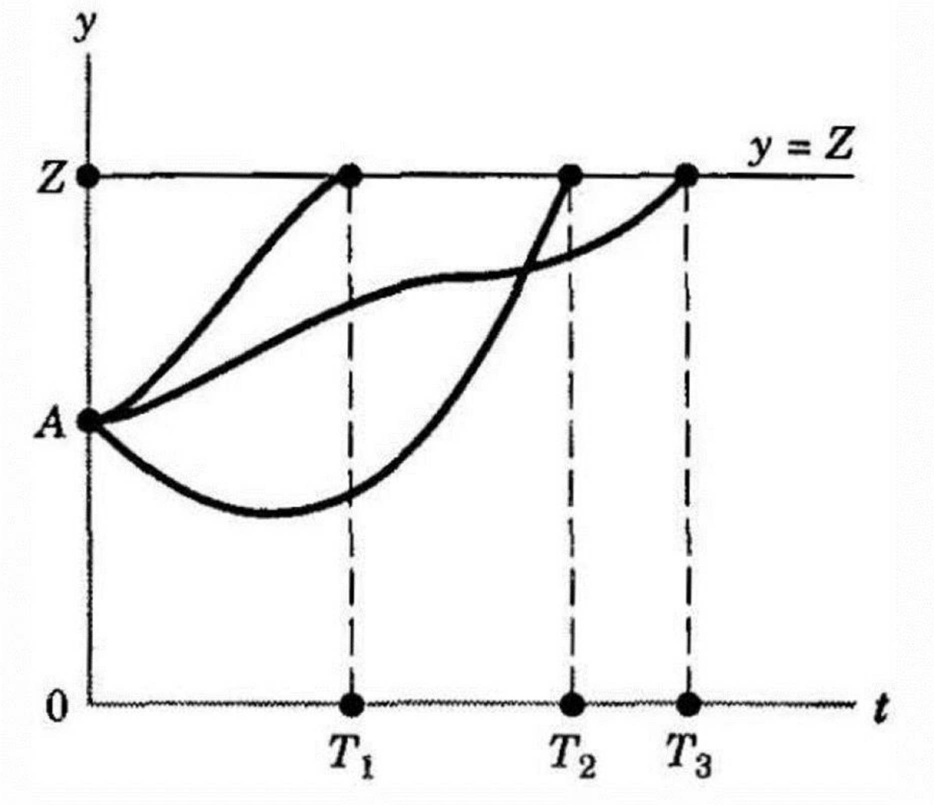
Figure 4a – Vertical terminal-line problem.



Horizontal terminal-line problem.

- The second type of variable terminal point occurs when the terminal state Z is stipulated, but the terminal time is free.
- In Fig. 4b, the horizontal line $y = Z$ constitutes the set of admissible terminal points. Each of these, depending on the path chosen, may be associated with a different terminal time, as exemplified by T_1, T_2, and T_3.
- This type of problem is commonly referred to as a fixed-endpoint problem.
- Alternatively, we may refer to the fixed-endpoint problem as the horizontal-terminal-line problem.

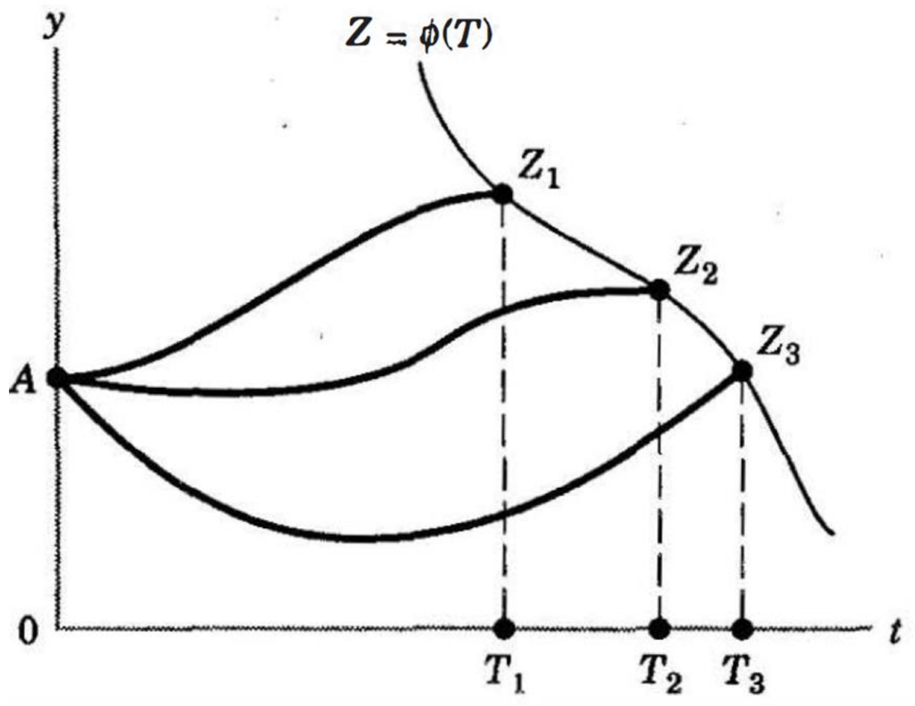
Figure 4b – Horizontal terminal-line problem.



Terminal curve problem

- In the third type of variable terminal point problem, neither the terminal time T nor the terminal state Z is individually preset, but the two are tied together via a constraint equation of the form $Z = \phi(T)$.
- In Fig. 4c, such an equation plots as a terminal curve (or, in higher dimension, a terminal surface) that associates a particular terminal time (say, T_1) with a corresponding terminal state (say, Z_1).
- Even though the problem leaves both T and Z flexible, the planner actually has only one degree of freedom in the choice of the terminal point.
- This type of problem is commonly referred to as terminal-curve (or terminal-surface) problem .

Figure 4c – Terminal curve problem



Transversality Condition

- The common feature of variable-terminal-point problems is that the planner has one more degree of freedom than in the fixed-terminal-point case.
- But this fact automatically implies that, in deriving the optimal solution, an extra condition is needed to pinpoint the exact path chosen.
- Compare the boundary conditions for the optimal path in the fixed- versus the variable-terminal-point cases. In the former, the optimal path must satisfy the boundary (initial and terminal) conditions:

(1) $\mathrm{y}(0) = A$ and $\mathrm{y}(T) = Z$; $(T$, A, and Z all given)

Transversality Condition

- In the variable-terminal-point case, the initial condition $y(0) = A$ still applies by assumption. But since T and Z are now variable, the terminal condition $y(T) = Z$ is no longer capable of pinpointing the optimal path for us.
- What is needed, therefore, is a terminal condition that can conclusively distinguish the optimal path from the other admissible paths.
- Such a condition is referred to as a transversality condition, because it normally appears as a description of how the optimal path crosses the terminal line or the terminal curve.

The integral form of functional

- Figure 1 suggests that three pieces of information are needed for arc identification:

1. the starting stage (time);
2. the starting state;
3. the direction in which the arc proceeds.

- With continuous time, since each arc is infinitesimal in length, these three items are represented by, respectively:

1. t;
2. $y(t)$;
3. $y'(t) = \frac{dy}{dt}$

- For instance, on a given path y_I, the arc associated with a specific point of time t_0 is characterized by a unique value $y_I(t_0)$ and a unique slope $y_I'(t_0)$. If there exists some function, F, that assigns a arc values to $arcs$, then the value of the said arc can be written as:

(2) $F[t_0, y_I(t_0), y_I'(t_0)]$

The integral form of functional

- It follows that the general expression for arc values is $F[t, y(t), y'(t)]$, and the path-value functional-the sum of arc values-can generally be written as the definite integral:

(3) $\mathrm{V}[y] = \int_0^T F[t, y(t), y'(t)]dt$

- The symbol $V[y]$ emphasizes that it is the variation in the y path (y_I versus y_{II}) that alters the magnitude of V. Each different y path consists of a different set of $arcs$ in the time interval $[0, T]$, which, through the $arc\text{-}value\text{-}assigning$ function F, takes a different set of arc values.
- The definite integral sums those arc values on each y path into a path value.

The integral form of functional

- If there are two state variables, y and z, in the problem, the arc values on both the y and z paths must be taken into account.
- The objective functional should then appear as:

(4) $\mathrm{V}[y,z]=\int_0^T F[t,y(t),z(t),y'(t),z'(t)]dt$

- A problem with an objective functional in the form of (3) or (4) constitutes the standard problem.
- For simplicity, we shall often suppress the time argument (t) for the state variables and write the integrand function more concisely as $F(t,y,y')$ or $F(t,y,z,y',z')$ F(t, y, z, y', z').

A macroeconomic example

- Let the social welfare of an economy at any time be measured by the utility from consumption, $U = U(C)$.
- Consumption is by definition that portion of output not saved (and not invested). If we adopt the production function $\mathrm{Q} = Q(K, L)$, and assume away depreciation, we can then write:

(5) $\quad \mathrm{C} = Q(K, L) - I = Q(K, L) - K'$

- where $K' \equiv I$ denotes net investment. This implies that the utility function can be rewritten as:

(6) $\quad U = U[Q(K, L) - K']$

A macroeconomic example

- If the societal goal is to maximize the sum of utility over a period $[0, T]$, then its objective functional takes the form:

  (7) $\mathrm{V}[y, z] = \int_0^T U[Q(K, L) - K']dt$

- This exemplifies the functional in (4), where the two state variables y and z refer in the present example to K and L, respectively.
- Note that while the integrand function of this example does contain both K and K' as arguments, the L variable appears only in its natural form unaccompanied by K'. Moreover, the t argument is absent from the F function, too.

The Calculus of Variations

- The usual problem can be represented by the following general formulation:

$$\text{Maximize ou minimize} \quad V[y] = \int_0^T F[t, y(t), y'(t)]dt$$

$$\text{(8)} \quad \text{Subject to} \quad y(0) = A \qquad (A\ given)$$

$$\text{and} \quad y(T) = Z \qquad (T\ and\ Z\ given)$$

- Such a problem, with an integral functional in a single state variable, with completely specified initial and terminal points, and with no constraints, is known as the fundamental problem (or simplest problem) of calculus of variations.

The Calculus of Variations

- In order to make such problems meaningful, it is necessary that the functional be integrable (i.e., the integral must be convergent).
- We shall assume this condition is met whenever we write an integral of the general form, as in (8).
- Furthermore, we shall assume that all the functions that appear in the problem are continuous and continuously differentiable.
- This assumption is needed because the basic methodology underlying the calculus of variations closely parallels that of the classical differential calculus.
- The main difference is that, instead of dealing with the differential dx that changes the value of $y = f(x)$, we will now deal with the "variation" of an entire curve $\mathrm{y}(t)$ that affects the value of the functional $V[y]$.