• Representation Theory (Groups)

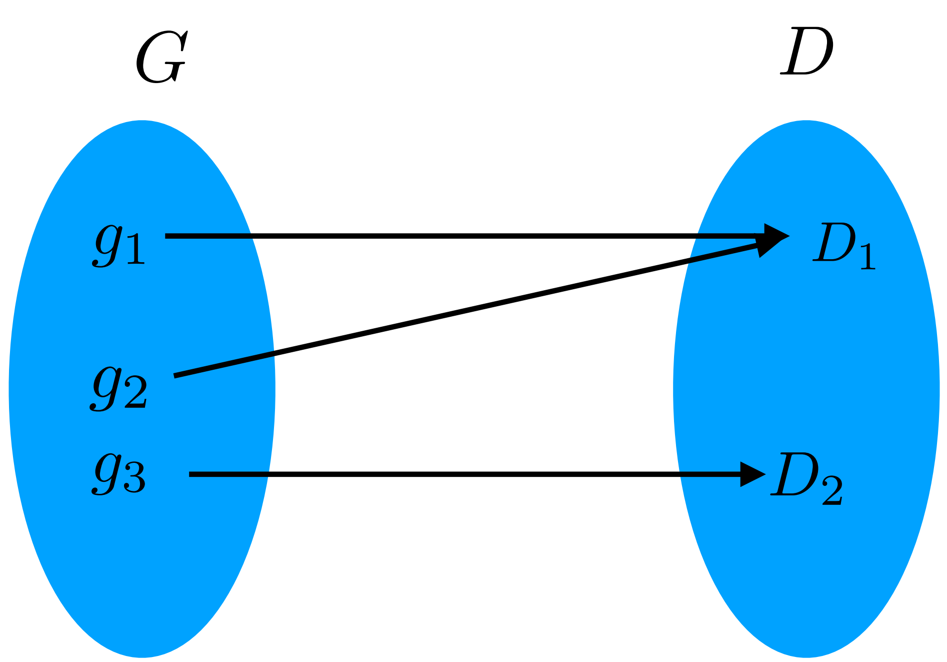
We need:

A vector space V, with vectors $\mid v\rangle$

Operators D acting on V

$$D : \qquad V \to V \qquad\qquad D \mid v\rangle = \mid v'\rangle$$

An homomorphism $G \to D$



$$D\,(g)\;D\,(g')\mid v\rangle = D\,(g\,g')\;\mid v\rangle$$

$$\text{for all } \mid v\rangle \in V$$

- Representation Theory (Lie Algebras)

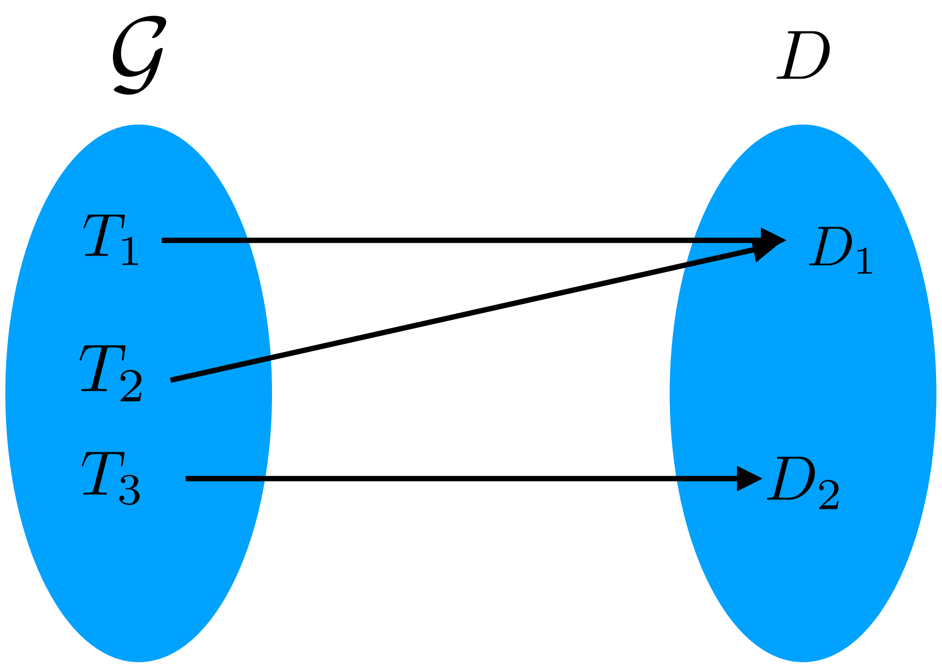
We need:

A vector space V, with vectors $\mid v\rangle$

Operators D acting on V

$$D: \qquad V \to V \qquad\qquad D \mid v\rangle = \mid v'\rangle$$

An homomorphism $\mathcal{G} \to D$



$$\left(D\left(T\right)\, D\left(T'\right) - D\left(T'\right)\, D\left(T\right)\right) \mid v\rangle = D\left(\left[T\, , \, T'\right]\right)\, \mid v\rangle$$

for all $\mid v\rangle \in V$

• Two Theorems

Theorem 3.1 *A finite dimensional representation of a compact Lie group is equivalent to a unitary one.*

Theorem 3.2 *A unitary representation can be decomposed into unitary irreducible representations.*

3.2 The notion of weights

We have defined in section 2.6 (see definition 2.12) the Cartan subalgebra of a semisimple Lie algebra as the maximal abelian subalgebra wich can be diagonalized simultaneously. Therefore we can take the basis of the representation space V as the eigenstates of the Cartan subalgebra generators. Then we have

$$H_i \mid \mu\rangle = \mu_i \mid \mu\rangle \qquad\qquad i = 1, 2, 3 ... r(\text{rank}) \tag{3.1}$$

The eigenvalues of the Cartan subalgebra generators constitute r-component vectors and they are called *weights*. Like the roots, the weights live in a r-dimensional Euclidean space. There can be more than one base state associated to a single weight. So the base states can be degenerated.

In section 2.8 we have seen that the operator $H_\alpha = 2\alpha \cdot H/\alpha^2$, has integer eigenvalues. Therefore from (3.1) we have

$$H_\alpha \mid \mu\rangle = \frac{2\alpha \cdot \mu}{\alpha^2} \mid \mu\rangle \tag{3.2}$$

and consenquently we have that

$$\frac{2\alpha \cdot \mu}{\alpha^2} \qquad \text{is an integer for any root } \alpha \tag{3.3}$$

Any vector μ satisfying this condition is a weight, and in fact this is the only condition a weight has to satisfy. From (2.148) we see that any root is a weight but the converse is not true. Notice that $\frac{2\alpha\cdot\mu}{\mu^2}$ does not have to be an integer and therefore the table 2.2 does not apply to the weights.

A weight is called *dominant* if it lies in the Fundamental Weyl Chamber or on its borders. Obviously a dominant weight has a non negative scalar product with any positive root. It is possible to find among the dominant weights, r weights λ_a, $a = 1, 2...r$, satisfying

$$\frac{2\lambda_a \cdot \alpha_b}{\alpha_b^2} = \delta_{ab} \qquad \text{for any simple root } \alpha_b \tag{3.4}$$

In orther words we can find r dominant weights which are orthogonal to all simple roots except one. These weights are called *fundamental weights*. They play an important role in representation theory as we will see below.

Consider now a simple root α_a and any weight μ. From (3.3) we have that

$$\frac{2\mu \cdot \alpha_a}{\alpha_a^2} = m_a = \text{integer} \tag{3.5}$$

Using (3.4) we have

$$\frac{2\alpha_a}{\alpha_a^2} \cdot \left(\mu - \sum_{a=1}^{r} m_a \lambda_a\right) = 0 \tag{3.6}$$

Since the simple roots constitute a basis of an r-dimensional Euclidean space we conclude that

$$\mu = \sum_{a=1}^{r} m_a \lambda_a \tag{3.7}$$

Therefore any weight can be written as a linear combination of the fundamental weights with integer coefficients. We now want to show that any vector formed by an integer linear combination of the fundamental weights is also a weight, i.e., it satisfies the condition (3.3). In order to do that we introduce the concept of *co-root* , which is a root devided by its squared lenght

$$\alpha^v \equiv \frac{\alpha}{\alpha^2} \tag{3.8}$$

Since

$$(\alpha^v)^2 = \frac{1}{\alpha^2} \tag{3.9}$$

and

$$\frac{2\alpha^v \cdot \beta^v}{(\alpha^v)^2} = \frac{2\alpha \cdot \beta}{\beta^2} \tag{3.10}$$

one sees that the co-roots satisfy all the properties of roots and consequently are also roots. However the co-roots of a given algebra $\mathcal{G}$ are the roots of another algebra $\mathcal{G}^v$, called the dual algebra to $\mathcal{G}$. The simply laced algebras, $su(N)$ (A_{N_1}), $so(2N)$ (D_N), E_6 , E_7 and E_8, together with the exceptional algebras G_2 and F_4 are self-dual algebras, in the sense that $\mathcal{G} = \mathcal{G}^v$. However $so(2N+1)$ (B_N) is the dual algebra to $sp(N)$ (C_N) and vice versa. The Cartan matrix of the dual algebra $\mathcal{G}^v$ is the transpose of the Cartan matrix of $\mathcal{G}$ since

$$(K_{ab})^v = \frac{2\alpha_a^v \cdot \alpha_b^v}{(\alpha_b^v)^2} = \frac{2\alpha_a \cdot \alpha_b}{\alpha_a^2} = K_{ba} \tag{3.11}$$

where we have used the fact that the simple co-roots are given by

$$\alpha_a^v = \frac{\alpha_a}{\alpha_a^2} \tag{3.12}$$

Any co-root can be written as a linear combination of the simple co-roots with integer coefficients all of the same sign. To show that we observe from theorem 2.7 that

$$\alpha^v = \frac{\alpha}{\alpha^2} = \sum_{a=1}^{r} n_a \frac{\alpha_a^2}{\alpha^2} \alpha_a^v \tag{3.13}$$

and from (3.4) we get

$$n_a = \frac{2\lambda_a \cdot \alpha}{\alpha_a^2} \tag{3.14}$$

Therefore

$$\alpha^v = \sum_{a=1}^{r} \frac{2\lambda_a \cdot \alpha}{\alpha^2} \alpha_a^v \equiv \sum_{a=1}^{r} m_a \alpha_a^v \tag{3.15}$$

since from (3.3) we have that $\frac{2\lambda_a \cdot \alpha}{\alpha^2}$ is an integer. In additon these integers are all of the same sign since all λ_a's lie on the Fundamental Weyl Chamber or on its border.

Let ν be a vector defined by

$$\nu = \sum_{a=1}^{r} k_a \lambda_a \tag{3.16}$$

where λ_a are the fundamental weights and k_a are arbitrary integers. Using (3.15) and (3.4) we get

$$\frac{2\alpha \cdot \nu}{\alpha^2} = 2\alpha^v \cdot \nu = \sum_{a,b} m_a k_b \frac{2\lambda_b \cdot \alpha_a}{\alpha_a^2} = \sum_{a} m_a k_a \tag{3.17}$$

Therefore ν is a weight. So we have shown that any integer linear combination of the fundamental weights is a weigtht and that all weights are of this form. Consequently the weights constitute a lattice Λ called the *weight lattice*. This quantized spectra of weights is a consequence of the fact that H_α has integer eigenvalues and is an important feature of representation theory of compact Lie algebras.

As we have said any root is a weight and consequently belong to Λ. We can also form a lattice by taking all vectors which are integer linear combinations of the simple roots. This lattice is called the *root lattice* and is denoted by Λ_r . All points in Λ_r are weights and therefore Λ_r is a sublattice of Λ. The weight lattice forms an abelian group under the addition of vectors. The root lattice is an invariant subgroup and consequently the coset space Λ/Λ_r has the structure of a group (see section 1.4). One can show that Λ/Λ_r corresponds to the center of the covering group corresponding to the algebra which weight lattice is Λ. We will show that all the weights of a given irreducible representation of a compact Lie algebra lie in the same coset.

Before giving some examples we would like to discuss the relation between the simple roots and the fundamental weights, which constitute two basis for the root (or weight) space. Since any root is a weight we have that the simple roots can be written as integer linear combination of the fundamental weights. Using (3.4) one gets that the integer coefficients are the entries of the Cartan matrix, i.e.

$$\alpha_a = \sum_b K_{ab} \lambda_b \tag{3.18}$$

and then

$$\lambda_a = \sum_b K^{-1}_{ab} \alpha_b \tag{3.19}$$

So the fundamental weights are not, in general, written as integer linear combination of the simple roots.

- Root Lattice Λ_r $$v_r = \sum_{a=1}^{r} n_a \, \alpha_a$$

- Weight Lattice Λ $$v = \sum_{a=1}^{r} m_a \, \lambda_a$$

They are both abelian groups under addition of vectors

Λ_r is an invariant subgroup of Λ

Factor group $\Lambda/\Lambda_r \to$ center of covering group

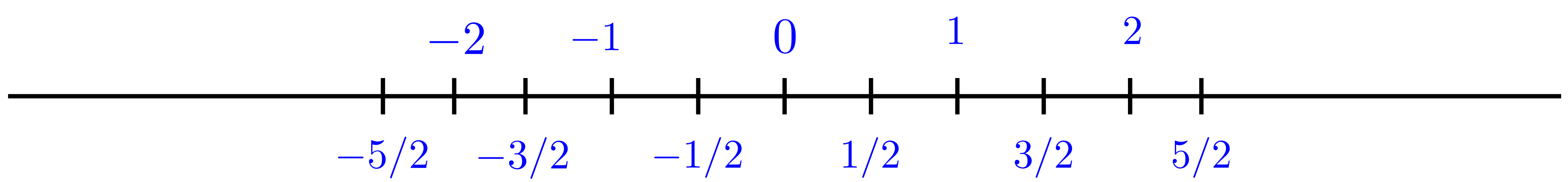
Example 3.1 *$SU(2)$ has only one simple root and consequently only one fundamental weight. Choosing a normalization such that $\alpha = 1$, we have that*

$$\frac{2\lambda \cdot \alpha}{\alpha^2} = 1 \qquad \text{and so} \qquad \lambda = \frac{1}{2} \tag{3.20}$$

Therefore the weight lattice of $SU(2)$ is formed by the integers and half integer numbers and the root lattice only by the integers. Then

$$\Lambda/\Lambda_r = \mathbb{Z}_2 \tag{3.21}$$

which is the center of $SU(2)$.



Example 3.2 *$SU(3)$ has two fundamental weights since it has rank two. They can be constructed solving (3.4) or equivalently (3.19). The Cartan matrix of $SU(3)$ and its inverse are given by (see example 2.13)*

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad K^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.22}$$

So, from (3.19), we get that fundamental weights are

$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) \qquad \lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) \tag{3.23}$$

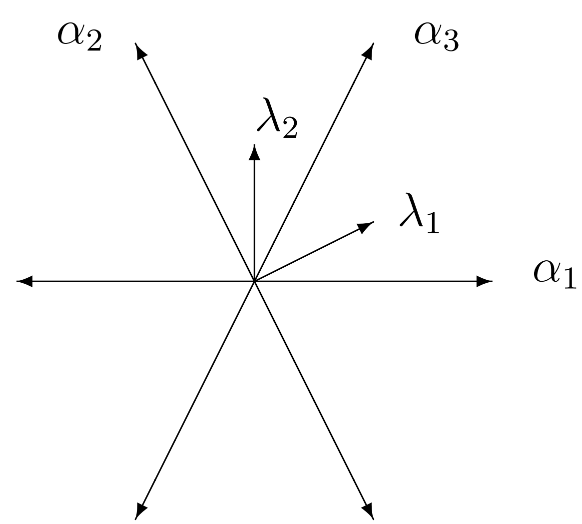


Figure 3.1: The fundamental weights of A_2 ($SU(3)$ or $SL(3)$)

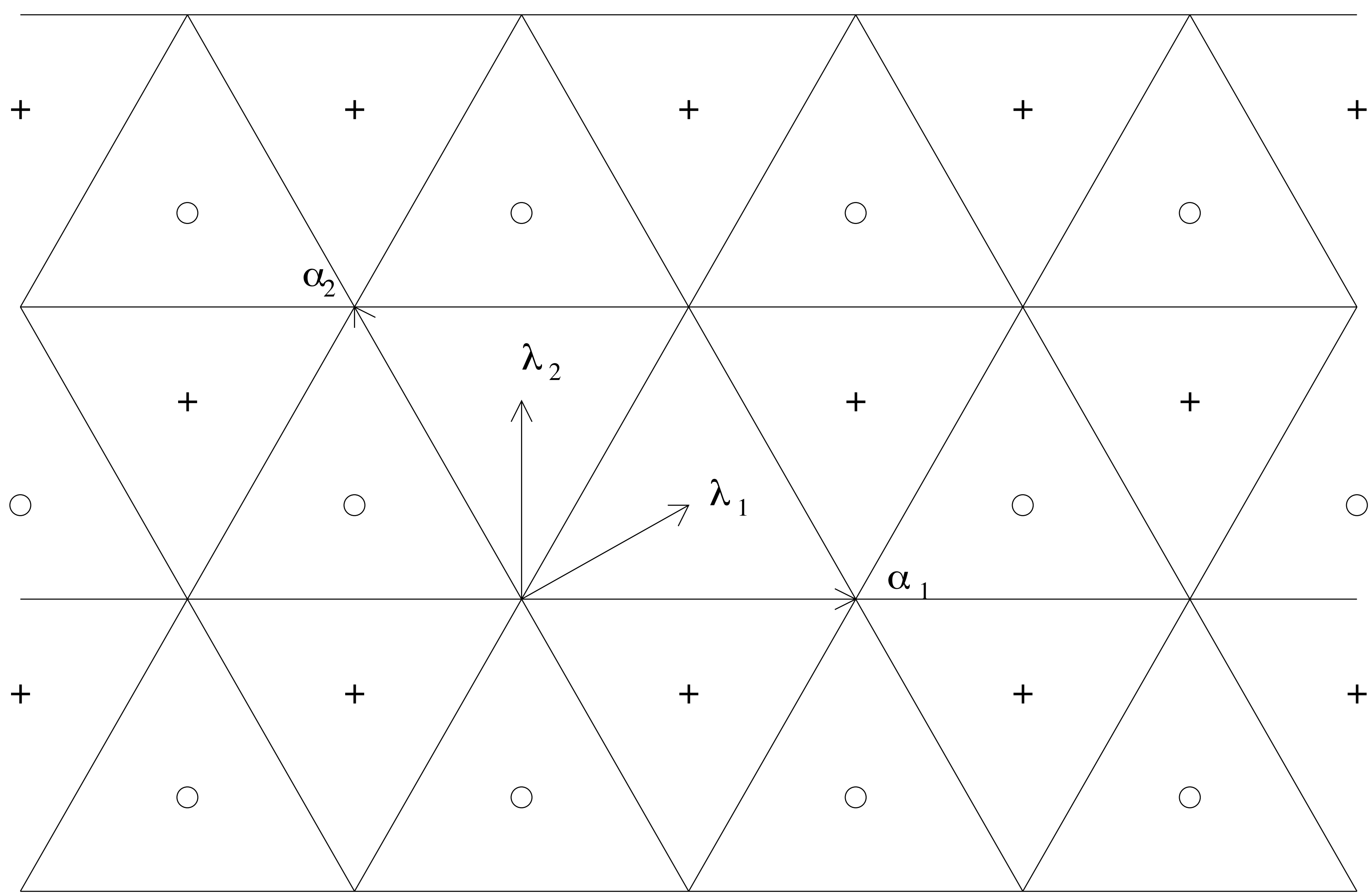
In example 2.10 we have seen that the simple roots of $SU(3)$ are given by $\alpha_1 = (1,0)$ and $\alpha_2 = \left(-1/2, \sqrt{3}/2\right)$. Therefore

$$\lambda_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right) \qquad \lambda_2 = \left(0, \frac{\sqrt{3}}{3}\right) \tag{3.24}$$

The vectors representing the fundamental weights are given in figure 3.1.

The root lattice, Λ_r , generated by the simple roots α_1 and α_2, corresponds to the points on the intersection of lines shown in the figure 3.2. The weight lattice, generated by the fundamental weights λ_1 and λ_2 , are all points of Λ_r plus the centroid of the triangles, shown by circles and plus signs on the figure 3.2.

The points of the weight lattice can be obtained from the origin, λ_1 and λ_2 by adding to them all points of the root lattice. Therefore the coset space Λ/Λ_r has three points which can be represented by 0, λ_1 and λ_2. Since $\lambda_1 + \lambda_2 = \alpha_1 + \alpha_2$ and $3\lambda_1 = 2\alpha_1 + \alpha_2$ lie in the same coset as 0, we see that Λ/Λ_r has the structure of the cyclic group $\mathbb{Z}_3$ which is the center of $SU(3)$.



3.3 The highest weight state

In a irreducible representation one can obtain all states of the representation by starting with a given state and applying sequences of step operators on it. If that was not possible the representation would have an invariant subspace and therefore would not be irreducible.

Consider a state with weight μ satisfying (3.1). The state defined by

$$\mid \mu' \rangle \equiv E_\alpha \mid \mu \rangle \tag{3.25}$$

satisfies

$$\begin{aligned} H_i \mid \mu' \rangle &= H_i E_\alpha \mid \mu \rangle \\ &= (E_\alpha H_i + [\, H_i \, , \, E_\alpha \,]) \mid \mu \rangle \\ &= (\mu_i + \alpha_i) \, E_\alpha \mid \mu \rangle \end{aligned} \tag{3.26}$$

and therefore it has weight $\mu + \alpha$. Therefore the state

$$E_{\alpha_1} E_{\alpha_2} \ldots E_{\alpha_n} \mid \mu \rangle \tag{3.27}$$

has weight $\mu + \alpha_1 + \ldots + \alpha_n$.

For this reason the weights in an irreducible representation differ by a sum of roots, and consequently they all lie in the same coset in Λ / Λ_r. Since that is the center of the covering group we see that the weights of an irreducible representation is associated to only one element of the center.

In a finite dimensional representation, the number of weights is finite, since this is at most the number of base states (remember the weights can be degenerated). Therefore, by applying sequences of step operators corresponding to positive roots on a given state we will eventually get zero. So, an irreducible finite dimensional representation possesses a state such that

$$E_\alpha \mid \lambda\rangle = 0 \qquad \text{for any } \alpha > 0 \tag{3.28}$$

This state is called the *highest weight state* of the representation, and λ is the *highest weight*. It is possible to show that there is only one highest weight in an irrep. and only one highest weight state associated to it. That is, the highest weight is *unique and non degenerate*.

All other states of the representation are obtained from the highest weight state by the application of a sequence of step operators corresponding to negative roots. The state defined by

$$| \, \mu \rangle \equiv E_{-\alpha_1} E_{-\alpha_2} \ldots E_{-\alpha_n} \mid \lambda \rangle \tag{3.29}$$

according to (3.26) has weight $\lambda - \alpha_1 - \alpha_2 \ldots - \alpha_n$. All the basis states are of the form (3.29). If one applies a positive step operator on the state (3.29) the resulting state of the representation can be written as a linear combination of states of the form (3.29). To see this, let β be a a positive root and α any of the negative roots appearing in (3.29). Then we have

$$E_\beta \mid \mu \rangle = \left(E_{-\alpha_1} E_\beta + [\, E_\beta \, , \, E_{-\alpha_1} \,] \right) E_{-\alpha_2} \ldots E_{-\alpha_n} \mid \lambda \rangle \tag{3.30}$$

In the cases where $\beta - \alpha_1$ is a negative root or it is not a root or even $\beta - \alpha_1 = 0$, we obtain that the second term on the r.h.s. of (3.30) is a state of the form of (3.29). In the case $\beta - \alpha_1$ is a positive root we contiunue the process until all positive step operators act directly on the highest state $\mid \lambda \rangle$, and consequently annihilate it. Therefore the state (3.30) is a linear combination of the states (3.29).

The weight lattice Λ is invariant by the Weyl group. If μ is a weight, and therefore satisfies (3.3), it follows that $\sigma_\beta(\mu)$ also satisfies (3.3) for any root β, and so is a weight. To show this we use the fact that $\sigma_\beta(x) \cdot \sigma_\beta(y) = x \cdot y$ and $\sigma_\beta^2 = 1$. Then (denoting $\gamma = \sigma_\beta(\alpha)$)

$$\frac{2\alpha \cdot \sigma_\beta(\mu)}{\alpha^2} = \frac{2\mu \cdot \sigma_\beta(\alpha)}{\sigma_\beta(\alpha)^2} = \frac{2\gamma \cdot \mu}{\gamma^2} = \text{integer} \tag{3.31}$$

However we can show that the set of weights of a given representation, which is a finite subset of Λ, is invariant by the Weyl group. The state defined by

$$\mid \bar{\mu}\rangle \equiv S_\alpha \mid \mu\rangle \tag{3.32}$$

$$S_\alpha = \exp(i\pi T_2(\alpha))$$

where $\mid \mu\rangle$ is a state of the representation and S_α is defined in (2.154), is also a state of the representation since it is obtained from $\mid \mu\rangle$ by the action of an operator of the representation. Using (2.155) we get

$$\begin{aligned} x \cdot H \mid \bar{\mu}\rangle &= S_\alpha S_\alpha^{-1} x \cdot H S_\alpha \mid \mu\rangle \\ &= S_\alpha \sigma_\alpha(x) \cdot H \mid \mu\rangle \\ &= \sigma_\alpha(x) \cdot \mu \mid \bar{\mu}\rangle \\ &= \sigma_\alpha(\mu) \cdot x \mid \bar{\mu}\rangle \end{aligned} \tag{3.33}$$

$$S_\alpha(x.H)S_\alpha^{-1} \quad = \quad \sigma_\alpha(x).H$$

Since the vector x is arbitrary we obtain that the state $\mid \bar{\mu}\rangle$ has, weight $\sigma_\alpha(\mu)$

$$H_i \mid \bar{\mu}\rangle = H_i S_\alpha \mid \mu\rangle = \sigma_\alpha(\mu)_i S_\alpha \mid \mu\rangle = \sigma_\alpha(\mu)_i \mid \bar{\mu}\rangle \tag{3.34}$$

Therefore if μ is a weight of the representation so is $\sigma_\alpha(\mu)$ for any root α. One can easily check that the root lattice Λ_r is also invariant by the Weyl reflections.

A consequence of the above result is that the highest weight λ of an irrep. is a dominant weight. By taking its Weyl reflection

$$\sigma_{\alpha}\left(\lambda\right) = \lambda - \frac{2\lambda \cdot \alpha}{\alpha^2}\alpha \tag{3.35}$$

one obtains that $2\lambda \cdot \alpha$ has to be non negative if α is a positive root, since $\sigma_{\alpha}\left(\lambda\right)$ is also a weight of the representation and consequenlty can not exceed λ by a multiple of a positive root. Therefore

$$\lambda \cdot \alpha \geq 0 \qquad \text{for any positive root } \alpha \tag{3.36}$$

and the highest weight λ is a dominant weight.

The highest weight λ can be used to label the representation. This is one of the consequences of the following theorem which we state without proof.

Theorem 3.3 *There exists a unique irreducible representation of a compact Lie algebra (up to equivalence) with highest weight state* $\mid \lambda\rangle$ *for each* λ *of the weight lattice in the Fundamental Weyl Chamber or on its border.*

The importance of this theorem is that it provides some sort of classification of all irreps. of a compact Lie algebra. All other reducible representations are constructed from these ones. The irreps. can be labelled by their highest weight λ as D^{λ} or $D^{(n_1,n_2,\ldots n_r)}$ where the n_a's are non-negative integers appearing in the expansion of λ in terms of the fundamental weights λ_a, i.e. $\lambda = \sum_{a=1}^{r} n_a\lambda_a$, and $n_a = \frac{2\lambda\cdot\alpha_a}{\alpha_a^2}$.

An irrep. is called a *fundamental representation* when its highest weight is a fundamental weight. Therefore the number of fundamental representations of a semisimple compact Lie algebra is equal to its rank.

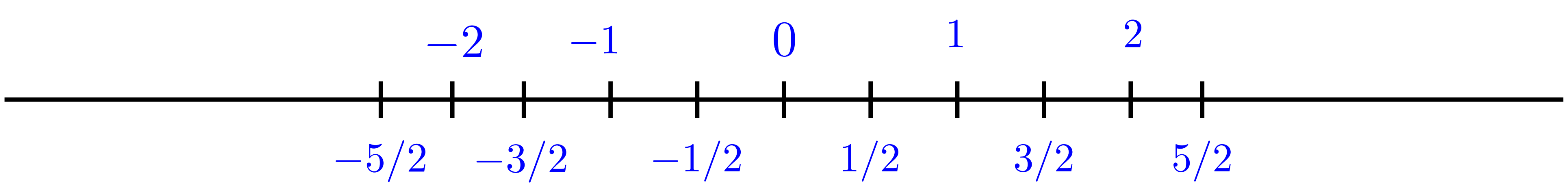
The highest weight of the adjoint representation is the highest positive root (see section 2.13). It follows that the weights of the adjoint representation are all roots of the algebra together with zero which is a weight r-fold degenerated ($r=$ rank).

We say a weight μ is a *minimal weight* if it satisfies

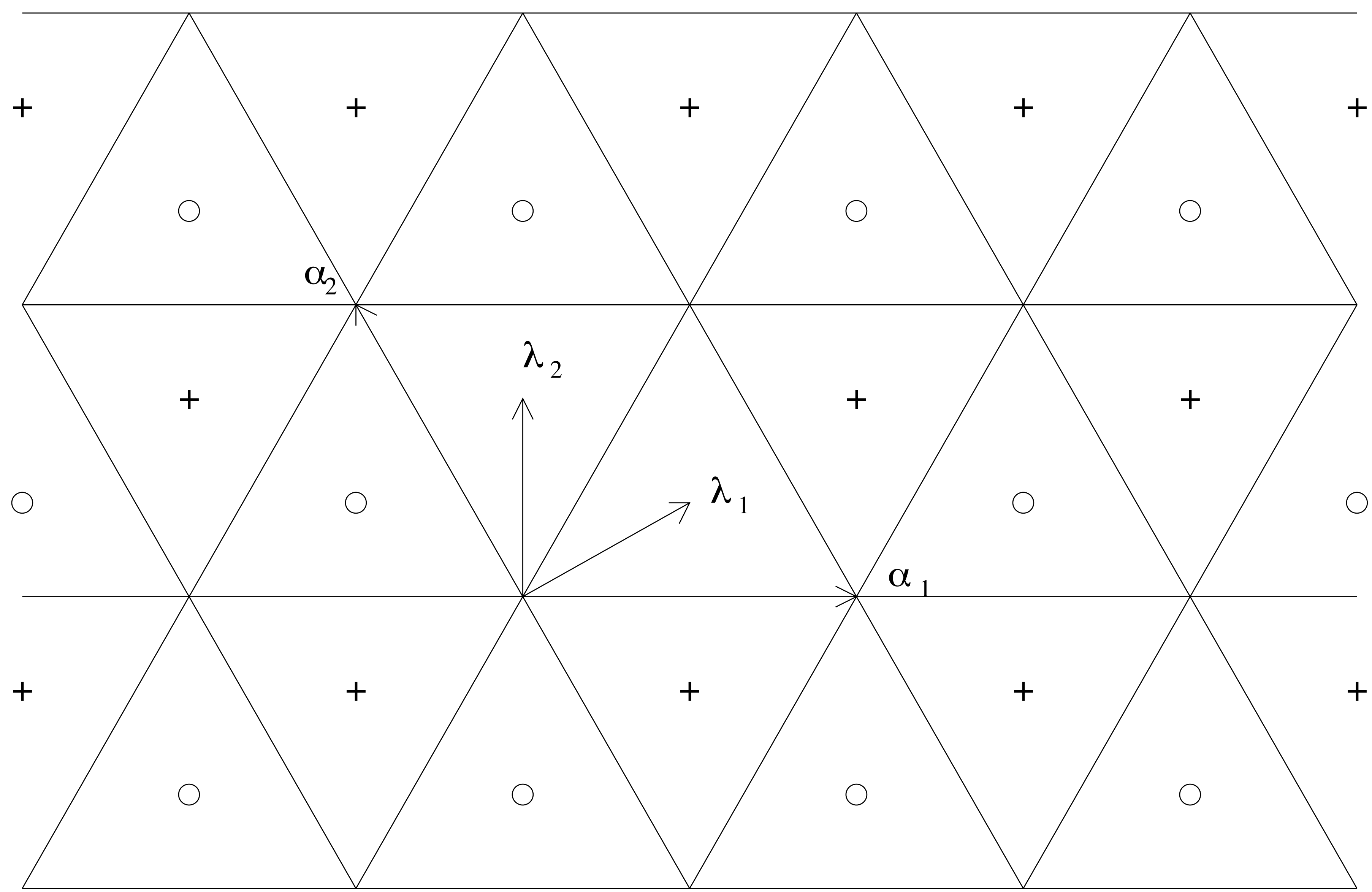
$$\frac{2\mu\cdot\alpha}{\alpha^2} = 0 \text{ or } \pm 1 \text{ for any root } \alpha \tag{3.37}$$

The representation for which the highest weight is minimal is said to be a *minimal representation*. These representations play an important role in grand unified theories (GUT) in the sense that the constituent fermions prefer, in general, to form multiplets in such minimal representations.

Example 3.3 *In the example 3.1 we have seen that the only fundamental weight of* $SU(2)$ *is* $\lambda = \frac{1}{2}$ *. Therefore the dominant weights of* $SU(2)$ *are the positive integers and half integers. Each one of these dominant weights corresponds to an irreducible representation of* $SU(2)$*. Then we have that* $\lambda = 0$ *corresponds to the scalar representation,* $\lambda = \frac{1}{2}$ *the spinorial rep. which is the fundamental rep. of* $SU(2)$ *(dim* $= 2$*),* $\lambda = 1$ *is the vectorial rep. which is the adjoint of* $SU(2)$ *(dim* $= 3$*) and so on.*



Example 3.4 *In the case of $SU(3)$ we have two fundamental representations with highest weights λ_1, and λ_2 (see example 3.2. They are respectively the triplet and antitriplet representations of $SU(3)$. The rep. with highest weight $\lambda_1 + \lambda_2 = \alpha_3$ is the adjoint. All representations with highest weight of the form with $\lambda = n_1\lambda_1 + n_2\lambda_2$, with n_1 and n_2 non negative integers are irreducible representations of SU(3).*



3.4 Weight strings and multiplicities

If we apply the step operator E_α or $E_{-\alpha}$, for a fixed root α, successively on a state of weight μ of a finite dimensional representation, we will eventually get zero. That means that there exist positive integer numbers p and q such that

$$E_\alpha \mid \mu + p\alpha\rangle \qquad \text{and} \qquad E_{-\alpha} \mid \mu - q\alpha\rangle \tag{3.38}$$

p and q are the greatest positive integers for which $\mu+p\alpha$ and $\mu-q\alpha$ are weights of the representation. One can show that all vectors of the form $\mu + n\alpha$ with n integer and $-q < n < p$, are weights of the representation. Therefore the weights form unbroken strings, called *weight strings* , of the form

$$\mu + p\alpha \; ; \mu + (p-1)\,\alpha \; ; \ldots \mu + \alpha \; ; \mu \; ; \mu - \alpha \; ; \ldots \mu - q\alpha \tag{3.39}$$

We have shown in the last section that the set of weights of a representation is invariant under the Weyl group. The effect of the action of the Weyl reflection σ_α on a weight is to add or subtract a multiple of the root α, since $\sigma_\alpha(\mu) = \mu - \frac{2\mu\cdot\alpha}{\alpha^2}\alpha$, and from (3.3) we have that $\frac{2\mu\cdot\alpha}{\alpha^2}$ is an integer. Therefore the weight string (3.39) is invariant by the Weyl reflection σ_α. In fact, σ_α reverses the string (3.39) and consenquently we have that

$$\sigma_\alpha(\mu + p\alpha) = \mu - q\alpha = \mu - \frac{2\mu\cdot\alpha}{\alpha^2}\alpha - p\alpha \tag{3.40}$$

and so

$$\frac{2\mu\cdot\alpha}{\alpha^2} = q - p \tag{3.41}$$

This result is similar to (2.187) which was obtained for root strings. However, notice that the possible values of $q-p$, in this case, are not restrict to the values given in (2.187) ($q-p$ can, in principle, have any integer value). In the case where μ is the highest weight of the representation we have that p is zero if α is a positive root, and q is zero if α is negative. The relation (3.41) provides a practical way of finding the weights of the representation. In some cases it is easier to find some weights of a given representation by taking successive Weyl reflections of the highest weight. However, this method does not provide, in general, all the weights of the representation.

Once the weights are known one has to calculate their multiplicities. There exists a formula, due to Kostant, which expresses the multiplicities directly as a sum over the elements of the Weyl group. However, it is not easy to use this formula in practice. There exists a recursive formula, called *Freudenthal's formula* , which is much easier to use. According to it the multiplicity $m(\mu)$ of a weight μ in an irreducible representation of highest weight λ is given recursively as (see sections 22.3 and 24.2 of [HUM 72])

$$\left((\lambda+\delta)^2-(\mu+\delta)^2\right)m(\mu)=2\sum_{\alpha>0}\sum_{n=1}^{p(\alpha)}\alpha\cdot(\mu+n\alpha)\,m(\mu+n\alpha) \tag{3.42}$$

where

$$\delta\equiv\frac{1}{2}\sum_{\alpha>0}\alpha \tag{3.43}$$

The first summation on the l.h.s. is over the positive roots and the second one over all positive integers n such that $\mu+n\alpha$ is a weight of the representation, and we have denoted by $p(\alpha)$ the highest value of n. By starting with $m(\lambda)=1$ one can use (3.43) to calculate the multiplicities of the weights from the higher ones to the lower ones.

If the states $\mid \mu\rangle_1$ and $\mid \mu\rangle_2$ have the same weight, i.e., μ is degenerated, then the weight $\sigma_\alpha(\mu)$ is also degenerate and has the same multiplicity as μ. Using (3.32) we obtain that the states

$$\mid \sigma_\alpha(\mu)\rangle_1 = S_\alpha \mid \mu\rangle_1 \qquad \text{and} \qquad \mid \sigma_\alpha(\mu)\rangle_2 = S_\alpha \mid \mu\rangle_2 \tag{3.44}$$

have weight $\sigma_\alpha(\mu)$ and their linear independence follows from the linear independence of $\mid \mu\rangle_1$ and $\mid \mu\rangle_2$. Indeed,

$$0 = x_1 \mid \sigma_\alpha(\mu)\rangle_1 + x_2 \mid \sigma_\alpha(\mu)\rangle_2 = S_\alpha \left(x_1 \mid \mu\rangle_1 + x_2 \mid \mu\rangle_2\right) \tag{3.45}$$

So, if $\mid \mu\rangle_1$ and $\mid \mu\rangle_2$ are linearly independent one gets that one must have $x_1 = x_2 = 0$ and so, $\mid \sigma_\alpha(\mu)\rangle_1$ and $\mid \sigma_\alpha(\mu)\rangle_2$ are also linearly independent.

Therefore all the weights of a representation which are conjugate under the Weyl group have the same multiplicity. This fact can be used to make the Freudenthal's formula more efficient in the calculation of the multiplicities.

Example 3.5 *Using the results of example 2.14 we have that the Cartan matrix of* $so(5)$ *ond its inverse are*

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad K^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \tag{3.46}$$

Then, using (3.19), we get that the fundamental weights of $so(5)$ *are*

$$\lambda_1 = \frac{1}{2}(2\alpha_1 + \alpha_2) \qquad \lambda_2 = \alpha_1 + \alpha_2 \tag{3.47}$$

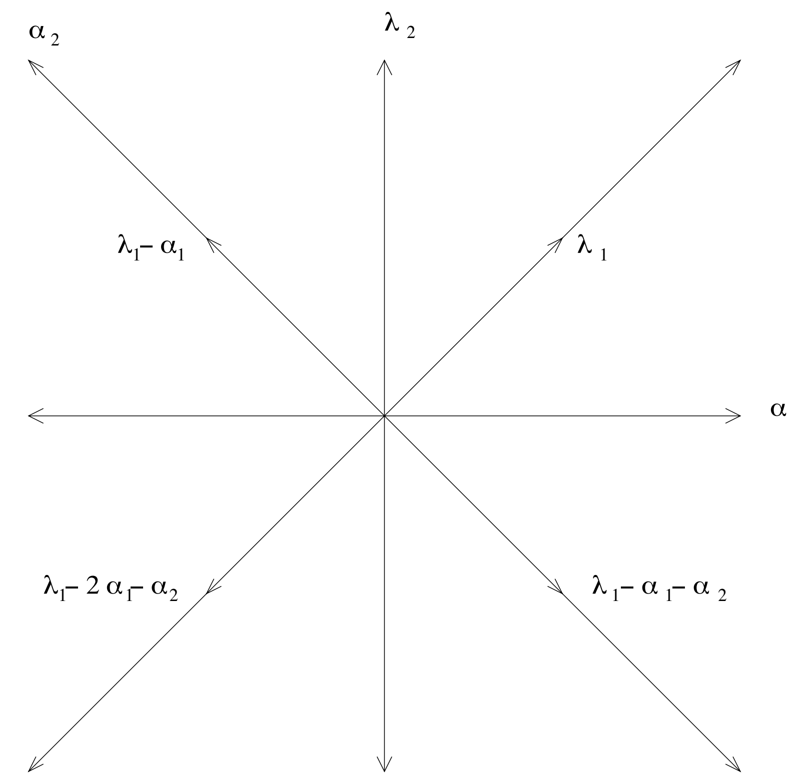
where α_1 *and* α_2 *are the simple roots of* $so(5)$*. Let us consider the fundamental representation with highest weight* λ_1*. The scalar products of* λ_1 *with the positive roots of* $so(5)$ *are*

$$\begin{aligned} \frac{2\lambda_1 \cdot \alpha_1}{\alpha_1^2} &= 1 \qquad \frac{2\lambda_1 \cdot \alpha_2}{\alpha_2^2} = 0 \\ \frac{2\lambda_1 \cdot (\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2)^2} &= 1 \qquad \frac{2\lambda_1 \cdot (2\alpha_1 + \alpha_2)}{(2\alpha_1 + \alpha_2)^2} = 1 \end{aligned} \tag{3.48}$$

Therefore using (3.41) (with $p = 0$ *since* λ_1 *is the highest weight) we get that*

$$\lambda_1 \; ; \quad (\lambda_1 - \alpha_1) \; ; \quad (\lambda_1 - \alpha_1 - \alpha_2) \; ; \quad (\lambda_1 - 2\alpha_1 - \alpha_2) \tag{3.49}$$

are weights of the representation. By taking Weyl reflections of these weights or using (3.41) further one can check that these are the only weights of the fundamental rep. with highest weight λ_1*.*

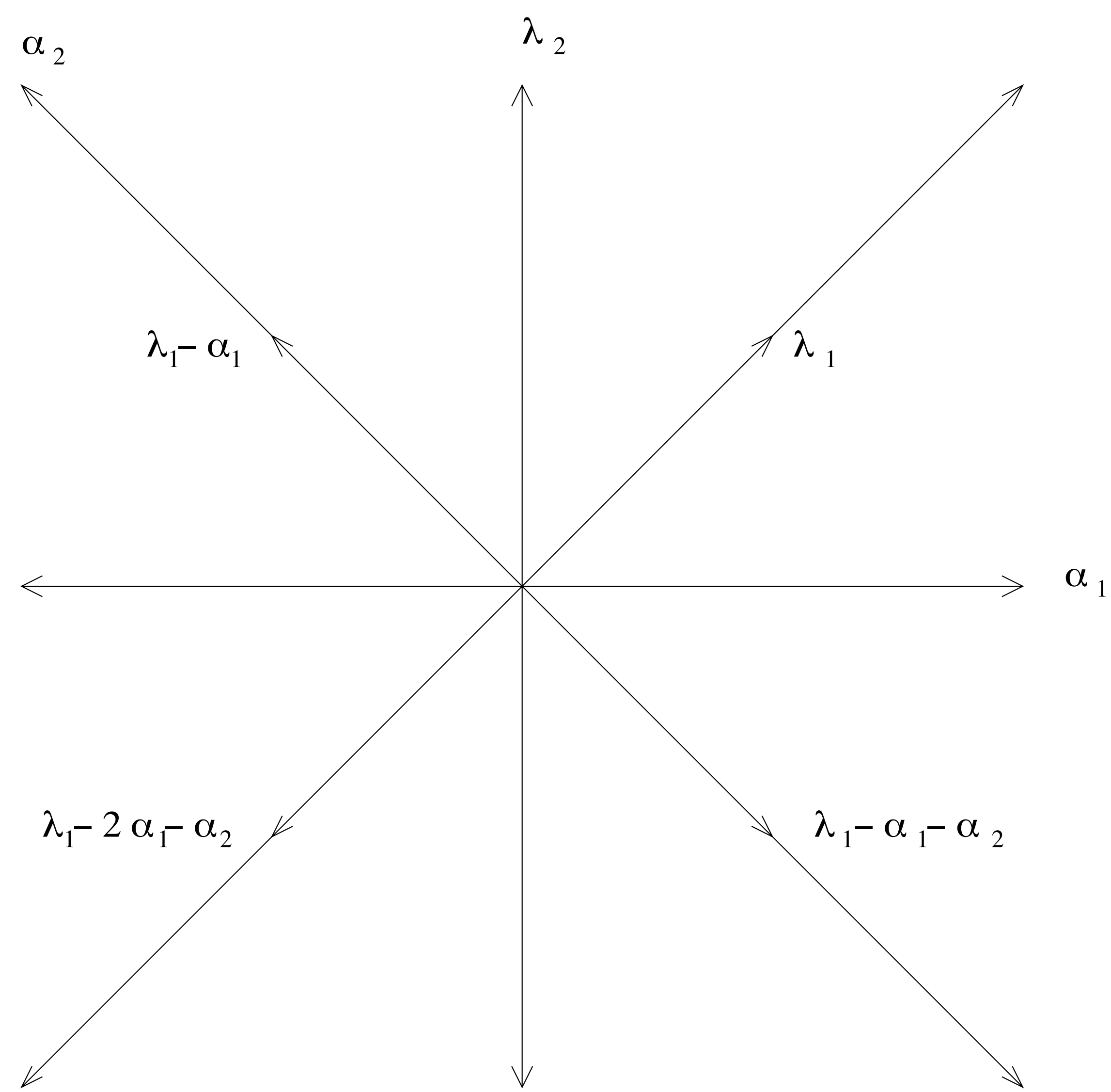


$$\frac{2\mu \cdot \alpha}{\alpha^2} = q - p \tag{3.41}$$

Since all weights are conjugate under the Weyl group they all have the same multiplicity as λ_1 *, which is one. Therefore they are not degenerate and the representation has dimension* 4*. This is the spinor representation of* $so(5)$*. One can check that the weights of the fundamental representation of* $so(5)$ *with highest weight* λ_2 *are*

$$\begin{aligned} &\lambda_2\ ; \quad \lambda_2 - \alpha_2 = \alpha_1\ ; \quad \lambda_2 - \alpha_1 - \alpha_2 = 0\ ; \\ &\lambda_2 - 2\alpha_1 - \alpha_2 = -\alpha_1\ ; \quad \lambda_2 - 2\alpha_1 - 2\alpha_2 = -\left(\alpha_1 + \alpha_2\right) \end{aligned} \tag{3.50}$$

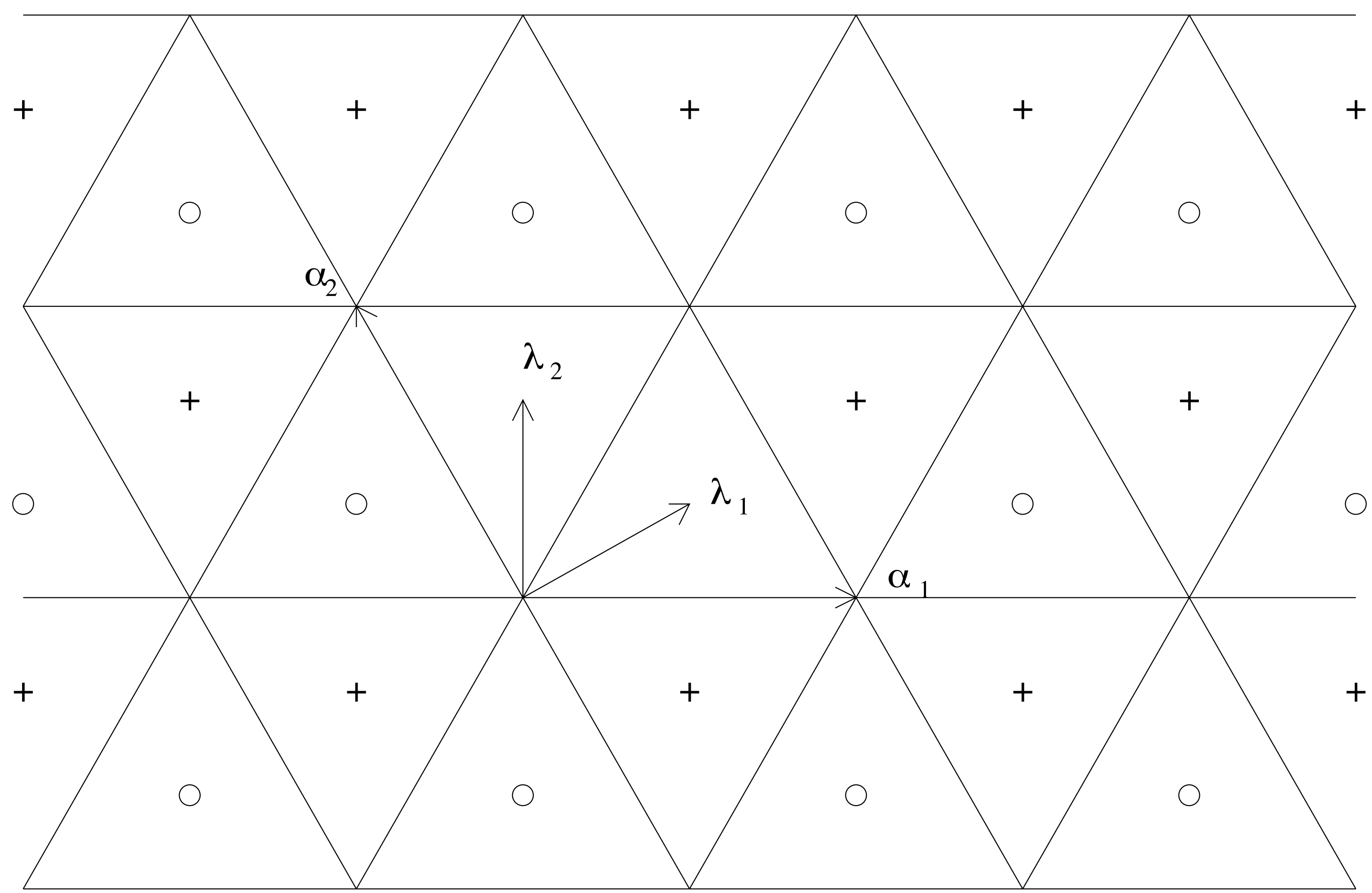
Again these weights are not degenerate and the representation has dimension 5*. This is the vector representation of* $so(5)$*.*



Example 3.6 *Consider the irrep. of su(3) with highest weight* $\lambda = \alpha_3 = \alpha_1 + \alpha_2$ *, i.e., the highest positive root. Using (3.41) and performing Weyl reflections one can check that the weights of such rep. are all roots plus the zero weight. Since the roots are conjugated to* $\alpha_3 = \lambda$ *under the Weyl group we conclude that they are non degenerated weights. The multiplicity of the zero weight can be calculated from the Freundenthal's formula. From (3.43) we have that, in this case,* $\delta = \alpha_3$ *and so from (3.42) we get*

$$\left(4\alpha_3^2 - \alpha_3^2\right) m\left(0\right) = 2\left(m\left(\alpha_1\right)\alpha_1^2 + m\left(\alpha_2\right)\alpha_2^2 + m\left(\alpha_3\right)\alpha_3^2\right) \tag{3.51}$$

Since $m(\alpha_1) = m(\alpha_2) = m(\alpha_3) = 1$ *and* $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$ *we obtain that* $m(0) = 2$*. So there are two states with zero weight and consequently the representation has dimension 8. This is the adjoint of su(3).*



$$\left(\left(\lambda+\delta\right)^2 - \left(\mu+\delta\right)^2\right) m\left(\mu\right) = 2\sum_{\alpha>0}\sum_{n=1}^{p(\alpha)} \alpha\cdot\left(\mu+n\alpha\right) m\left(\mu+n\alpha\right)$$

$$\delta \equiv \frac{1}{2}\sum_{\alpha>0}\alpha$$

3.5 The weight δ

$$\delta \equiv \frac{1}{2} \sum_{\alpha>0} \alpha$$

A vector which plays an important role in the representation theory of Lie algebras is the vector δ defined in (3.43). It is half of the sum of all positive roots. In same cases δ is a root, but in general that is not so. However δ is always a dominant weight of the algebra. In other to show that we need some results which we now prove.

Let α_a be a simple root and let β be a positive root non proportional to α_a. If we write $\beta = \sum_{b=1}^{r} n_b \alpha_b$ we have that $n_b \neq 0$ for some $b \neq a$. Now, the coefficient of α_b in $\sigma_{\alpha_a}(\beta)$ is still n_b, and consequently $\sigma_{\alpha_a}(\beta)$ has at least one positive coefficient. So, $\sigma_{\alpha_a}(\beta)$ is a positive root, and it is different from α_a, since α_a is the image of $-\alpha_a$ under σ_{α_a}. Therefore we have proved the following lemma.

Lemma 3.1 *If α_a is a simple root, then σ_{α_a} permutes the positive roots other than α_a.*

From this lemma it follows that

$$\sigma_{\alpha_a}(\delta) = \delta - \alpha_a \tag{3.52}$$

and consequently

$$\frac{2\delta \cdot \alpha_a}{\alpha_a^2} = 1 \qquad \text{for any simple root } \alpha_a \tag{3.53}$$

From the definition (3.43) it follows that δ is a vector on the root (or weight) space and therefore can be written in terms of the simple roots or the fundamental weights. Writing

$$\delta = \sum_{b=1}^{r} x_b \lambda_b \tag{3.54}$$

we get from (3.4) and (3.53) that

$$\frac{2\delta \cdot \alpha_a}{\alpha_a^2} = 1 = \sum_{b=1}^{r} x_b \frac{2\lambda_b \cdot \alpha_a}{\alpha_a^2} = x_a \tag{3.55}$$

So we have shown that

$$\delta = \sum_{b=1}^{r} \lambda_b \tag{3.56}$$

and consequently δ is a dominant weight.

3.6 Casimir operators

Let $\Gamma^{s_1 s_2 \dots s_n}$ be a tensor invariant under the adjoint representation of a Lie group G. By that we mean

$$\Gamma^{s_1 s_2 \dots s_n} = d^{s_1}_{s'_1}(g)\, d^{s_2}_{s'_2}(g) \dots d^{s_n}_{s'_n}(g)\, \Gamma^{s'_1 s'_2 \dots s'_n} \tag{3.57}$$

for any $g \in G$, and where $d^{s_j}_{s'_j}(g)$ is the matrix representing g in the adjoint representation, i.e. $g T_s g^{-1} = T_{s'} d^{s'}_{s}(g)$ (see (2.31)).

Consider now a representation D of G and construct the operator

$$C_n^{(D)} \equiv \Gamma^{s_1 s_2 \dots s_n}\, D(T_{s_1})\, D(T_{s_2}) \dots D(T_{s_n}) \tag{3.58}$$

Notice that such operator can only be defined on a given representation since it involves the product of operators and not Lie brackets of the generators.

We then have

$$\begin{aligned} D(g)\, C_n^{(D)} &= \Gamma^{s_1 s_2 \dots s_n}\, D\left(g T_{s_1} g^{-1}\right) D\left(g T_{s_2} g^{-1}\right) \dots D\left(g T_{s_n} g^{-1}\right) D(g) \\ &= d^{s'_1}_{s_1}(g) \dots d^{s'_n}_{s_n}(g)\, \Gamma^{s_1 \dots s_n} D\left(T_{s'_1}\right) \dots D\left(T_{s'_n}\right) D(g) \\ &= \Gamma^{s'_1 \dots s'_n} D\left(T_{s'_1}\right) \dots D\left(T_{s'_n}\right) D(g) \\ &= C_n^{(D)} D(g) \end{aligned} \tag{3.59}$$

So, we have shown that $C_n^{(D)}$ commutes with any matrix of the representation

$$\left[\, C_n^{(D)}\, ,\, D(g) \right] = 0 \tag{3.60}$$

We are interested in operators that can not be reduced to lower orders. That implies that the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ has to be totally symmetric. Indeed, suppose that $\Gamma^{s_1 s_2 \ldots s_n}$ has an antisymmetric part in the indices s_j and s_{j+1}. Then we write

$$\begin{aligned} D\left(T_{s_j}\right) D\left(T_{s_{j+1}}\right) &= \frac{1}{2}\{D\left(T_{s_j}\right) , D\left(T_{s_{j+1}}\right)\} + \frac{1}{2}\left[D\left(T_{s_j}\right), D\left(T_{s_{j+1}}\right)\right] \\ &= \frac{1}{2}\{D\left(T_{s_j}\right) , D\left(T_{s_{j+1}}\right)\} + f^t_{s_j s_{j+1}} D\left(T_t\right) \end{aligned} \tag{3.61}$$

and so, $C_n^{(D)}$ will have terms involving the product of $(n-1)$ operators. Therefore, by totally symmetrizing the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ we get operators $C_n^{(D)}$ which are monomials of order n in $D\left(T_s\right)$'s. Such operators are called *Casimir operators*, and n is called their *order*. They play an important role in representation theory. From Schur's lemma 1.1 it follows that in an irreducible representation the Casimir operators have to be proportional to the identity.

One way of constructing tensors which are invariant under the adjoint representation, is by considering traces of products of generators in a given representation D', since

$$\mathrm{Tr}\left(D'\left(T_{s_1} T_{s_2} \ldots T_{s_n}\right)\right) = \mathrm{Tr}\left(D'\left(g T_{s_1} g^{-1} g T_{s_2} g^{-1} \ldots g T_{s_n} g^{-1}\right)\right) \tag{3.62}$$

Then taking

$$\Gamma_{s_1 s_2 \ldots s_n} \equiv \frac{1}{n!} \sum_{\text{permutations}} \mathrm{Tr}\left(D'\left(T_{s_1} T_{s_2} \ldots T_{s_n}\right)\right) \tag{3.63}$$

we get Casimir operators. However, one finds that after the symetrization procedure very few tensors of the form above survive. It follows that a semisimple Lie algebra of rank r possesses r invariant Casimir operators functionally independent. Their orders, for the simple Lie algebras, are given in table 3.1.

A_r	$SU(r+1)$	2, 3, 4, $\ldots$ $r+1$
B_r	$SO(2r+1)$	2, 4, 6, $\ldots$ $2r$
C_r	$Sp(r)$	2, 4, 6 $\ldots$ $2r$
D_r	$SO(2r)$	2, 4, 6 $\ldots$ $2r-2$, r
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6

3.6.1 The Quadratic Casimir operator

Notice from table 3.1 that all simple Lie groups have a quadratic Casimir operator. That is because all such groups have an invariant symmetric tensor of order two which is the Killing form (see section 2.4)

$$\eta_{st} = \mathrm{Tr}\left(d\left(T_s\right) d\left(T_t\right)\right) \tag{3.64}$$

and

$$C_2^{(D)} \equiv \eta^{st} D\left(T_s\right) D\left(T_t\right) \tag{3.65}$$

where η^{st} is the inverse of η_{st}.

Using the normalization (2.134) of the Killing form, we have that the Casimir operator in the Cartan-Weyl basis is given by

$$C_2^{(D)} = \sum_{i=1}^{r} D\left(H_i\right) D\left(H_i\right) + \sum_{\alpha>0} \frac{\alpha^2}{2}\left(D\left(E_\alpha\right) D\left(E_{-\alpha}\right) + D\left(E_{-\alpha}\right) D\left(E_\alpha\right)\right) \tag{3.66}$$

Since the Casimir operator commutes with all generators, we have from the Schur's lemma 1.1 that in an irreducible representation it must be proportional to the unit matrix. Denoting by λ the highest weight of the irreducible representation D we have

$$\begin{aligned} C_2^{(D)} \mid \lambda\rangle &= \left(\sum_{i=1}^{r} \lambda_i^2 + \sum_{\alpha>0} \frac{\alpha^2}{2}\left[D\left(E_\alpha\right), D\left(E_{-\alpha}\right)\right]\right) \mid \lambda\rangle \\ &= \left(\lambda^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} H_\alpha^2\right) \mid \lambda\rangle \\ &= \left(\lambda^2 + \sum_{\alpha>0} \alpha \cdot \lambda\right) \mid \lambda\rangle \end{aligned} \tag{3.67}$$

where we have used (3.28) and (2.125). So, if D, with highest weight λ, is irreducible, we can write using (3.43) that

$$C_2^{(D)} = \lambda \cdot (\lambda + 2\delta)\, \mathbb{1} = \left((\lambda+\delta)^2 - \delta^2\right) \mathbb{1} \tag{3.68}$$

where $\mathbb{1}$ is the unit matrix in the representation D under consideration.

Example 3.7 *In the case of $SU(2)$ the quadratic operator is J^2 , i.e., the square of the angular momentum. Indeed, from example 3.1 we have that $\alpha = 1$, and then $\delta = 1/2$ and therefore $C_2^{(D)} = \lambda(\lambda+1)$. Since λ is a positive integer or half integer we see that these are really the eigenvalues of J^2.*