MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

Prof. Dr. Luis Carlos de Castro Santos

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO

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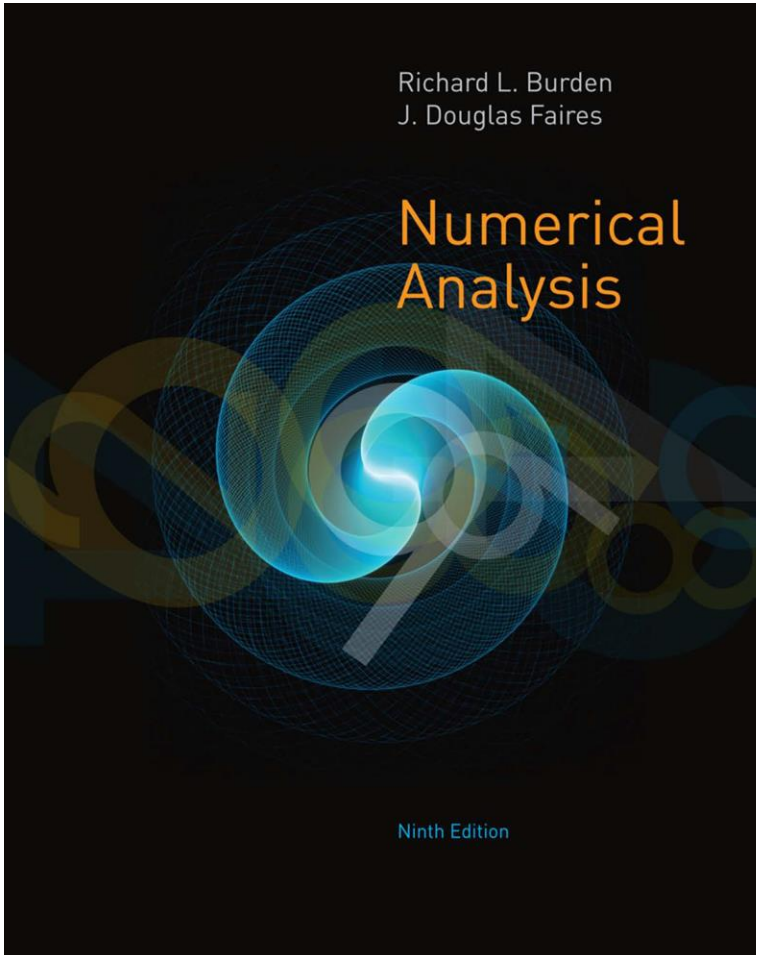


Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

7.2 Eigenvalues and Eigenvectors

Definition 7.12

If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

Definition 7.13

If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ.

2.6 Zeros of Polynomials and Müller's Method

A *polynomial of degree n* has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i's, called the *coefficients* of P, are constants and $a_n \neq 0$. The zero function, $P(x) = 0$ for all values of x, is considered a polynomial but is assigned no degree.

Theorem 2.16

(Fundamental Theorem of Algebra)

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then $P(x) = 0$ has at least one (possibly complex) root. ■

Corollary 2.17

If $P(x)$ is a polynomial of degree $n \geq 1$ with real or complex coefficients, then there exist unique constants $x_1, x_2, \ldots, x_k$, possibly complex, and unique positive integers $m_1, m_2, \ldots, m_k$, such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n(x - x_1)^{m_1}(x - x_2)^{m_2} \cdots (x - x_k)^{m_k}. \qquad \blacksquare$$

By Corollary 2.17 the collection of zeros of a polynomial is unique and, if each zero x_i is counted as many times as its multiplicity m_i, a polynomial of degree n has exactly n zeros.

Corollary 2.18

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$ for $i = 1, 2, \ldots, k$, then $P(x) = Q(x)$ for all values of x. $\blacksquare$

Horner's Method

To use Newton's method to locate approximate zeros of a polynomial $P(x)$, we need to evaluate $P(x)$ and $P'(x)$ at specified values. Since $P(x)$ and $P'(x)$ are both polynomials, computational efficiency requires that the evaluation of these functions be done in the nested manner discussed in Section 1.2. Horner's method incorporates this nesting technique, and, as a consequence, requires only n multiplications and n additions to evaluate an arbitrary nth-degree polynomial.

Polynomials and Nested Brackets

Polynomials can be rewritten using brackets within brackets. This is known as nested form.

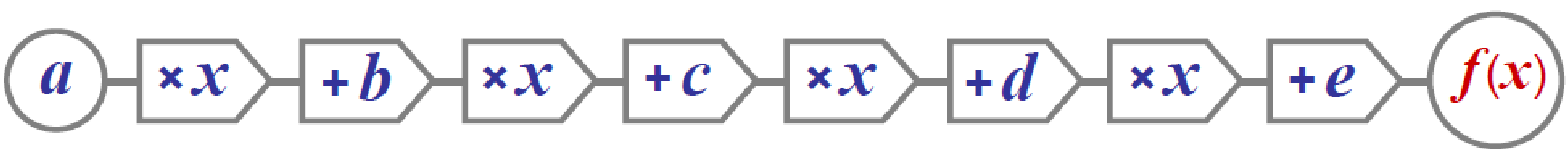
Example

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$= (ax^3 + bx^2 + cx + d)x + e$$

$$= ((ax^2 + bx + c)x + d)x + e$$

$$= (((ax + b)x + c)x + d)x + e$$



Theorem 2.19 (Horner's Method)

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Define $b_n = a_n$ and

$$b_k = a_k + b_{k+1} x_0, \quad \text{for } k = n-1, n-2, \ldots, 1, 0.$$

Then $b_0 = P(x_0)$. Moreover, if

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1,$$

then

$$P(x) = (x - x_0) Q(x) + b_0. \qquad \blacksquare$$

Proof By the definition of $Q(x)$,

$$\begin{aligned}(x-x_0)Q(x)+b_0 &= (x-x_0)(b_n x^{n-1}+\cdots+b_2x+b_1)+b_0\\ &= (b_nx^n+b_{n-1}x^{n-1}+\cdots+b_2x^2+b_1x)\\ &\quad -(b_nx_0x^{n-1}+\cdots+b_2x_0x+b_1x_0)+b_0\\ &= b_nx^n+(b_{n-1}-b_nx_0)x^{n-1}+\cdots+(b_1-b_2x_0)x+(b_0-b_1x_0).\end{aligned}$$

By the hypothesis, $b_n=a_n$ and $b_k-b_{k+1}x_0=a_k$, so

$$(x-x_0)Q(x)+b_0=P(x) \quad \text{and} \quad b_0=P(x_0). \qquad \blacksquare\ \blacksquare\ \blacksquare$$

Example 2

Use Horner's method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

Solution When we use hand calculation in Horner's method, we first construct a table, which suggests the *synthetic division* name that is often applied to the technique. For this problem, the table appears as follows:

	Coefficient of x^4	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant term
$x_0 = -2$	$a_4 = 2$	$a_3 = 0$	$a_2 = -3$	$a_1 = 3$	$a_0 = -4$
		$b_4x_0 = -4$	$b_3x_0 = 8$	$b_2x_0 = -10$	$b_1x_0 = 14$
	$b_4 = 2$	$b_3 = -4$	$b_2 = 5$	$b_1 = -7$	$b_0 = 10$

So,

$$P(x) = (x + 2)(2x^3 - 4x^2 + 5x - 7) + 10.$$

$b_n = a_n$ and

$$b_k = a_k + b_{k+1}x_0, \quad \text{for } k = n - 1, n - 2, \ldots, 1, 0.$$

An additional advantage of using the Horner (or synthetic-division) procedure is that, since

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1,$$

differentiating with respect to x gives

$$P'(x) = Q(x) + (x - x_0)Q'(x) \quad \text{and} \quad P'(x_0) = Q(x_0). \tag{2.16}$$

When the Newton-Raphson method is being used to find an approximate zero of a polynomial, $P(x)$ and $P'(x)$ can be evaluated in the same manner.

2.3 Newton's Method and Its Extensions

Newton's (or the *Newton-Raphson*) **method** is one of the most powerful and well-known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's method.

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \tag{2.7}$$

Example 3

Find an approximation to a zero of

$$P(x) = 2x^4 - 3x^2 + 3x - 4,$$

using Newton's method with $x_0 = -2$ and synthetic division to evaluate $P(x_n)$ and $P'(x_n)$ for each iterate x_n.

Solution With $x_0 = -2$ as an initial approximation, we obtained $P(-2)$ in Example 1 by

$$\begin{array}{r|rrrrrl} x_0 = -2 & 2 & 0 & -3 & 3 & -4 & \\ & & -4 & 8 & -10 & 14 & \\ \hline & 2 & -4 & 5 & -7 & 10 & = P(-2). \end{array}$$

Using Theorem 2.19 and Eq. (2.16),

$$Q(x) = 2x^3 - 4x^2 + 5x - 7 \quad \text{and} \quad P'(-2) = Q(-2),$$

so $P'(-2)$ can be found by evaluating $Q(-2)$ in a similar manner:

$$\begin{array}{r|rrrrl} x_0 = -2 & 2 & -4 & 5 & -7 & \\ & & -4 & 16 & -42 & \\ \hline & 2 & -8 & 21 & -49 & = Q(-2) = P'(-2) \end{array}$$

and

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = x_0 - \frac{P(x_0)}{Q(x_0)} = -2 - \frac{10}{-49} \approx -1.796.$$

Repeating the procedure to find x_2 gives

$$
\begin{array}{r|rrrrrr}
-1.796 & 2 & 0 & -3 & 3 & -4 & \\
 & & -3.592 & 6.451 & -6.197 & 5.742 & \\
\hline
 & 2 & -3.592 & 3.451 & -3.197 & 1.742 & = P(x_1) \\
 & & -3.592 & 12.902 & -29.368 & & \\
\hline
 & 2 & -7.184 & 16.353 & -32.565 & = Q(x_1) & = P'(x_1).
\end{array}
$$

So $P(-1.796) = 1.742$, $P'(-1.796) = Q(-1.796) = -32.565$, and

$$x_2 = -1.796 - \frac{1.742}{-32.565} \approx -1.7425.$$

In a similar manner, $x_3 = -1.73897$, and an actual zero to five decimal places is -1.73896.

Note that the polynomial $Q(x)$ depends on the approximation being used and changes from iterate to iterate. ■

ALGORITHM
2.7

Horner's

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at x_0:

INPUT degree n; coefficients $a_0, a_1, \ldots, a_n$; x_0.

OUTPUT $y = P(x_0)$; $z = P'(x_0)$.

Step 1 Set $y = a_n$; (*Compute b_n for P.*)
$z = a_n$. (*Compute b_{n-1} for Q.*)

Step 2 For $j = n-1, n-2, \ldots, 1$
set $y = x_0 y + a_j$; (*Compute b_j for P.*)
$z = x_0 z + y$. (*Compute b_{j-1} for Q.*)

Step 3 Set $y = x_0 y + a_0$. (*Compute b_0 for P.*)

Step 4 OUTPUT (y, z);
STOP.

O output é acoplado ao método de Newton

EXERCISE SET 2.6

1. Find the approximations to within 10^{-4} to all the real zeros of the following polynomials using Newton's method.

   a. $f(x) = x^3 - 2x^2 - 5$

   b. $f(x) = x^3 + 3x^2 - 1$

   c. $f(x) = x^3 - x - 1$

   d. $f(x) = x^4 + 2x^2 - x - 3$

   e. $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$

   f. $f(x) = x^5 - x^4 + 2x^3 - 3x^2 + x - 4$

Considere implementar em Python

Fim...

AULA 14