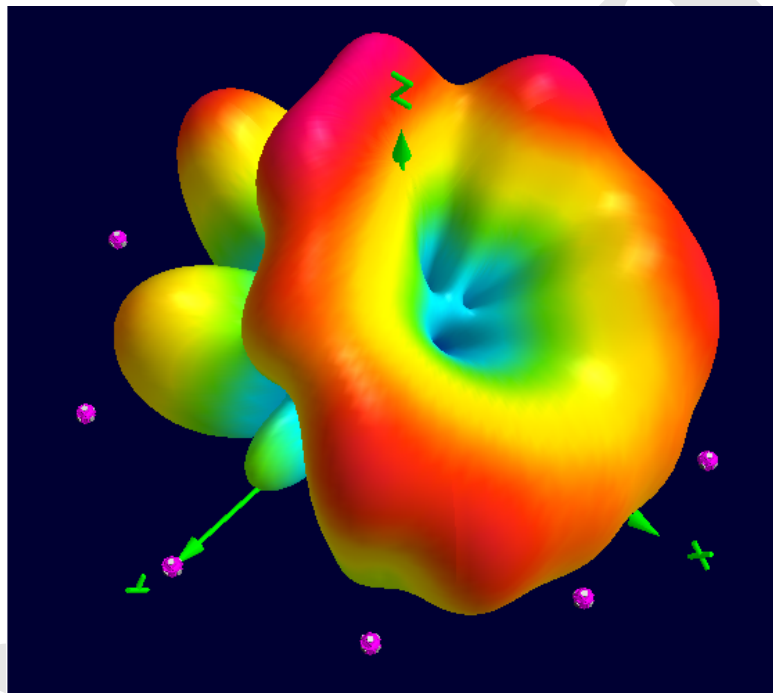


BO THIDÉ

ELECTROMAGNETIC FIELD THEORY



SECOND EDITION

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ELECTROMAGNETIC FIELD THEORY

SECOND EDITION

Bo Thidé

Swedish Institute of Space Physics
Uppsala, Sweden

and

Department of Physics and Astronomy
Uppsala University, Sweden

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Electromagnetic Field Theory
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The cover graphics illustrates the linear momentum radiation pattern of a radio beam endowed with orbital angular momentum, generated by an array of tri-axial antennas. This graphics illustration was prepared by JOHAN SJÖHOLM and KRISTOFFER PALMER as part of their undergraduate Diploma Thesis work in Engineering Physics at Uppsala University 2006–2007.

To the memory of professor
LEV MIKHAILOVICH ERUKHIMOV (1936–1997)
dear friend, great physicist, poet
and a truly remarkable man.

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PREFACE TO THE SECOND EDITION

This second edition of the book ELECTROMAGNETIC FIELD THEORY is a major revision of the first edition that was published on the Internet (www.plasma.uu.se/CED/Book). The main reasons for trying to improve the presentation and to add more material is that this new edition is now being made available in printed form by Dover Publications and is to be used in an extended Classical Electrodynamics course at Uppsala University, at the last-year undergraduate, master, and beginning post-graduate/doctoral level. Hopefully, this means that the book will find new uses in Academia and elsewhere.

The revision includes a slight reordering of the chapters. First, we describe the properties of electromagnetism when the charges and currents are located in otherwise free space, a space that is free of matter (vacuum) and external fields (*e.g.*, gravitation). Only when the fundamental properties of the electromagnetic fields in free space have been established with sufficient rigour and completeness do we go on to show how the fields interact with matter. In the author's opinion, this approach is preferable as it avoids the formal logical inconsistency of discussing, very early in the book, the effect on the electric and magnetic fields when conductors and dielectrics are present (and *vice versa*), before constitutive relations and physical models for the electromagnetic properties of matter, including conductors and dielectrics, have been derived from first principles. Curved-space effects are not treated.

In addition to the Maxwell-Lorentz equations, which postulate the behaviour of electromagnetic fields due to electric charges and currents on a microscopic classical scale, chapter 1 also introduces Dirac's symmetrised equations that incorporate the effects of magnetic charges and currents. In chapter 2, stronger emphasis is now put on the axiomatic foundation of electrodynamics as provided by the Maxwell-Lorentz equations. Chapter 4 is new and deals with symmetries and conserved quantities in a more rigorous, profound and detailed way than in the first edition. For instance, the presentation of the theory of electromagnetic angular momentum and other observables (constants of motion) has been substantially expanded and put on a more firm basis. chapter 9 is a complete rewrite that combines material that was scattered more or less all over the first edition. It also contains new material on wave propagation in plasma and other media. When, in chapter 9, the macroscopic Maxwell equations are introduced, the inherent approximations in the derived field quantities are clearly pointed out. The collection of formulae in appendix F has been augmented quite substantially. In appendix M, the treatment of dyadic products and tensors has been expanded.

I want to express my warm gratitude to professor CESARE BARBIERI and his entire group, particularly FABRIZIO TAMBURINI, at the Department of Astronomy, University of Padova, for stimulating discussions and the generous hospitality bestowed upon me during several shorter and longer visits in 2008, 2009, and 2010 that made it possible to prepare the current major revision of the book. In this breathtakingly beautiful northern Italy, intellectual titan GALILEO GALILEI worked for eighteen years and gave birth to modern physics, astronomy and science as we know it today, by sweeping away Aristotelian dogmas, misconceptions and mere superstition, thus most profoundly changing our conception of the world and our place in it. In the process, Galileo's new ideas transformed society and mankind forever. It is hoped that this book may contribute in some small, humble way to further these, once upon a time, mind-boggling—and even dangerous—ideas of intellectual freedom and enlightenment.

Thanks are also due to JOHAN SJÖHOLM, KRISTOFFER PALMER, MARCUS ERIKSSON, and JOHAN LINDBERG who during their work on their Diploma theses suggested improvements and additions and to HOLGER THEN and STAFFAN YNGVE for carefully checking some lengthy calculations and to the numerous undergraduate students, who have been exposed to various draft versions of this second edition. In particular, I would like to mention BRUNO STRANDBERG.

This book is dedicated to my son MATTIAS, my daughter KAROLINA, my four grandsons MAX, ALBIN, FILIP and OSKAR, my high-school physics teacher, STAFFAN RÖSBY, and my fellow members of the CAPELLA PEDAGOGICA UPSALIENSIS.

Padova, Italy
February, 2011

BO THIDÉ
www.physics.irfu.se/~bt

PREFACE TO THE FIRST EDITION

Of the four known fundamental interactions in nature—gravitational, strong, weak, and electromagnetic—the latter has a special standing in the physical sciences. Not only does it, together with gravitation, permanently make itself known to all of us in our everyday lives. Electrodynamics is also by far the most accurate physical theory known, tested on scales running from sub-nuclear to galactic, and electromagnetic field theory is the prototype of all other field theories.

This book, *ELECTROMAGNETIC FIELD THEORY*, which tries to give a modern view of classical electrodynamics, is the result of a more than thirty-five year long love affair. In the autumn of 1972, I took my first advanced course in electrodynamics at the Department of Theoretical Physics, Uppsala University. Soon I joined the research group there and took on the task of helping the late professor PER OLOF FRÖMAN, who was to become my Ph.D. thesis adviser, with the preparation of a new version of his lecture notes on the Theory of Electricity. This opened my eyes to the beauty and intricacy of electrodynamics and I simply became intrigued by it. The teaching of a course in Classical Electrodynamics at Uppsala University, some twenty odd years after I experienced the first encounter with the subject, provided the incentive and impetus to write this book.

Intended primarily as a textbook for physics and engineering students at the advanced undergraduate or beginning graduate level, it is hoped that the present book will be useful for research workers too. It aims at providing a thorough treatment of the theory of electrodynamics, mainly from a classical field-theoretical point of view. The first chapter is, by and large, a description of how Classical Electrodynamics was established by JAMES CLERK MAXWELL as a fundamental theory of nature. It does so by introducing electrostatics and magnetostatics and demonstrating how they can be unified into one theory, classical electrodynamics, summarised in Lorentz's microscopic formulation of the Maxwell equations. These equations are used as an axiomatic foundation for the treatment in the remainder of the book, which includes modern formulation of the theory; electromagnetic waves and their propagation; electromagnetic potentials and gauge transformations; analysis of symmetries and conservation laws describing the electromagnetic counterparts of the classical concepts of force, momentum and energy, plus other fundamental properties of the electromagnetic field; radiation phenomena; and covariant Lagrangian/Hamiltonian field-theoretical methods for electromagnetic fields, particles and interactions. Emphasis has been put on modern electrodynamics concepts while the mathematical tools used, some of them presented in an Appendix, are essentially the same kind of vector and tensor

analysis methods that are used in intermediate level textbooks on electromagnetics but perhaps a bit more advanced and far-reaching.

The aim has been to write a book that can serve both as an advanced text in Classical Electrodynamics and as a preparation for studies in Quantum Electrodynamics and Field Theory, as well as more applied subjects such as Plasma Physics, Astrophysics, Condensed Matter Physics, Optics, Antenna Engineering, and Wireless Communications.

The current version of the book is a major revision of an earlier version, which in turn was an outgrowth of the lecture notes that the author prepared for the four-credit course Electrodynamics that was introduced in the Uppsala University curriculum in 1992, to become the five-credit course Classical Electrodynamics in 1997. To some extent, parts of those notes were based on lecture notes prepared, in Swedish, by my friend and Theoretical Physics colleague BENGT LUNDBORG, who created, developed and taught an earlier, two-credit course called Electromagnetic Radiation at our faculty. Thanks are due not only to Bengt Lundborg for providing the inspiration to write this book, but also to professor CHRISTER WAHLBERG, and professor GÖRAN FÄLDT, both at the Department of Physics and Astronomy, Uppsala University, for insightful suggestions, to professor JOHN LEARNED, Department of Physics and Astronomy, University of Hawaii, for decisive encouragement at the early stage of this book project, to professor GERARDUS T'HOOFT, for recommending this book on his web page 'How to become a *good* theoretical physicist', and professor CECILIA JARLSKOG, Lund University, for pointing out a couple of errors and ambiguities.

I am particularly indebted to the late professor VITALIY LAZAREVICH GINZBURG, for his many fascinating and very elucidating lectures, comments and historical notes on plasma physics, electromagnetic radiation and cosmic electrodynamics while cruising up and down the Volga and Oka rivers in Russia at the ship-borne Russian-Swedish summer schools that were organised jointly by late professor LEV MIKHAILOVICH ERUKHIMOV and the author during the 1990's, and for numerous deep discussions over the years.

Helpful comments and suggestions for improvement from former PhD students TOBIA CARROZZI, ROGER KARLSSON, and MATTIAS WALDENVIK, as well as ANDERS ERIKSSON at the Swedish Institute of Space Physics in Uppsala and who have all taught Uppsala students on the material covered in this book, are gratefully acknowledged. Thanks are also due to the late HELMUT KOPKA, for more than twenty-five years a close friend and space physics colleague working at the Max-Planck-Institut für Aeronomie, Lindau, Germany, who not only taught me the practical aspects of the use of high-power electromagnetic radiation for studying space, but also some of the delicate aspects of typesetting in \TeX and \LaTeX .

In an attempt to encourage the involvement of other scientists and students in the making of this book, thereby trying to ensure its quality and scope to make it useful in higher university education anywhere in the world, it was produced as a World-Wide Web (WWW) project. This turned out to be a rather successful move. By making an electronic version of the book freely downloadable on the Internet, comments have been received from fellow physicists around the world. To judge from WWW 'hit' statistics, it seems that the book serves as a frequently used Internet resource. This way it is hoped that it will be particularly useful for students and research-

PREFACE TO THE FIRST EDITION

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ers working under financial or other circumstances that make it difficult to procure a printed copy of the book. I would like to thank all students and Internet users who have downloaded and commented on the book during its life on the World-Wide Web.

Uppsala, Sweden
December, 2008

BO THIDÉ
www.physics.irfu.se/~bt

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1

FOUNDATIONS OF CLASSICAL ELECTRODYNAMICS

The classical theory of electromagnetism deals with electric and magnetic fields and interactions caused by distributions of electric charges and currents. This presupposes that the concepts of localised electric charges and currents assume the validity of certain mathematical limiting processes in which it is considered possible for the charge and current distributions to be localised in infinitesimally small volumes of space.¹ Clearly, this is in contradistinction to electromagnetism on an atomistic scale, where charges and currents have to be described in a nonlocal quantum formalism. However, the limiting processes used in the classical domain, which, crudely speaking, assume that an elementary charge has a continuous distribution of charge density, will yield results that agree perfectly with experiments on non-atomistic scales, small or large.

It took the genius of JAMES CLERK MAXWELL to consistently unify the two distinct theories *electricity* and *magnetism* into a single super-theory, *electromagnetism* or *classical electrodynamics* (CED), and to realise that optics is a sub-field of this super-theory. Early in the 20th century, HENDRIK ANTOON LORENTZ took the electrodynamics theory further to the microscopic scale and also laid the foundation for the special theory of relativity, formulated in its full extent by ALBERT EINSTEIN in 1905. In the 1930's PAUL ADRIEN MAURICE DIRAC expanded electrodynamics to a more symmetric form, including magnetic as well as electric charges. With his relativistic quantum mechanics and field quantisation concepts, he also paved the way for the development of *quantum electrodynamics* (QED) for which RICHARD PHILLIPS FEYNMAN, JULIAN SEYMOUR SCHWINGER, and SIN-ITIRO TOMONAGA were

¹ Accepting the mere existence of an electrically charged particle requires some careful thinking. In his excellent book *Classical Charged Particles*, FRITZ ROHRlich writes

‘To what extent does it make sense to talk about an electron, say, in classical terms? These and similar questions clearly indicate that ignoring philosophy in physics means not understanding physics. For there is no theoretical physics without some philosophy; not admitting this fact would be self-deception.’

awarded the Nobel Prize in Physics in 1965. Around the same time, physicists such as SHELDON GLASHOW, ABDUS SALAM, and STEVEN WEINBERG were able to unify electrodynamics with the weak interaction theory, creating yet another super-theory, *electroweak theory*, an achievement which rendered them the Nobel Prize in Physics 1979. The modern theory of strong interactions, *quantum chromodynamics (QCD)*, is heavily influenced by QED.

In this introductory chapter we start with the force interactions in classical electrostatics and classical magnetostatics and introduce the static electric and magnetic fields to find two uncoupled systems of equations for them. Then we see how the conservation of electric charge and its relation to electric current leads to the dynamic connection between electricity and magnetism and how the two can be unified into classical electrodynamics. This theory is described by a system of coupled dynamic field equations—the microscopic Maxwell equations introduced by Lorentz—which we take as the axiomatic foundation for the theory of electromagnetic fields.

At the end of this chapter we present Dirac's symmetrised form of the Maxwell equations by introducing (hypothetical) magnetic charges and magnetic currents into the theory. Whereas not identified unambiguously as free entities in experiments yet, magnetic charges and currents make the theory much more appealing, for instance by allowing for duality transformations in a most natural way. Besides, in practical work, such as in antenna engineering, magnetic currents have proved to be a very useful concept. We shall make use of these symmetrised equations throughout the book.

1.1 Electrostatics

² The physicist and philosopher PIERRE DUHEM (1861–1916) once wrote:

‘The whole theory of electrostatics constitutes a group of abstract ideas and general propositions, formulated in the clear and concise language of geometry and algebra, and connected with one another by the rules of strict logic. This whole fully satisfies the reason of a French physicist and his taste for clarity, simplicity and order...’

The theory which describes physical phenomena related to the interaction between stationary electric charges or charge distributions in a finite space with stationary boundaries is called *electrostatics*. For a long time, electrostatics, under the name *electricity*, was considered an independent physical theory of its own, alongside other physical theories such as Magnetism, Mechanics, Optics, and Thermodynamics.²

1.1.1 Coulomb's law

It has been found experimentally that in classical electrostatics the interaction between stationary, electrically charged bodies can be described in terms of two-body mechanical forces. In the simple case depicted in figure 1.1 on the facing

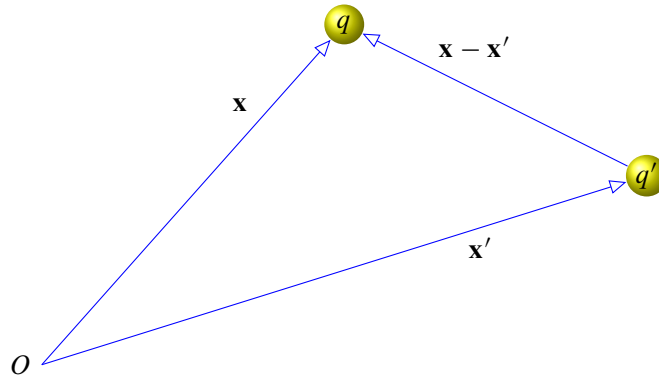


Figure 1.1: Coulomb's law describes how a static electric charge q , located at a point \mathbf{x} relative to the origin O , experiences an electrostatic force from a static electric charge q' located at \mathbf{x}' .

page, the force \mathbf{F} acting on the electrically charged particle with charge q located at \mathbf{x} , due to the presence of the charge q' located at \mathbf{x}' in an otherwise empty space, is given by *Coulomb's law*.³ This law postulates that \mathbf{F} is directed along the line connecting the two charges, repulsive for charges of equal signs and attractive for charges of opposite signs, and therefore can be formulated mathematically as

$$\mathbf{F}(\mathbf{x}) = \frac{qq'}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\frac{qq'}{4\pi\epsilon_0} \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{qq'}{4\pi\epsilon_0} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \quad (1.1)$$

where, in the last step, formula (F.119) on page 201 was used. In *SI units*, which we shall use throughout, the force \mathbf{F} is measured in Newton (N), the electric charges q and q' in Coulomb (C) or Ampere-seconds (A s), and the length $|\mathbf{x} - \mathbf{x}'|$ in metres (m). The constant $\epsilon_0 = 10^7/(4\pi c^2) \approx 8.8542 \times 10^{-12}$ Farad per metre (F m^{-1}) is the *vacuum permittivity* and $c \approx 2.9979 \times 10^8 \text{ m s}^{-1}$ is the speed of light in vacuum.⁴ In *CGS units*, $\epsilon_0 = 1/(4\pi)$ and the force is measured in dyne, electric charge in statcoulomb, and length in centimetres (cm).

1.1.2 The electrostatic field

Instead of describing the electrostatic interaction in terms of a 'force action at a distance', it turns out that for many purposes it is useful to introduce the concept of a field. Therefore we describe the electrostatic interaction in terms of a static vectorial *electric field* \mathbf{E}^{stat} defined by the limiting process

$$\mathbf{E}^{\text{stat}} \stackrel{\text{def}}{=} \lim_{q \rightarrow 0} \frac{\mathbf{F}}{q} \quad (1.2)$$

where \mathbf{F} is the electrostatic force, as defined in equation (1.1), from a net electric charge q' on the test particle with a small electric net electric charge q . Since the

³ CHARLES-AUGUSTIN DE COULOMB (1736–1806) was a French physicist who in 1775 published three reports on the forces between electrically charged bodies.

⁴ The notation c for speed comes from the Latin word 'celeritas' which means 'swiftness'. This notation seems to have been introduced by WILHELM EDUARD WEBER (1804–1891), and RUDOLF KOHLRAUSCH (1809–1858) and c is therefore sometimes referred to as *Weber's constant*. In all his works from 1907 and onward, ALBERT EINSTEIN (1879–1955) used c to denote the speed of light in vacuum.

⁵ In the preface to the first edition of the first volume of his book *A Treatise on Electricity and Magnetism*, first published in 1873, James Clerk Maxwell describes this in the following almost poetic manner:

‘For instance, Faraday, in his mind’s eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.’

⁶ In fact, vacuum exhibits a *quantum mechanical non-linearity* due to *vacuum polarisation effects* manifesting themselves in the momentary creation and annihilation of electron-positron pairs, but classically this non-linearity is negligible.

purpose of the limiting process is to assure that the test charge q does not distort the field set up by q' , the expression for \mathbf{E}^{stat} does not depend explicitly on q but only on the charge q' and the relative position vector $\mathbf{x} - \mathbf{x}'$. This means that we can say that any net electric charge produces an electric field in the space that surrounds it, regardless of the existence of a second charge anywhere in this space.⁵ However, in order to experimentally detect a charge, a second (test) charge that senses the presence of the first one, must be introduced.

Using equation (1.1) on page 3 and equation (1.2) on the previous page, and formula (F.118) on page 201, we find that the electrostatic field \mathbf{E}^{stat} at the *observation point* \mathbf{x} (also known as the *field point*), due to a field-producing electric charge q' at the *source point* \mathbf{x}' , is given by

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{q'}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\frac{q'}{4\pi\epsilon_0} \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{q'}{4\pi\epsilon_0} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \quad (1.3)$$

In the presence of several field producing discrete electric charges q'_i , located at the points \mathbf{x}'_i , $i = 1, 2, 3, \dots$, respectively, in an otherwise empty space, the assumption of linearity of vacuum⁶ allows us to superimpose their individual electrostatic fields into a total electrostatic field

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_i q'_i \frac{\mathbf{x} - \mathbf{x}'_i}{|\mathbf{x} - \mathbf{x}'_i|^3} \quad (1.4)$$

If the discrete electric charges are small and numerous enough, we can, in a continuum limit, assume that the total charge q' from an extended volume to be built up by local infinitesimal elemental charges dq' , each producing an elemental electric field

$$d\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{dq'}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \quad (1.5)$$

By introducing the *electric charge density* ρ , measured in C m^{-3} in SI units, at the point \mathbf{x}' within the volume element $d^3x' = dx'_1 dx'_2 dx'_3$ (measured in m^3), the elemental charge can be expressed as $dq' = d^3x' \rho(\mathbf{x}')$, and the elemental electrostatic field as

$$d\mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{d^3x' \rho(\mathbf{x}')}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \quad (1.6)$$

Integrating this over the entire source volume V' , we obtain

$$\begin{aligned} \mathbf{E}^{\text{stat}}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \nabla \int_{V'} d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \end{aligned} \quad (1.7)$$

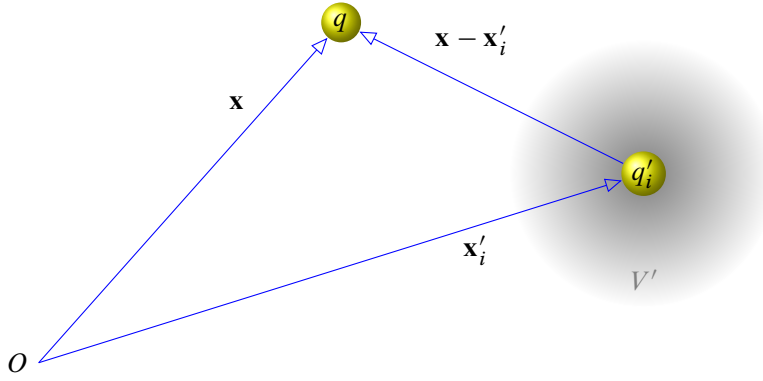


Figure 1.2: Coulomb's law for a distribution of individual charges q_i' localised within a volume V' of limited extent.

where we used formula (F.118) on page 201 and the fact that $\rho(\mathbf{x}')$ does not depend on the unprimed (field point) coordinates on which ∇ operates.

We emphasise that under the assumption of linear superposition, equation (1.7) on the facing page is valid for an arbitrary distribution of electric charges, including discrete charges, in which case ρ is expressed in terms of Dirac delta distributions:⁷

$$\rho(\mathbf{x}') = \sum_i q_i' \delta(\mathbf{x}' - \mathbf{x}_i') \quad (1.8)$$

as illustrated in figure 1.2. Inserting this expression into expression (1.7) on the facing page we recover expression (1.4) on the preceding page, as we should.

According to *Helmholtz's theorem*, a well-behaved vector field is completely known if one knows its divergence and curl.⁸ Taking the divergence of the general \mathbf{E}^{stat} expression for an arbitrary electric charge distribution, equation (1.7) on the facing page, and using the representation of the Dirac delta distribution given by formula (F.121) on page 201, one finds that

$$\begin{aligned} \nabla \cdot \mathbf{E}^{\text{stat}}(\mathbf{x}) &= \nabla \cdot \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \nabla \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \frac{1}{\epsilon_0} \int_{V'} d^3x' \rho(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') = \frac{\rho(\mathbf{x})}{\epsilon_0} \end{aligned} \quad (1.9)$$

which is the differential form of *Gauss's law of electrostatics*.

Since, according to formula (F.107) on page 201, $\nabla \times [\nabla \alpha(\mathbf{x})] \equiv \mathbf{0}$ for any

⁷ Since, by definition, the integral

$$\begin{aligned} &\int_{V'} d^3x' \delta(\mathbf{x}' - \mathbf{x}_i') \\ &\equiv \int_{V'} d^3x' \delta(x' - x_i') \\ &\quad \times \delta(y' - y_i') \delta(z' - z_i') = 1 \end{aligned}$$

is dimensionless, and x has the dimension m , the 3D Dirac delta distribution $\delta(\mathbf{x}' - \mathbf{x}_i')$ must have the dimension m^{-3} .

⁸ HERMANN LUDWIG FERDINAND VON HELMHOLTZ (1821–1894) was a physicist, physician and philosopher who contributed to wide areas of science, ranging from electrodynamics to ophthalmology.

\mathbb{R}^3 scalar field $\alpha(\mathbf{x})$, we immediately find that in electrostatics

$$\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \nabla \times \left(\nabla \int_{V'} d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = \mathbf{0} \quad (1.10)$$

i.e., that \mathbf{E}^{stat} is an *irrotational* field.

To summarise, electrostatics can be described in terms of two vector partial differential equations

$$\nabla \cdot \mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0} \quad (1.11a)$$

$$\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0} \quad (1.11b)$$

representing four scalar partial differential equations.

1.2 Magnetostatics

Whereas electrostatics deals with static electric charges, *magnetostatics* deals with static electric currents, *i.e.*, electric charges moving with constant speeds, and the interaction between these currents. Here we shall discuss this theory in some detail.

1.2.1 Ampère's law

Experiments on the interaction between two small electric current loops have shown that they interact via a mechanical force, much the same way that electric charges interact. In figure 1.3 on the next page, let \mathbf{F} denote such a force acting on a small loop C , with tangential line element $d\mathbf{l}$, located at \mathbf{x} and carrying a current I in the direction of $d\mathbf{l}$, due to the presence of a small loop C' , with tangential line element $d\mathbf{l}'$, located at \mathbf{x}' and carrying a current I' in the direction of $d\mathbf{l}'$ in otherwise empty space. According to *Ampère's law* this force is given by the expression⁹

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= \frac{\mu_0 II'}{4\pi} \oint_C d\mathbf{l} \times \left(\oint_{C'} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \right) \\ &= -\frac{\mu_0 II'}{4\pi} \oint_C d\mathbf{l} \times \left[\oint_{C'} d\mathbf{l}' \times \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \end{aligned} \quad (1.12)$$

In SI units, $\mu_0 = 4\pi \times 10^{-7} \approx 1.2566 \times 10^{-6} \text{ H m}^{-1}$ is the *vacuum permeability*. From the definition of ϵ_0 and μ_0 (in SI units) we observe that

$$\epsilon_0 \mu_0 = \frac{10^7}{4\pi c^2} (\text{F m}^{-1}) \times 4\pi \times 10^{-7} (\text{H m}^{-1}) = \frac{1}{c^2} (\text{s}^2 \text{m}^{-2}) \quad (1.13)$$

⁹ ANDRÉ-MARIE AMPÈRE (1775–1836) was a French mathematician and physicist who, only a few days after he learned about the findings by the Danish physicist and chemist HANS CHRISTIAN ØRSTED (1777–1851) regarding the magnetic effects of electric currents, presented a paper to the Académie des Sciences in Paris, introducing the law that now bears his name.

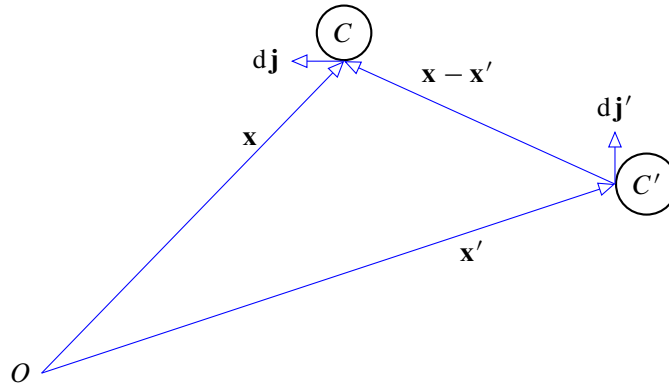


Figure 1.3: Ampère's law describes how a small loop C , carrying a static electric current density element $d\mathbf{j}$ at \mathbf{x} , experiences a magnetostatic force from a small loop C' , carrying a static electric current density element $d\mathbf{j}'$ located at \mathbf{x}' . The loops can have arbitrary shapes as long as they are simple and closed.

which is a most useful relation.

At first glance, equation (1.12) on the facing page may appear asymmetric in terms of the loops and therefore be a force law that does not obey Newton's third law. However, by applying the vector triple product 'bac-cab' formula (F.73) on page 200, we can rewrite (1.12) as

$$\begin{aligned} \mathbf{F}(\mathbf{x}) = & -\frac{\mu_0 I I'}{4\pi} \oint_{C'} d\mathbf{l}' \oint_C d\mathbf{l} \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ & - \frac{\mu_0 I I'}{4\pi} \oint_C \oint_{C'} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{l} \cdot d\mathbf{l}' \end{aligned} \quad (1.14)$$

Since the integrand in the first integral is an exact differential, this integral vanishes and we can rewrite the force expression, formula (1.12) on the facing page, in the following symmetric way

$$\mathbf{F}(\mathbf{x}) = -\frac{\mu_0 I I'}{4\pi} \oint_C \oint_{C'} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{l} \cdot d\mathbf{l}' \quad (1.15)$$

which clearly exhibits the expected interchange symmetry between loops C and C' .

1.2.2 The magnetostatic field

In analogy with the electrostatic case, we may attribute the magnetostatic interaction to a static vectorial *magnetic field* \mathbf{B}^{stat} . The elemental \mathbf{B}^{stat} from the elemental current element $d\mathbf{I}' = I' d\mathbf{l}'$ is defined as

$$d\mathbf{B}^{\text{stat}}(\mathbf{x}) \stackrel{\text{def}}{=} \frac{\mu_0}{4\pi} d\mathbf{I}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{\mu_0 I'}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \quad (1.16)$$

which expresses the small element $d\mathbf{B}^{\text{stat}}(\mathbf{x})$ of the static magnetic field set up at the field point \mathbf{x} by a small line current element $d\mathbf{I}' = I'd\mathbf{l}'$ of static current I' directed along the line element $d\mathbf{l}'$ at the source point \mathbf{x}' . The SI unit for the magnetic field, sometimes called the *magnetic flux density* or *magnetic induction*, is Tesla (T).

If we generalise expression (1.16) on the previous page to an integrated steady state *electric current density* $\mathbf{j}(\mathbf{x})$, measured in A m^{-2} in SI units, we can write $I'd\mathbf{l} = d\mathbf{I}' = d^3x' \mathbf{j}(\mathbf{x}')$, and we obtain *Biot-Savart's law*

$$\begin{aligned}\mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -\frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}(\mathbf{x}') \times \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \frac{\mu_0}{4\pi} \nabla \times \int_{V'} d^3x' \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\end{aligned}\quad (1.17)$$

where we used formula (F.118) on page 201, formula (F.97) on page 200, and the fact that $\mathbf{j}(\mathbf{x}')$ does not depend on the unprimed coordinates on which ∇ operates. Comparing equation (1.7) on page 4 with equation (1.17) above, we see that there exists a close analogy between the expressions for \mathbf{E}^{stat} and \mathbf{B}^{stat} but that they differ in their vectorial characteristics. With this definition of \mathbf{B}^{stat} , equation (1.12) on page 6 may be written

$$\mathbf{F}(\mathbf{x}) = I \oint_C d\mathbf{l} \times \mathbf{B}^{\text{stat}}(\mathbf{x}) \quad (1.18)$$

In order to assess the properties of \mathbf{B}^{stat} , we determine its divergence and curl. Taking the divergence of both sides of equation (1.17) above and utilising formula (F.106) on page 201, we obtain

$$\nabla \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \cdot \left(\nabla \times \int_{V'} d^3x' \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = 0 \quad (1.19)$$

since, according to formula (F.106) on page 201, $\nabla \cdot (\nabla \times \mathbf{a})$ vanishes for any vector field $\mathbf{a}(\mathbf{x})$.

With the use of the operator ‘bac-cab’ rule, formula (F.103) on page 201, the curl of equation (1.17) above can be written

$$\begin{aligned}\nabla \times \mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \nabla \times \left(\nabla \times \int_{V'} d^3x' \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}(\mathbf{x}') \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &\quad + \frac{\mu_0}{4\pi} \int_{V'} d^3x' [\mathbf{j}(\mathbf{x}') \cdot \nabla'] \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)\end{aligned}\quad (1.20)$$

In the first of the two integrals on the right-hand side, we use the representation of the Dirac delta function given in formula (F.121) on page 201, and integrate the second integral by parts, utilising formula (F.94) on page 200, as follows:

$$\begin{aligned}
 & \int_{V'} d^3x' [\mathbf{j}(\mathbf{x}') \cdot \nabla'] \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
 &= \hat{\mathbf{x}}_k \int_{V'} d^3x' \nabla' \cdot \left\{ \mathbf{j}(\mathbf{x}') \left[\frac{\partial}{\partial x'_k} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \right\} \\
 &\quad - \int_{V'} d^3x' [\nabla' \cdot \mathbf{j}(\mathbf{x}')] \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
 &= \hat{\mathbf{x}}_k \int_{S'} d^2x' \hat{\mathbf{n}}' \cdot \mathbf{j}(\mathbf{x}') \frac{\partial}{\partial x'_k} \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
 &\quad - \int_{V'} d^3x' [\nabla' \cdot \mathbf{j}(\mathbf{x}')] \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)
 \end{aligned} \tag{1.21}$$

We note that the first integral in the result, obtained by applying Gauss's theorem, vanishes when integrated over a large sphere far away from the localised source $\mathbf{j}(\mathbf{x}')$, and that the second integral vanishes because $\nabla \cdot \mathbf{j} = 0$ for stationary currents (no charge accumulation in space). The net result is simply

$$\nabla \times \mathbf{B}^{\text{stat}}(\mathbf{x}) = \mu_0 \int_{V'} d^3x' \mathbf{j}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.22}$$

1.3 Electrodynamics

As we saw in the previous sections, the laws of electrostatics and magnetostatics can be summarised in two pairs of time-independent, uncoupled partial differential equations, namely the *equations of classical electrostatics*

$$\nabla \cdot \mathbf{E}^{\text{stat}}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0} \tag{1.23a}$$

$$\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) = \mathbf{0} \tag{1.23b}$$

and the *equations of classical magnetostatics*

$$\nabla \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = 0 \tag{1.24a}$$

$$\nabla \times \mathbf{B}^{\text{stat}}(\mathbf{x}) = \mu_0 \mathbf{j}(\mathbf{x}) \tag{1.24b}$$

Since there is nothing *a priori* which connects \mathbf{E}^{stat} directly with \mathbf{B}^{stat} , we must consider classical electrostatics and classical magnetostatics as two separate and mutually independent physical theories.

However, when we include time-dependence, these theories are unified into a single super-theory, *classical electrodynamics*. This unification of the theories of electricity and magnetism can be inferred from two empirically established facts:

1. Electric charge is a conserved quantity and electric current is a transport of electric charge. As we shall see, this fact manifests itself in the equation of continuity and, as a consequence, in *Maxwell's displacement current*.
2. A change in the magnetic flux through a loop will induce an electromotive force electric field in the loop. This is the celebrated *Faraday's law of induction*.

1.3.1 The indestructibility of electric charge

Let $\mathbf{j}(t, \mathbf{x})$ denote the time-dependent electric current density. In the simplest case it can be defined as $\mathbf{j} = \mathbf{v}\rho$ where \mathbf{v} is the velocity of the electric charge density ρ .¹⁰

¹⁰ A more accurate model is to assume that the individual charge elements obey some distribution function that describes their local variation of velocity in space and time.

The *electric charge conservation law* can be formulated in the *equation of continuity for electric charge*

$$\frac{\partial \rho(t, \mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}(t, \mathbf{x}) = 0 \quad (1.25)$$

which states that the time rate of change of electric charge $\rho(t, \mathbf{x})$ is balanced by a divergence in the electric current density $\mathbf{j}(t, \mathbf{x})$, *i.e.* the influx of charge.

1.3.2 Maxwell's displacement current

We recall from the derivation of equation (1.22) on the preceding page that there we used the fact that in magnetostatics $\nabla \cdot \mathbf{j}(\mathbf{x}) = 0$. In the case of non-stationary sources and fields, we must, in accordance with the continuity equation (1.25) above, set $\nabla \cdot \mathbf{j}(t, \mathbf{x}) = -\partial \rho(t, \mathbf{x}) / \partial t$. Doing so, and formally repeating the steps in the derivation of equation (1.22) on the preceding page, we would obtain the formal result

$$\begin{aligned} \nabla \times \mathbf{B}(t, \mathbf{x}) &= \mu_0 \int_{V'} d^3x' \mathbf{j}(t, \mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \\ &\quad + \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int_{V'} d^3x' \rho(t, \mathbf{x}') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \mu_0 \mathbf{j}(t, \mathbf{x}) + \mu_0 \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}(t, \mathbf{x}) \end{aligned} \quad (1.26)$$

¹¹ Later, we will need to consider this generalisation and formal identification further.

where, in the last step, we have assumed that a generalisation of equation (1.7) on page 4 to time-varying fields allows us to make the identification¹¹

$$\begin{aligned}
 & \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial t} \int_{V'} d^3x' \rho(t, \mathbf{x}') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
 &= \frac{\partial}{\partial t} \left[-\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho(t, \mathbf{x}') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \\
 &= \frac{\partial}{\partial t} \left[-\frac{1}{4\pi\epsilon_0} \nabla \int_{V'} d^3x' \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] = \frac{\partial}{\partial t} \mathbf{E}(t, \mathbf{x})
 \end{aligned} \tag{1.27}$$

The result is Maxwell's source equation for the \mathbf{B} field

$$\nabla \times \mathbf{B}(t, \mathbf{x}) = \mu_0 \left(\mathbf{j}(t, \mathbf{x}) + \frac{\partial}{\partial t} \epsilon_0 \mathbf{E}(t, \mathbf{x}) \right) = \mu_0 \mathbf{j}(t, \mathbf{x}) + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \mathbf{E}(t, \mathbf{x}) \tag{1.28}$$

where $\epsilon_0 \partial \mathbf{E}(t, \mathbf{x}) / \partial t$ is the famous *displacement current*. This, at the time, unobserved current was introduced by Maxwell, in a stroke of genius, in order to make also the right-hand side of this equation divergence-free when $\mathbf{j}(t, \mathbf{x})$ is assumed to represent the density of the total electric current. This total current which can be split up into ‘ordinary’ conduction currents, polarisation currents and magnetisation currents. This will be discussed in subsection 9.1 on page 170. The displacement current behaves like a current density flowing in free space. As we shall see later, its existence has far-reaching physical consequences as it predicts that such physical observables as electromagnetic energy, linear momentum, and angular momentum can be transmitted over very long distances, even through empty space.

1.3.3 Electromotive force

If an electric field $\mathbf{E}(t, \mathbf{x})$ is applied to a conducting medium, a current density $\mathbf{j}(t, \mathbf{x})$ will be set up in this medium. But also mechanical, hydrodynamical and chemical processes can give rise to electric currents. Under certain physical conditions, and for certain materials, one can sometimes assume that a linear relationship exists between the electric current density \mathbf{j} and \mathbf{E} . This approximation is called *Ohm's law*:¹²

$$\mathbf{j}(t, \mathbf{x}) = \sigma \mathbf{E}(t, \mathbf{x}) \tag{1.29}$$

where σ is the *electric conductivity* (S m^{-1}). In the case of an anisotropic conductor, σ is a tensor.

We can view Ohm's law, equation (1.29) above, as the first term in a Taylor expansion of a general law $\mathbf{j}[\mathbf{E}(t, \mathbf{x})]$. This general law incorporates *non-linear*

¹² In semiconductors this approximation is in general applicable only for a limited range of \mathbf{E} .

effects such as *frequency mixing* and *frequency conversion*. Examples of media which are highly non-linear are semiconductors and plasma. We draw the attention to the fact that even in cases when the linear relation between \mathbf{E} and \mathbf{j} is a good approximation, we still have to use Ohm's law with care. The conductivity σ is, in general, time-dependent (*temporal dispersive media*) but then it is often the case that equation (1.29) on the preceding page is valid for each individual Fourier (spectral) component of the field.

If the current is caused by an applied electric field $\mathbf{E}(t, \mathbf{x})$, this electric field will exert work on the charges in the medium and, unless the medium is superconducting, there will be some energy loss. The time rate at which this energy is expended is $\mathbf{j} \cdot \mathbf{E}$ per unit volume (W m^{-3}). If \mathbf{E} is irrotational (conservative), \mathbf{j} will decay away with time. Stationary currents therefore require that an electric field due to an *electromotive force (EMF)* is present. In the presence of such a field \mathbf{E}^{emf} , Ohm's law, equation (1.29) on the previous page, takes the form

$$\mathbf{j} = \sigma(\mathbf{E}^{\text{stat}} + \mathbf{E}^{\text{emf}}) \quad (1.30)$$

The electromotive force is defined as

$$\mathcal{E} = \oint_C d\mathbf{l} \cdot (\mathbf{E}^{\text{stat}} + \mathbf{E}^{\text{emf}}) \quad (1.31)$$

where $d\mathbf{l}$ is a tangential line element of the closed loop C .¹³

¹³ The term 'electromotive force' is something of a misnomer since \mathcal{E} represents a voltage, *i.e.*, its SI dimension is V.

1.3.4 Faraday's law of induction

In subsection 1.1.2 we derived the differential equations for the electrostatic field. Specifically, on page 6 we derived equation (1.10) stating that $\nabla \times \mathbf{E}^{\text{stat}} = \mathbf{0}$ and hence that \mathbf{E}^{stat} is a *conservative field* (it can be expressed as a gradient of a scalar field). This implies that the closed line integral of \mathbf{E}^{stat} in equation (1.31) above vanishes and that this equation becomes

$$\mathcal{E} = \oint_C d\mathbf{l} \cdot \mathbf{E}^{\text{emf}} \quad (1.32)$$

It has been established experimentally that a non-conservative EMF field is produced in a closed circuit C at rest if the magnetic flux through this circuit varies with time. This is formulated in *Faraday's law* which, in Maxwell's generalised form, reads

$$\begin{aligned} \mathcal{E}(t) &= \oint_C d\mathbf{l} \cdot \mathbf{E}(t, \mathbf{x}) = -\frac{d}{dt} \Phi_m(t) \\ &= -\frac{d}{dt} \int_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{B}(t, \mathbf{x}) = -\int_S d^2x \, \hat{\mathbf{n}} \cdot \frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x}) \end{aligned} \quad (1.33)$$

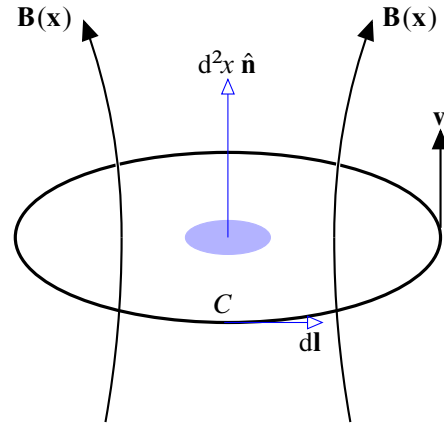


Figure 1.4: A loop C which moves with velocity \mathbf{v} in a spatially varying magnetic field $\mathbf{B}(\mathbf{x})$ will sense a varying magnetic flux during the motion.

where Φ_m is the *magnetic flux* and S is the surface encircled by C which can be interpreted as a generic stationary ‘loop’ and not necessarily as a conducting circuit. Application of Stokes’ theorem on this integral equation, transforms it into the differential equation

$$\nabla \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x}) \quad (1.34)$$

which is valid for arbitrary variations in the fields and constitutes the Maxwell equation which explicitly connects electricity with magnetism.

Any change of the magnetic flux Φ_m will induce an EMF. Let us therefore consider the case, illustrated in figure 1.4, when the ‘loop’ is moved in such a way that it encircles a magnetic field which varies during the movement. The total time derivative is evaluated according to the well-known operator formula

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla \quad (1.35)$$

which follows immediately from the multivariate chain rule for the differentiation of an arbitrary differentiable function $f(t, \mathbf{x}(t))$. Here, $d\mathbf{x}/dt$ describes a chosen path in space. We shall choose the flow path which means that $d\mathbf{x}/dt = \mathbf{v}$ and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (1.36)$$

where, in a continuum picture, \mathbf{v} is the fluid velocity. For this particular choice, the *convective derivative* $d\mathbf{x}/dt$ is usually referred to as the *material derivative* and is denoted $D\mathbf{x}/Dt$.

Applying the rule (1.36) to Faraday’s law, equation (1.33) on the preceding page, we obtain

$$\mathcal{E}(t) = -\frac{d}{dt} \int_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{B} = -\int_S d^2x \, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \int_S d^2x \, \hat{\mathbf{n}} \cdot (\mathbf{v} \cdot \nabla) \mathbf{B} \quad (1.37)$$

Furthermore, taking the divergence of equation (1.34) on the preceding page, we see that

$$\nabla \cdot \frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x}) = \frac{\partial}{\partial t} \nabla \cdot \mathbf{B}(t, \mathbf{x}) = -\nabla \cdot [\nabla \times \mathbf{E}(t, \mathbf{x})] = 0 \quad (1.38)$$

where in the last step formula (F.106) on page 201 was used. Since this is true $\forall t$, we conclude that

$$\nabla \cdot \mathbf{B}(t, \mathbf{x}) = 0 \quad (1.39)$$

also for time-varying fields; this is in fact one of the Maxwell equations. Using this result and formula (F.98) on page 200, we find that

$$\nabla \times (\mathbf{B} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{B} \quad (1.40)$$

since, during spatial differentiation, \mathbf{v} is to be considered as constant. This allows us to rewrite equation (1.37) on the previous page in the following way:

$$\begin{aligned} \mathcal{E}(t) &= \oint_C \mathbf{dl} \cdot \mathbf{E}^{\text{emf}} = -\frac{d}{dt} \int_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{B} \\ &= -\int_S d^2x \, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \int_S d^2x \, \hat{\mathbf{n}} \cdot \nabla \times (\mathbf{B} \times \mathbf{v}) \end{aligned} \quad (1.41)$$

With Stokes' theorem applied to the last integral, we finally get

$$\mathcal{E}(t) = \oint_C \mathbf{dl} \cdot \mathbf{E}^{\text{emf}} = -\int_S d^2x \, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} - \oint_C \mathbf{dl} \cdot (\mathbf{B} \times \mathbf{v}) \quad (1.42)$$

or, rearranging the terms,

$$\oint_C \mathbf{dl} \cdot (\mathbf{E}^{\text{emf}} - \mathbf{v} \times \mathbf{B}) = -\int_S d^2x \, \hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \quad (1.43)$$

where \mathbf{E}^{emf} is the field which is induced in the 'loop', *i.e.*, in the moving system. The application of Stokes' theorem 'in reverse' on equation (1.43) yields

$$\nabla \times (\mathbf{E}^{\text{emf}} - \mathbf{v} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.44)$$

An observer in a fixed frame of reference measures the electric field

$$\mathbf{E} = \mathbf{E}^{\text{emf}} - \mathbf{v} \times \mathbf{B} \quad (1.45)$$

and an observer in the moving frame of reference measures the following *Lorentz force* on a charge q

$$\mathbf{F} = q\mathbf{E}^{\text{emf}} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \quad (1.46)$$

corresponding to an 'effective' electric field in the 'loop' (moving observer)

$$\mathbf{E}^{\text{emf}} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad (1.47)$$

Hence, we conclude that for a stationary observer, the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.48)$$

is indeed valid even if the 'loop' is moving.

1.3.5 The microscopic Maxwell equations

We are now able to collect the results from the above considerations and formulate the equations of classical electrodynamics valid for arbitrary variations in time and space of the coupled electric and magnetic fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$. The equations are, in *SI units*,¹⁴

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (1.49a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.49b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.49c)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (1.49d)$$

In these equations $\rho = \rho(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric charge density, with contributions from free as well as induced (polarisation) charges. Likewise, $\mathbf{j} = \mathbf{j}(t, \mathbf{x})$ represents the total, possibly both time and space dependent, electric current density, with contributions from conduction currents (motion of free charges) as well as all atomistic (polarisation and magnetisation) currents. As they stand, the equations therefore incorporate the classical interaction between all electric charges and currents, free or bound, in the system and are called *Maxwell's microscopic equations*. They were first formulated by Lorentz and therefore another name often used for them is the *Maxwell-Lorentz equations*. Together with the appropriate *constitutive relations*, which relate ρ and \mathbf{j} to the fields, and the initial and boundary conditions pertinent to the physical situation at hand, they form a system of well-posed partial differential equations which completely determine \mathbf{E} and \mathbf{B} .

1.3.6 Dirac's symmetrised Maxwell equations

If we look more closely at the microscopic Maxwell equations (1.49), we see that they exhibit a certain, albeit not complete, symmetry. Dirac therefore made the *ad hoc* assumption that there exist *magnetic monopoles* represented by a *magnetic charge density*, which we denote by $\rho^m = \rho^m(t, \mathbf{x})$, and a *magnetic current density*, which we denote by $\mathbf{j}^m = \mathbf{j}^m(t, \mathbf{x})$.¹⁵

With these new hypothetical physical entities included in the theory, and with the electric charge density denoted ρ^e and the electric current density denoted \mathbf{j}^e , the Maxwell equations will be symmetrised into the following two scalar and

¹⁴ In *CGS units* the microscopic Maxwell equations are

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

in *Heaviside-Lorentz units* (one of several *natural units*)

$$\nabla \cdot \mathbf{E} = \rho$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

and in *Planck units* (another set of natural units)

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}$$

¹⁵ JULIAN SEYMOUR SCHWINGER (1918–1994) once put it:

‘... there are strong theoretical reasons to believe that magnetic charge exists in nature, and may have played an important role in the development of the Universe. Searches for magnetic charge continue at the present time, emphasising that electromagnetism is very far from being a closed object’.

The magnetic monopole was first postulated by PIERRE CURIE (1859–1906) and inferred from experiments in 2009.

two coupled vectorial partial differential equations (SI units):

$$\nabla \cdot \mathbf{E} = \frac{\rho^e}{\varepsilon_0} \quad (1.50a)$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{j}^m - \frac{\partial \mathbf{B}}{\partial t} \quad (1.50b)$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^m \quad (1.50c)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^e + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1.50d)$$

We shall call these equations *Dirac's symmetrised Maxwell equations* or the *electromagnetodynamic equations*.

Taking the divergence of (1.50b), we find that

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) - \mu_0 \nabla \cdot \mathbf{j}^m \equiv 0 \quad (1.51)$$

where we used the fact that, according to formula (F.106) on page 201, the divergence of a curl always vanishes. Using (1.50c) to rewrite this relation, we obtain the *equation of continuity for magnetic charge*

$$\frac{\partial \rho^m}{\partial t} + \nabla \cdot \mathbf{j}^m = 0 \quad (1.52)$$

which has the same form as that for the electric charges (electric monopoles) and currents, equation (1.25) on page 10.

EXAMPLE 1.1 ▷Faraday's law derived from the assumed conservation of magnetic charge

POSTULATE 1.1 (INDESTRUCTIBILITY OF MAGNETIC CHARGE) *Magnetic charge exists and is indestructible in the same way that electric charge exists and is indestructible.*

In other words, we *postulate* that there exists an equation of continuity for magnetic charges:

$$\frac{\partial \rho^m(t, \mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}^m(t, \mathbf{x}) = 0 \quad (1.53)$$

Use this postulate and Dirac's symmetrised form of Maxwell's equations to derive Faraday's law.

The assumption of the existence of magnetic charges suggests a Coulomb-like law for magnetic fields:

$$\begin{aligned} \mathbf{B}^{\text{stat}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int_{V'} d^3x' \rho^m(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\frac{\mu_0}{4\pi} \int_{V'} d^3x' \rho^m(\mathbf{x}') \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{\mu_0}{4\pi} \nabla \int_{V'} d^3x' \frac{\rho^m(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \end{aligned} \quad (1.54)$$

[cf. equation (1.7) on page 4 for \mathbf{E}^{stat}] and, if magnetic currents exist, a Biot-Savart-like

law for electric fields [cf. equation (1.17) on page 8 for \mathbf{B}^{stat}]:

$$\begin{aligned}\mathbf{E}^{\text{stat}}(\mathbf{x}) &= -\frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}^{\text{m}}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}^{\text{m}}(\mathbf{x}') \times \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{\mu_0}{4\pi} \nabla \times \int_{V'} d^3x' \frac{\mathbf{j}^{\text{m}}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\end{aligned}\quad (1.55)$$

Taking the curl of the latter and using the operator ‘bac-cab’ rule, formula (F.98) on page 200, we find that

$$\begin{aligned}\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) &= -\frac{\mu_0}{4\pi} \nabla \times \left(\nabla \times \int_{V'} d^3x' \frac{\mathbf{j}^{\text{m}}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = \\ &= \frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}^{\text{m}}(\mathbf{x}') \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) - \frac{\mu_0}{4\pi} \int_{V'} d^3x' [\mathbf{j}^{\text{m}}(\mathbf{x}') \cdot \nabla'] \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)\end{aligned}\quad (1.56)$$

Comparing with equation (1.20) on page 8 for \mathbf{E}^{stat} and the evaluation of the integrals there, we realise that

$$\nabla \times \mathbf{E}^{\text{stat}}(\mathbf{x}) = -\mu_0 \int_{V'} d^3x' \mathbf{j}^{\text{m}}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') = -\mu_0 \mathbf{j}^{\text{m}}(\mathbf{x}) \quad (1.57)$$

by analogy.

It is intriguing to note that if we assume that formula (1.55) above is valid also for time-varying magnetic currents, then, with the use of the representation of the Dirac delta function, equation (F.121) on page 201, the equation of continuity for magnetic charge, equation (1.52) on the preceding page, and the assumption of the generalisation of equation (1.54) on the facing page to time-dependent magnetic charge distributions, we obtain, at least formally,

$$\begin{aligned}\nabla \times \mathbf{E}(t, \mathbf{x}) &= -\mu_0 \int_{V'} d^3x' \mathbf{j}^{\text{m}}(t, \mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') \\ &\quad - \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \int_{V'} d^3x' \rho^{\text{m}}(t, \mathbf{x}') \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\mu_0 \mathbf{j}^{\text{m}}(t, \mathbf{x}) - \frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x})\end{aligned}\quad (1.58)$$

[cf. equation (1.26) on page 10] which we recognise as equation (1.50b) on the facing page. A transformation of this electromagnetodynamic result by rotating into the ‘electric realm’ of charge space, thereby letting \mathbf{j}^{m} tend to zero, yields the electrodynamic equation (1.50b) on the preceding page, *i.e.*, the Faraday law in the ordinary Maxwell equations. This process would also provide an alternative interpretation of the term $\partial \mathbf{B} / \partial t$ as a *magnetic displacement current*, dual to the *electric displacement current* [cf. equation (1.28) on page 11].

By postulating the indestructibility of a hypothetical magnetic charge, and assuming a direct extension of results from statics to dynamics, we have been able to replace Faraday’s experimental results on electromotive forces and induction in loops as a foundation for the Maxwell equations by a more fundamental one. At first sight, this result seems to be in conflict with the concept of retardation. Therefore a more detailed analysis of it is required. This analysis is left to the reader.

—End of example 1.1◀

1.4 Bibliography

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2

ELECTROMAGNETIC FIELDS AND WAVES

As a first step in the study of the dynamical properties of the classical electromagnetic field, we shall in this chapter, as alternatives to the first-order Maxwell-Lorentz equations, derive a set of second-order differential equations for the fields \mathbf{E} and \mathbf{B} . It turns out that these second-order equations are wave equations for \mathbf{E} and \mathbf{B} , indicating that electromagnetic wave modes are very natural and common manifestations of classical electrodynamics.¹

But before deriving these alternatives to the Maxwell-Lorentz equations, we shall discuss the mathematical techniques of making use of complex variables to represent real physical observables in order to simplify the mathematical work involved. In this chapter we will also describe how to make use of the single spectral component (Fourier component) technique, which simplifies the algebra, at the same time as it clarifies the physical content.

2.1 Axiomatic classical electrodynamics

In chapter 1 we described the historical route which led to the formulation of the microscopic Maxwell equations. From now on we shall consider these equations as *postulates*, *i.e.*, as the *axiomatic foundation of classical electrodynamics*.² As such, they describe, in scalar and vector-differential-equation form, the behaviour in time $t \in \mathbb{R}^1$ and in space $\mathbf{x} \in \mathbb{R}^3$ of the electric and magnetic fields

¹ In 1864, in a lecture at the Royal Society of London, JAMES CLERK MAXWELL (1831–1879) himself said:

‘We have strong reason to conclude that light itself—including radiant heat and other radiation, if any—is an electromagnetic disturbance in the form of waves propagated through the electro-magnetic field according to electro-magnetic laws.’

² FRITZ ROHRLICH writes in *Classical Charged Particles* that

‘A physical theory, in the narrow sense of the word, is a logical structure based on assumptions and definitions which permits one to predict the outcome of a maximum number of different experiments on the basis of a minimum number of postulates. One usually desires the postulates (or axioms) to be as self-evident as possible; one demands simplicity, beauty, and even elegance.’

$\mathbf{E}(t, \mathbf{x}) \in \mathbb{R}^3$ and $\mathbf{B}(t, \mathbf{x}) \in \mathbb{R}^3$, respectively [cf. equations (1.49) on page 15]:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \in \mathbb{R}^1 \quad (\text{Gauss's law}) \quad (2.1a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \in \mathbb{R}^3 \quad (\text{Faraday's law}) \quad (2.1b)$$

$$\nabla \cdot \mathbf{B} = 0 \in \mathbb{R}^1 \quad (\text{No magnetic charges}) \quad (2.1c)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \in \mathbb{R}^3 \quad (\text{Maxwell's law}) \quad (2.1d)$$

We reiterate that in these equations $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$ are the *total* charge and current densities, respectively. Hence, they are considered microscopic in the sense that *all* charges and currents, including the intrinsic ones in matter, such as bound charges in atoms and molecules, as well as magnetisation currents in magnetic material, are included (but macroscopic in the sense that quantum effects are neglected). These charge and current densities may have prescribed arbitrary time and space dependencies and be considered the sources of the fields, but they may also be generated by the fields. Despite these dual rôles, we shall refer to them as the *source terms* of the microscopic Maxwell equations and the two equations where they appear as the *Maxwell-Lorentz source equations*.

2.2 Complex notation and physical observables

In order to simplify the mathematical treatment, we shall frequently allow the mathematical variables representing the fields, the charge and current densities, and other physical quantities to be analytically continued into the complex domain. However, when we use such a *complex notation* we must be very careful how to interpret the results derived within this notation. This is because every *physical observable* is, by definition, real-valued.³ Consequently, the mathematical expression for the observable under consideration must also be real-valued to be physically meaningful.

If a physical scalar variable, or a component of a physical vector or tensor, is represented mathematically by the complex-valued number ψ , *i.e.*, $\psi \in \mathbb{C}$, then in classical electrodynamics (in fact, in classical physics as a whole), one makes the identification $\psi^{\text{observable}} = \text{Re} \{ \psi^{\text{mathematical}} \}$. Therefore, it is always understood that one must take the real part of a complex mathematical variable in order for it to represent a classical physical observable.⁴

For mathematical convenience and ease of calculation, we shall in the following regularly use—and tacitly assume—complex notation, stating explicitly when we do not. One convenient property of the complex notation is that differentiations often become trivial to perform. However, care must always be

³ A physical observable is something that can, at least in principle, be ultimately reduced to an input to the human sensory system. In other words, physical observables quantify (our perception of) the physical reality and as such they must, of course, be described by real-valued quantities.

⁴ This is at variance with quantum physics, where $\psi^{\text{observable}} = |\psi^{\text{mathematical}}|$. Letting $*$ denote complex conjugation, the real part can be written $\text{Re} \{ \psi \} = \frac{1}{2}(\psi + \psi^*)$, *i.e.*, as the arithmetic mean of ψ and its complex conjugate ψ^* . Similarly, the magnitude can be written $|\psi| = (\psi \psi^*)^{1/2}$, *i.e.*, as the geometric mean of ψ and ψ^* . Under certain conditions, also the imaginary part corresponds to a physical observable.

exercised. A typical situation being when products of two or more quantities are calculated since, for instance, for two complex-valued variables ψ_1 and ψ_2 we know that $\text{Re}\{\psi_1\psi_2\} \neq \text{Re}\{\psi_1\}\text{Re}\{\psi_2\}$. On the other hand, $(\psi_1\psi_2)^* = \psi_1^*\psi_2^*$.

2.2.1 Physical observables and averages

As just mentioned, it is important to be aware of the limitations of complex representation of real observables when evaluating products of mathematical variables that represent physical quantities. Let us, for example, consider two physical vector fields $\mathbf{a}(t, \mathbf{x})$ and $\mathbf{b}(t, \mathbf{x})$ that are represented by their Fourier components, $\mathbf{a}_0(\mathbf{x})\exp(-i\omega t)$ and $\mathbf{b}_0(\mathbf{x})\exp(-i\omega t)$, *i.e.*, by vectors in (a domain of) 3D complex space \mathbb{C}^3 . Furthermore, let \circ be a binary operator for these vectors, representing either the scalar product operator (\cdot), the vector product operator (\times), the dyadic product operator (juxtaposition), or the outer tensor operator (\otimes). Then we make the interpretation

$$\mathbf{a}(t, \mathbf{x}) \circ \mathbf{b}(t, \mathbf{x}) = \text{Re}\{\mathbf{a}\} \circ \text{Re}\{\mathbf{b}\} = \text{Re}\{\mathbf{a}_0(\mathbf{x})e^{-i\omega t}\} \circ \text{Re}\{\mathbf{b}_0(\mathbf{x})e^{-i\omega t}\} \quad (2.2)$$

We can express the real part of the complex vector \mathbf{a} as

$$\text{Re}\{\mathbf{a}\} = \text{Re}\{\mathbf{a}_0(\mathbf{x})e^{-i\omega t}\} = \frac{1}{2}[\mathbf{a}_0(\mathbf{x})e^{-i\omega t} + \mathbf{a}_0^*(\mathbf{x})e^{i\omega t}] \quad (2.3)$$

and similarly for \mathbf{b} . Hence, the physically acceptable interpretation of the scalar product of two complex vectors, representing classical physical observables, is

$$\begin{aligned} \mathbf{a}(t, \mathbf{x}) \circ \mathbf{b}(t, \mathbf{x}) &= \text{Re}\{\mathbf{a}_0(\mathbf{x})e^{-i\omega t}\} \circ \text{Re}\{\mathbf{b}_0(\mathbf{x})e^{-i\omega t}\} \\ &= \frac{1}{2}[\mathbf{a}_0(\mathbf{x})e^{-i\omega t} + \mathbf{a}_0^*(\mathbf{x})e^{i\omega t}] \circ \frac{1}{2}[\mathbf{b}_0(\mathbf{x})e^{-i\omega t} + \mathbf{b}_0^*(\mathbf{x})e^{i\omega t}] \\ &= \frac{1}{4}(\mathbf{a}_0 \circ \mathbf{b}_0^* + \mathbf{a}_0^* \circ \mathbf{b}_0 + \mathbf{a}_0 \circ \mathbf{b}_0 e^{-2i\omega t} + \mathbf{a}_0^* \circ \mathbf{b}_0^* e^{2i\omega t}) \\ &= \frac{1}{2}\text{Re}\{\mathbf{a}_0 \circ \mathbf{b}_0^*\} + \frac{1}{2}\text{Re}\{\mathbf{a}_0 \circ \mathbf{b}_0 e^{-2i\omega t}\} \\ &= \frac{1}{2}\text{Re}\{\mathbf{a}_0 e^{-i\omega t} \circ \mathbf{b}_0^* e^{i\omega t}\} + \frac{1}{2}\text{Re}\{\mathbf{a}_0 e^{-i\omega t} \circ \mathbf{b}_0 e^{-i\omega t}\} \\ &= \frac{1}{2}\text{Re}\{\mathbf{a}(t, \mathbf{x}) \circ \mathbf{b}^*(t, \mathbf{x})\} + \frac{1}{2}\text{Re}\{\mathbf{a}(t, \mathbf{x}) \circ \mathbf{b}(t, \mathbf{x})\} \end{aligned} \quad (2.4)$$

In physics, we are often forced to measure the *temporal average* (*cycle average*) of a physical observable. We use the notation $\langle \cdots \rangle_t$, for such an average

and find that the average of the product of the two physical quantities represented by \mathbf{a} and \mathbf{b} can be expressed as

$$\begin{aligned}\langle \mathbf{a} \circ \mathbf{b} \rangle_t &\equiv \langle \text{Re} \{ \mathbf{a} \} \circ \text{Re} \{ \mathbf{b} \} \rangle_t = \frac{1}{2} \text{Re} \{ \mathbf{a} \circ \mathbf{b}^* \} = \frac{1}{2} \text{Re} \{ \mathbf{a}^* \circ \mathbf{b} \} \\ &= \frac{1}{2} \text{Re} \{ \mathbf{a}_0 \circ \mathbf{b}_0^* \} = \frac{1}{2} \text{Re} \{ \mathbf{a}_0^* \circ \mathbf{b}_0 \}\end{aligned}\quad (2.5)$$

This is because the oscillating function $\exp(-2i\omega t)$ in equation (2.4) on the preceding page vanishes when averaged in time over a complete period $2\pi/\omega$ (or over infinitely many periods), and, therefore, $\langle \mathbf{a}(t, \mathbf{x}) \circ \mathbf{b}(t, \mathbf{x}) \rangle_t$ gives no contribution.

2.2.2 Maxwell equations in Majorana representation

It is sometimes convenient to introduce the *complex-field six-vector*, also known as the *Riemann-Silberstein vector*

$$\mathbf{G}(t, \mathbf{x}) = \mathbf{E}(t, \mathbf{x}) + ic\mathbf{B}(t, \mathbf{x}) \quad (2.6)$$

where $\mathbf{G} \in \mathbb{C}^3$ even if $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$. Expressed in this vector, the Maxwell equations (2.1) on page 20 transform into

$$\nabla \cdot \mathbf{G} = \frac{\rho}{\varepsilon_0} \quad (2.7a)$$

$$\nabla \times \mathbf{G} = ic \left(\mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{G}}{\partial t} \right) \quad (2.7b)$$

In regions where $\rho = 0$ and $\mathbf{j} = \mathbf{0}$ these equations reduce to

$$\nabla \cdot \mathbf{G} = 0 \quad (2.8a)$$

$$\nabla \times \mathbf{G} = \frac{ic}{c} \frac{\partial \mathbf{G}}{\partial t} \quad (2.8b)$$

which, the help of the *linear momentum operator*,

$$\hat{\mathbf{p}} = -i\hbar \nabla \quad (2.9)$$

can be rewritten

$$\hat{\mathbf{p}} \cdot \mathbf{G} = 0 \quad (2.10a)$$

$$i\hbar \frac{\partial \mathbf{G}}{\partial t} = c i \hat{\mathbf{p}} \times \mathbf{G} \quad (2.10b)$$

The first equation is the *transversality condition* $\mathbf{p} = \hbar \mathbf{k} \perp \mathbf{G}$ where we anticipate the quantal relation between the *field momentum* \mathbf{p} and the *wave vector* \mathbf{k} and the second equation describes the dynamics.

Using formula (F.88) on page 200, we can rewrite equation (2.10b) on the preceding page as

$$i\hbar \frac{\partial \mathbf{G}}{\partial t} = c \mathbf{S} \cdot \hat{\mathbf{p}} \mathbf{G} \quad (2.11)$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \hat{\mathbf{x}}_1 + \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \hat{\mathbf{x}}_2 + \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{\mathbf{x}}_3 \quad (2.12)$$

By introducing the Hamiltonian-like operator

$$\hat{H} = c \mathbf{S} \cdot \hat{\mathbf{p}} \quad (2.13)$$

the Maxwell-Lorentz equations can be written as

$$i\hbar \frac{\partial \mathbf{G}}{\partial t} = \hat{H} \mathbf{G} \quad (2.14)$$

i.e., as a Schrödinger/Pauli/Dirac-like equation. This formulation of the free-space electromagnetic field equations is known as the *Majorana representation* of the Maxwell-Lorentz equations or the *Majorana formalism*.⁵

⁵ It so happens that ETTORE MAJORANA (1906-1938) used the definition $\mathbf{G} = \mathbf{E} - ic\mathbf{B}$, but this is no essential difference from the definition (2.6). One may say that Majorana used the other branch of $\sqrt{-1}$ as the imaginary unit.

▷ Products of Riemann-Silberstein vectors with themselves for $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$

EXAMPLE 2.1

One fundamental property of \mathbb{C}^3 , to which the Riemann-Silberstein vector $\mathbf{G} = \mathbf{E} + ic\mathbf{B}$ belongs, is that inner (scalar) products in this space are invariant under rotations just as they are in \mathbb{R}^3 . However, as discussed in example M.4 on page 215, the inner (scalar) product in \mathbb{C}^3 can be defined in two different ways. Considering the special case of the scalar product of \mathbf{G} with itself, assuming that $\mathbf{E} \in \mathbb{R}^3$ and $\mathbf{B} \in \mathbb{R}^3$, we have the following possibilities of defining various products of \mathbf{G} with itself:

1. The inner (scalar) product defined as \mathbf{G} scalar multiplied with itself

$$\mathbf{G} \cdot \mathbf{G} = (\mathbf{E} + ic\mathbf{B}) \cdot (\mathbf{E} + ic\mathbf{B}) = E^2 - c^2 B^2 + 2ic\mathbf{E} \cdot \mathbf{B} \quad (2.15)$$

Since ‘length’ is a scalar quantity that is invariant under rotations, we find that

$$E^2 - c^2 B^2 = \text{Const} \quad (2.16a)$$

$$\mathbf{E} \cdot \mathbf{B} = \text{Const} \quad (2.16b)$$

2. The inner (scalar) product defined as \mathbf{G} scalar multiplied with the complex conjugate of itself

$$\mathbf{G} \cdot \mathbf{G}^* = (\mathbf{E} + ic\mathbf{B}) \cdot (\mathbf{E} - ic\mathbf{B}) = \mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B} \equiv E^2 + c^2 B^2 \quad (2.17)$$

which is also an invariant scalar quantity. As we shall see in chapter 4, this quantity is proportional to the *electromagnetic field energy density*.

3. As with any vector, the cross product of \mathbf{G} with itself vanishes:

$$\begin{aligned}\mathbf{G} \times \mathbf{G} &= (\mathbf{E} + ic\mathbf{B}) \times (\mathbf{E} + ic\mathbf{B}) \\ &= \mathbf{E} \times \mathbf{E} - c^2\mathbf{B} \times \mathbf{B} + ic(\mathbf{E} \times \mathbf{B}) + ic(\mathbf{B} \times \mathbf{E}) \\ &= \mathbf{0} + \mathbf{0} + ic(\mathbf{E} \times \mathbf{B}) - ic(\mathbf{E} \times \mathbf{B}) = \mathbf{0}\end{aligned}\quad (2.18)$$

4. The cross product of \mathbf{G} with the complex conjugate of itself does, however, not vanish. Instead it is

$$\begin{aligned}\mathbf{G} \times \mathbf{G}^* &= (\mathbf{E} + ic\mathbf{B}) \times (\mathbf{E} - ic\mathbf{B}) \\ &= \mathbf{E} \times \mathbf{E} + c^2\mathbf{B} \times \mathbf{B} - ic(\mathbf{E} \times \mathbf{B}) + ic(\mathbf{B} \times \mathbf{E}) \\ &= \mathbf{0} + \mathbf{0} - ic(\mathbf{E} \times \mathbf{B}) - ic(\mathbf{E} \times \mathbf{B}) = -2ic(\mathbf{E} \times \mathbf{B})\end{aligned}\quad (2.19)$$

that is proportional to the electromagnetic energy flux, to be introduced in chapter 4.

5. The dyadic product of \mathbf{G} with itself is

$$\mathbf{G}\mathbf{G} = \mathbf{E}\mathbf{E} - c^2\mathbf{B}\mathbf{B} + ic(\mathbf{E}\mathbf{B} + \mathbf{B}\mathbf{E}) \quad (2.20)$$

or, in component form,

$$(\mathbf{G}\mathbf{G})_{ij} = E_i E_j - c^2 B_i B_j + ic(E_i B_j + B_i E_j) \quad (2.21)$$

6. The dyadic product of \mathbf{G}^* with itself is

$$\mathbf{G}^*\mathbf{G}^* = \mathbf{E}\mathbf{E} - c^2\mathbf{B}\mathbf{B} - ic(\mathbf{E}\mathbf{B} + \mathbf{B}\mathbf{E}) = (\mathbf{G}\mathbf{G})^* \quad (2.22)$$

7. The dyadic product of \mathbf{G} with its own complex conjugate from the right is

$$\mathbf{G}\mathbf{G}^* = \mathbf{E}\mathbf{E} + c^2\mathbf{B}\mathbf{B} - ic(\mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E}) \quad (2.23)$$

and from the left it is

$$\mathbf{G}^*\mathbf{G} = \mathbf{E}\mathbf{E} + c^2\mathbf{B}\mathbf{B} + ic(\mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E}) = (\mathbf{G}\mathbf{G}^*)^* \quad (2.24)$$

—End of example 2.1◀

2.3 The wave equations for \mathbf{E} and \mathbf{B}

The Maxwell-Lorentz equations (2.1) on page 20 are four first-order *coupled differential equations* (both \mathbf{E} and \mathbf{B} appear in the same equations). Two of the equations are scalar [equation (2.1a) and equation (2.1c)], and two are in 3D Euclidean vector form [equation (2.1b) and equation (2.1d)], representing three scalar equations each. Hence, the Maxwell equations represent eight $(1 + 1 + 3 + 3)$ scalar coupled first-order partial differential equations. However, it is well known from the theory of differential equations that a set of first-order, coupled partial differential equations can be transformed into a smaller set of second order partial differential equations, which sometimes become decoupled in the process. It turns out that in our case we will obtain one second-order differential equation for \mathbf{E} and one second-order differential equation for \mathbf{B} . These second-order partial differential equations are, as we shall see, *wave equations*, and we

shall discuss the implications of them. For certain media, the **B** wave field can be easily obtained from the solution of the **E** wave equation but in general this is not the case.

To bring the first-order differential equations (2.1) on page 20 into second order one needs, of course, to operate on them with first-order differential operators. If we apply the curl vector operator ($\nabla \times$) to both sides of the two Maxwell vector equations, equation (2.1b) and equation (2.1d), assuming that our physical quantities vary in such a regular way that temporal and spatial differentiation commute, we obtain the second order differential equations

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \quad (2.25a)$$

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \quad (2.25b)$$

As they stand, these second-order partial differential equations still appear to be coupled. However, by using the Maxwell equations once again we can formally decouple them into

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial \mathbf{j}}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2.26a)$$

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{j} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (2.26b)$$

Using to the operator triple product ‘bac-cab’ rule equation (F.103) on page 201, which gives

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (2.27)$$

when applied to **E** and similarly to **B**, Gauss’s law equation (2.1a) on page 20, and then rearranging the terms, recalling that $\varepsilon_0 \mu_0 = 1/c^2$ where c is the speed of light in free (empty) space, we obtain the two inhomogeneous *vector wave equations*

$$\square^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\frac{\nabla \rho}{\varepsilon_0} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \quad (2.28a)$$

$$\square^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{j} \quad (2.28b)$$

where \square^2 is the *d’Alembert operator*, defined according to formula (M.109) on page 223.

These are the general wave equations for the electromagnetic fields, generated in regions where there exist sources $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$ of any kind. Simple everyday examples of such regions are electric conductors (*e.g.*, radio and TV transmitter antennas) or plasma (*e.g.*, the Sun and its surrounding corona). In

⁶ Clearly, if the current density in the RHS of equation (2.28b) is a function of \mathbf{E} , as is the case if Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ is applicable, the coupling is not removed.

principle, the sources ρ and \mathbf{j} can still cause the wave equations to be coupled, but in many important situations this is not the case.⁶

We notice that outside the source region, *i.e.*, in free space where $\rho = 0$ and $\mathbf{j} = \mathbf{0}$, the inhomogeneous wave equations (2.28) on the preceding page simplify to the well-known uncoupled, homogeneous wave equations

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \mathbf{0} \quad (2.29a)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mathbf{0} \quad (2.29b)$$

These equations describe how the fields that were generated in the source region, *e.g.*, a transmitting antenna, propagate as *vector waves* through free space. Once these waves impinge upon another region which can sustain charges and/or currents for a long enough time, *e.g.*, a receiving antenna or other electromagnetic sensors, the fields interact with the charges and the currents in this second region in accordance with equations (2.28) on the previous page.

EXAMPLE 2.2 ▷ Wave polarisation

⁷ A single plane wave is a mathematical idealisation. In reality, a wave appears as a building block of *wave packets*, *i.e.*, superpositions of a (possibly infinite) number of individual plane waves with different properties. *E.g.*, a radio beam from a transmitting antenna is a superposition (Fourier sum or integral) of many plane waves with slightly different angles relative to a fixed, given axis or a plane.

Since electromagnetic waves are vector waves they exhibit *wave polarisation*. Let us consider a single *plane wave* that propagates in free space⁷ *i.e.*, the electric and magnetic field vectors are restricted to a two-dimensional plane that is perpendicular the propagation direction. Let us choose this to be the $x_1 x_2$ plane and the *propagation vector* (wave vector) to be along the x_3 axis. A generic temporal Fourier mode of the electric field vector \mathbf{E} with (angular) frequency ω is given by the real-valued expression

$$\mathbf{E}(t, \mathbf{x}) = E_1 \cos(\omega t - kx_3 + \delta_1) \hat{\mathbf{x}}_1 + E_2 \cos(\omega t - kx_3 + \delta_2) \hat{\mathbf{x}}_2 \quad (2.30)$$

In complex notation we can write this as

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= E_1 e^{i\delta_1} e^{i(kx_3 - \omega t)} \hat{\mathbf{x}}_1 + E_2 e^{i\delta_2} e^{i(kx_3 - \omega t)} \hat{\mathbf{x}}_2 \\ &= (E_1 e^{i\delta_1} \hat{\mathbf{x}}_1 + E_2 e^{i\delta_2} \hat{\mathbf{x}}_2) e^{i(kx_3 - \omega t)} \\ &= (E_1 \hat{\mathbf{x}}_1 + E_2 e^{i\delta_0} \hat{\mathbf{x}}_2) e^{i(kx_3 - \omega t + \delta_1)} \end{aligned} \quad (2.31)$$

where $\delta_0 = \delta_2 - \delta_1$. When this phase δ_0 vanishes, the electric field oscillates along a line directed at an angle $\arctan(E_2/E_1)$ relative to the \mathbf{x}_1 axis. This is called *linear wave polarisation*. When $\delta_0 \neq 0$ the wave is in general in a state of *elliptical wave polarisation*.

For the special cases $\delta_0 = \pm\pi/2$ and $E_1 = E_2 = E_0$ the wave can, in complex notation, be described as

$$\mathbf{E}(t, \mathbf{x}) = E_0 e^{i(kx_3 - \omega t + \delta_1)} (\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \quad (2.32)$$

As shown in example M.1 on page 207, this describes a rotation. This rotation is called

circular wave polarisation. The helical base vectors

$$\hat{\mathbf{h}}_{\pm} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \quad (2.33)$$

which are fixed unit vectors, allow us to write equation (2.32) on the preceding page

$$\mathbf{E}(t, \mathbf{x}) = \sqrt{2}E_0 e^{i(kx_3 - \omega t + \delta_1)} \hat{\mathbf{h}}_{\pm} \quad (2.34)$$

We use the convention that $\hat{\mathbf{h}}_+$ represents *left-hand circular polarisation* and $\hat{\mathbf{h}}_-$ *right-hand circular polarisation*.⁸ Left-hand (right-hand) circular polarised waves are said to have *positive helicity* (*negative helicity*).

⁸ In physics, two different conventions are used. The handedness refers to the rotation in space of the electric field vector, either when viewed as the wave propagates away from the observer or toward the observer.

—End of example 2.2◁

▷Wave equations expressed in terms of Riemann-Silberstein vectors

EXAMPLE 2.3

Using the Maxwell equations expressed in the Riemann-Silberstein vector $\mathbf{G} = \mathbf{E} + i\mathbf{B}$, i.e., equations (2.7) on page 22, one can perform similar steps as was used in deriving equations (2.28) on page 25 and one finds that

$$\square^2 \mathbf{G} = \frac{1}{c^2} \frac{\partial^2 \mathbf{G}}{\partial t^2} - \nabla^2 \mathbf{G} = -\frac{\nabla \rho}{\epsilon_0} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} + i\mu_0 c \nabla \times \mathbf{j} \quad (2.35)$$

Taking the real and imaginary parts of this equation, assuming that $\mathbf{E}, \mathbf{B} \in \mathbb{R}^3$, we recover the wave equations (2.26) on page 25, as expected.

—End of example 2.3◁

2.3.1 The time-independent wave equations for \mathbf{E} and \mathbf{B}

Often one can assume that the temporal dependency of \mathbf{E} and \mathbf{B} and of the sources ρ and \mathbf{j} is well-behaved enough that it can be represented by the sum of a finite number N of *temporal spectral components* (*temporal Fourier components*), or, in other words, in the form of a *temporal Fourier series*. In such situations it is sufficient to study the properties of one arbitrary member of this set of spectral components $\{\omega_n : n = 0, 1, 2, \dots, N-1\}$, i.e.,

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_n(\mathbf{x}) \cos(\omega_n t) = \mathbf{E}_n(\mathbf{x}) \operatorname{Re} \{e^{-i\omega_n t}\} \equiv \mathbf{E}_n(\mathbf{x}) e^{-i\omega_n t} \quad (2.36a)$$

$$\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_n(\mathbf{x}) \cos(\omega_n t) = \mathbf{B}_n(\mathbf{x}) \operatorname{Re} \{e^{-i\omega_n t}\} \equiv \mathbf{B}_n(\mathbf{x}) e^{-i\omega_n t} \quad (2.36b)$$

where $\mathbf{E}_n, \mathbf{B}_n \in \mathbb{R}^3$ and $\omega_n, t \in \mathbb{R}^1$ is assumed. This is because the Maxwell-Lorentz equations are linear, implying that the general solution is obtained by a weighted linear superposition (summation) of the result for each such spectral component, where the weight of each spectral component is given by its *Fourier amplitude*, $\mathbf{E}_n(\mathbf{x})$, and $\mathbf{B}_n(\mathbf{x})$, respectively.

In a physical system, a temporal spectral component is identified uniquely by its *angular frequency* ω_n . A wave containing only a finite number of temporal

⁹ When subtle classical and quantum radiation effects are taken into account, one finds that all emissions suffer an unavoidable, intrinsic *line broadening*. Also, simply because the Universe has existed for only about 13.5 billion years, which is a finite time, no signals in the Universe can be observed to have a spectral width that is smaller than the inverse of this age.

spectral components is called a *time-harmonic wave*. In the limit when only one single frequency is present, we talk about a *monochromatic wave*. Strictly speaking, purely monochromatic waves do not exist.⁹

By inserting the temporal spectral component equation (2.36a) on the previous page into equation (2.28a) on page 25 one finds that for an arbitrary component the following *time-independent wave equation* is obtained:

$$\nabla^2 \mathbf{E}_n e^{-i\omega_n t} + \frac{\omega_n^2}{c^2} \mathbf{E}_n e^{-i\omega_n t} + i\omega_n \mu_0 \mathbf{j}_n e^{-i\omega_n t} = \frac{\nabla \rho_n}{\epsilon_0} e^{-i\omega_n t} \quad (2.37)$$

After dividing out the common factor $\exp(-i\omega_n t)$, this becomes

$$\nabla^2 \mathbf{E}_n + \frac{\omega_n^2}{c^2} \mathbf{E}_n + i\omega_n \mu_0 \mathbf{j}_n = \frac{\nabla \rho_n}{\epsilon_0} \quad (2.38)$$

and similarly for \mathbf{B} . Solving this equation, multiplying the solution obtained by $\exp(-i\omega_n t)$ and summing over all N such Fourier (spectral) components with frequencies $\omega_n, n = 0, 1, 2, \dots, N - 1$ present in the sources, and hence in the fields, the complete solution of the original wave equation is obtained. This is a consequence of the *superposition principle* which is valid as long as nonlinear effects can be neglected.

In the limit of very many frequency components, the Fourier sum goes over into a *Fourier integral*. To illustrate this generic case, let us introduce the *Fourier transform* of $\mathbf{E}(t, \mathbf{x})$

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{E}(t, \mathbf{x}) e^{i\omega t} \quad (2.39a)$$

and the corresponding *inverse Fourier transform*

$$\mathbf{E}(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \mathbf{E}_\omega(\mathbf{x}) e^{-i\omega t} \quad (2.39b)$$

where the amplitude $\mathbf{E}_\omega(\mathbf{x}) \in \mathbb{C}^3$ is a continuous function of (angular) frequency $\omega \in \mathbb{R}^1$ and of $\mathbf{x} \in \mathbb{R}^3$.

We see that the Fourier transform of $\partial \mathbf{E}(t, \mathbf{x}) / \partial t$ becomes

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left(\frac{\partial \mathbf{E}(t, \mathbf{x})}{\partial t} \right) e^{i\omega t} \\ &= \frac{1}{2\pi} \underbrace{\left[\mathbf{E}(t, \mathbf{x}) e^{i\omega t} \right]_{-\infty}^{\infty}}_{=0 \text{ since } \mathbf{E} \rightarrow 0, t \rightarrow \pm\infty} - i\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{E}(t, \mathbf{x}) e^{i\omega t} \\ &= -i\omega \mathbf{E}_\omega(\mathbf{x}) \end{aligned} \quad (2.40)$$

and that, consequently,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left(\frac{\partial^2 \mathbf{E}(t, \mathbf{x})}{\partial t^2} \right) e^{i\omega t} = -\omega^2 \mathbf{E}_\omega(\mathbf{x}) \quad (2.41)$$

Fourier transforming equation (2.28a) on page 25 and using (2.40) and (2.41), we thus obtain

$$\nabla^2 \mathbf{E}_\omega + \frac{\omega^2}{c^2} \mathbf{E}_\omega + i\omega\mu_0 \mathbf{j}_\omega = \frac{\nabla \rho_\omega}{\varepsilon_0} \quad (2.42)$$

which is mathematically identical to equation (2.38) on the facing page. A subsequent inverse Fourier transformation of the solution \mathbf{E}_ω of this equation leads to the same result as is obtained from the solution of equation (2.38) on page 28. Hence, by considering just one temporal Fourier component we obtain results which are identical to those that we would have obtained by employing the machinery of Fourier transforms and Fourier integrals. Hence, under the assumption of linearity (superposition principle) there is usually no need for the formal forward and inverse Fourier transform technique.

What was said above in general terms about temporal spectral components is true also for *spatial spectral components* (*spatial Fourier components*) only that we must use a three-dimensional Fourier representation

$$\mathbf{E}_{\mathbf{k}}(t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3x \mathbf{E}(t, \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \quad (2.43a)$$

$$\mathbf{E}(t, \mathbf{x}) = \int_{-\infty}^{\infty} d^3k \mathbf{E}_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2.43b)$$

Since we always assume that the real part shall be taken (if necessary), we can pick an arbitrary pair samples of the spatial amplitudes in equation (2.36a) and equation (2.36b) on page 27, denote the members of this pair by \mathbf{E}_0 and \mathbf{B}_0 , respectively, and then represent them as the Fourier modes

$$\mathbf{E}_0(\mathbf{x}) = \mathbf{e}_0 \text{Re} \{ e^{i\mathbf{k}_0 \cdot \mathbf{x}} \} = \mathbf{E}_{\omega, \mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2.44a)$$

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{b}_0 \text{Re} \{ e^{i\mathbf{k}_0 \cdot \mathbf{x}} \} = \mathbf{B}_{\omega, \mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2.44b)$$

respectively, where \mathbf{k}_0 is the *wave vector* (measured in m^{-1}) of mode 0; in the last step we introduced a complex notation and also dropped the mode number since the formulæ are valid for any mode n .

Now, since

$$\frac{\partial}{\partial t} e^{-i\omega t} = -i\omega e^{-i\omega t} \quad (2.45a)$$

and

$$\nabla e^{i\mathbf{k} \cdot \mathbf{x}} = \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} e^{ik_j x_j} = \hat{\mathbf{x}}_i ik_j \delta_{ij} e^{ik_j x_j} = ik_i x_i e^{ik_j x_j} = i\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2.45b)$$

we see that for each spectral component in equations (2.44), temporal and spatial differential operators turn into algebraic operations according to the following

scheme:

$$\frac{\partial}{\partial t} \rightarrow -i\omega \quad (2.46a)$$

$$\nabla \rightarrow i\mathbf{k} \quad (2.46b)$$

$$\nabla \cdot \rightarrow i\mathbf{k} \cdot \quad (2.46c)$$

$$\nabla \times \rightarrow i\mathbf{k} \times \quad (2.46d)$$

We note that

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E}_{\parallel} \quad (2.47a)$$

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}_{\perp} \quad (2.47b)$$

$$\nabla \cdot \mathbf{B} = i\mathbf{k} \cdot \mathbf{B} = i\mathbf{k} \cdot \mathbf{B}_{\parallel} \quad (2.47c)$$

$$\nabla \times \mathbf{B} = i\mathbf{k} \times \mathbf{B} = i\mathbf{k} \times \mathbf{B}_{\perp} \quad (2.47d)$$

$$\nabla \rho = i\mathbf{k} \rho \quad (2.47e)$$

$$\nabla \cdot \mathbf{j} = i\mathbf{k} \cdot \mathbf{j} = i\mathbf{k} \cdot \mathbf{j}_{\parallel} \quad (2.47f)$$

$$\nabla \times \mathbf{j} = i\mathbf{k} \times \mathbf{j} = i\mathbf{k} \times \mathbf{j}_{\perp} \quad (2.47g)$$

Hence, with respect to the wave vector \mathbf{k} , the $\nabla \cdot$ operator projects out the spatially longitudinal component, and the $\nabla \times$ operator projects out the spatially transverse component of the field vector in question. Put in another way,

$$(\nabla \rho)_{\perp} = 0 \quad (2.48a)$$

$$\nabla \cdot \mathbf{E}_{\perp} = 0 \quad (2.48b)$$

$$\nabla \times \mathbf{E}_{\parallel} = \mathbf{0} \quad (2.48c)$$

and so on for the other observables.

As seen from equations (2.46) above, the Fourier transform of a function of time t is a function of angular frequency ω , and the Fourier transform of a function of the position vector \mathbf{x} is a function of the wave vector \mathbf{k} . One says that ω is a *reciprocal space* to t and that \mathbf{k} spans a space that is reciprocal to \mathbf{x} .

2.4 Bibliography

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3

ELECTROMAGNETIC POTENTIALS AND GAUGES

As described in chapter 1, the concepts of electric and magnetic fields were introduced such that they are intimately related to the mechanical forces between charges and currents given by Coulomb's law and Ampère's law, respectively. Just as in mechanics, it turns out that in electrodynamics it is often more convenient to express the theory in terms of potentials rather than in terms of the electric and magnetic fields (Coulomb and Ampère forces) themselves. This is particularly true for problems related to radiation and relativity. As we shall see in chapter 7, the potentials play a central rôle in the formulation of relativistically covariant electromagnetism. At the quantum level, electrodynamics is almost exclusively formulated in terms of potentials rather than electric and magnetic fields.

In this chapter we introduce and study the properties of such potentials and shall find that they exhibit some remarkable properties that elucidate the fundamental aspects of electromagnetism, lead naturally to the special theory of relativity and pave the way for gauge field theories.

3.1 The electrostatic scalar potential

As we saw in equation (1.10) on page 6, the time-independent electric (electrostatic) field $\mathbf{E}^{\text{stat}}(\mathbf{x})$ is irrotational. Hence, it may be expressed in terms of the

gradient of a scalar field. If we denote this scalar field by $-\phi^{\text{stat}}(\mathbf{x})$, we get

$$\mathbf{E}^{\text{stat}}(\mathbf{x}) = -\nabla\phi^{\text{stat}}(\mathbf{x}) \quad (3.1)$$

Taking the divergence of this and using equation (1.9) on page 5, we obtain *Poisson's equation*

$$\nabla^2\phi^{\text{stat}}(\mathbf{x}) = -\nabla \cdot \mathbf{E}^{\text{stat}}(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\epsilon_0} \quad (3.2)$$

A comparison with the definition of \mathbf{E}^{stat} , namely equation (1.7) on page 4, shows that this equation has the solution

$$\phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \alpha \quad (3.3)$$

where the integration is taken over all source points \mathbf{x}' at which the charge density $\rho(\mathbf{x}')$ is non-zero and α is an arbitrary scalar function that is not dependent on \mathbf{x} , *e.g.*, a constant. The scalar function $\phi^{\text{stat}}(\mathbf{x})$ in equation (3.3) above is called the *electrostatic scalar potential*.

3.2 The magnetostatic vector potential

Consider the equations of magnetostatics (1.24) on page 9. According to formula (F.106) on page 201 any vector field \mathbf{a} has the property that $\nabla \cdot (\nabla \times \mathbf{a}) \equiv 0$ and in the derivation of equation (1.19) on page 8 in magnetostatics we found that $\nabla \cdot \mathbf{B}^{\text{stat}}(\mathbf{x}) = 0$. We therefore realise that we can always write

$$\mathbf{B}^{\text{stat}}(\mathbf{x}) = \nabla \times \mathbf{A}^{\text{stat}}(\mathbf{x}) \quad (3.4)$$

where $\mathbf{A}^{\text{stat}}(\mathbf{x})$ is called the *magnetostatic vector potential*.

We saw above that the electrostatic potential (as any scalar potential) is not unique since we may, without changing the physics, add to it a quantity whose spatial gradient vanishes. A similar arbitrariness is true also for the magnetostatic vector potential.

In the magnetostatic case, we may start from Biot-Savart's law as expressed by equation (1.17) on page 8. Identifying this expression with equation (3.4) above allows us to define the static vector potential as

$$\mathbf{A}^{\text{stat}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{a}(\mathbf{x}) \quad (3.5)$$

where $\mathbf{a}(\mathbf{x})$ is an arbitrary vector field whose curl vanishes. According to formula (F.107) on page 201 such a vector field can always be written as the gradient of a scalar field.

▷ Multipole expansion of the electrostatic potential

EXAMPLE 3.1

The integral in the electrostatic potential formula

$$\phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (3.6)$$

where ρ is the charge density introduced in equation (1.9) on page 5, is not always possible to evaluate analytically. However, for a charge distribution source $\rho(\mathbf{x}')$ that is well localised in a small volume V' around \mathbf{x}_0 , a series expansion of the integrand in such a way that the dominant contributions are contained in the first few terms can be made. *E.g.*, if we Taylor expand the inverse distance $1/|\mathbf{x} - \mathbf{x}'|$ with respect to the point $\mathbf{x}' = \mathbf{x}_0$ we obtain

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{|(\mathbf{x} - \mathbf{x}_0) - (\mathbf{x}' - \mathbf{x}_0)|} \\ &= \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1=1}^3 \cdots \sum_{i_n=1}^3 \\ &\quad \times \frac{\partial^n \frac{1}{|\mathbf{x} - \mathbf{x}_0|}}{\partial x_{i_1} \cdots \partial x_{i_n}} [-(x'_{i_1} - x_{0i_1})] \cdots [-(x'_{i_n} - x_{0i_n})] \\ &= \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \sum_{n=1}^{\infty} \sum_{n_1+n_2+n_3=n} \sum_{n_i \geq 0} \frac{(-1)^n}{n_1! n_2! n_3!} \\ &\quad \times \frac{\partial^n \frac{1}{|\mathbf{x} - \mathbf{x}_0|}}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} (x'_1 - x_{01})^{n_1} (x'_2 - x_{02})^{n_2} (x'_3 - x_{03})^{n_3} \end{aligned} \quad (3.7)$$

Inserting this into the integral in formula (3.6), we obtain the expansion

$$\begin{aligned} \phi^{\text{stat}}(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \int_{V'} d^3x' \rho(\mathbf{x}') + \sum_{n=1}^{\infty} \sum_{n_1+n_2+n_3=n} \sum_{n_i \geq 0} \frac{(-1)^n}{n_1! n_2! n_3!} \right. \\ &\quad \times \left. \frac{\partial^n \frac{1}{|\mathbf{x} - \mathbf{x}_0|}}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \int_{V'} d^3x' (x'_1 - x_{01})^{n_1} (x'_2 - x_{02})^{n_2} (x'_3 - x_{03})^{n_3} \rho(\mathbf{x}') \right] \end{aligned} \quad (3.8)$$

Clearly, the first integral in this expansion is nothing but the *static net charge*

$$q = \int_{V'} d^3x' \rho(\mathbf{x}') \quad (3.9)$$

If we introduce the *electrostatic dipole moment vector*

$$\mathbf{d}(\mathbf{x}_0) = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \rho(\mathbf{x}') \quad (3.10)$$

with components p_i , $i = 1, 2, 3$, and the *electrostatic quadrupole moment tensor*

$$\mathbf{Q}(\mathbf{x}_0) = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0) \rho(\mathbf{x}') \quad (3.11)$$

with components Q_{ij} , $i, j = 1, 2, 3$, and use the fact that

$$\frac{\partial \frac{1}{|\mathbf{x} - \mathbf{x}_0|}}{\partial x_i} = - \frac{x_i - x_{0i}}{|\mathbf{x} - \mathbf{x}_0|^3} \quad (3.12)$$

and that

$$\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\mathbf{x} - \mathbf{x}_0|} = \frac{3(x_i - x_{0i})(x_j - x_{0j}) - |\mathbf{x} - \mathbf{x}_0|^2 \delta_{ij}}{|\mathbf{x} - \mathbf{x}_0|^5} \quad (3.13)$$

then we can write the first three terms of the expansion of equation (3.6) on the previous page as

$$\begin{aligned} \phi^{\text{stat}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} \mathbf{d} \cdot \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \right. \\ \left. + \frac{1}{|\mathbf{x} - \mathbf{x}_0|^3} Q_{ij} \left(\frac{3}{2} \frac{(x_i - x_{0i})(x_j - x_{0j})}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{2} \delta_{ij} \right) + \dots \right] \end{aligned} \quad (3.14)$$

where Einstein's summation convention over i and j is implied. We see that at large distances from a localised charge distribution, the electrostatic potential can, to the lowest order, be approximated by the (Coulomb) potential from a single point charge q located at the moment point \mathbf{x}_0 . We also see that

$$\begin{aligned} \mathbf{d}(\mathbf{x}_0) &= \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \rho(\mathbf{x}') = \int_{V'} d^3x' \mathbf{x}' \rho(\mathbf{x}') - \mathbf{x}_0 \int_{V'} d^3x' \rho(\mathbf{x}') \\ &= \int_{V'} d^3x' \mathbf{x}' \rho(\mathbf{x}') - \mathbf{x}_0 q \end{aligned} \quad (3.15)$$

from which we draw the conclusion that if $q \neq 0$, it is always possible to choose the moment point \mathbf{x}_0 such that $\mathbf{d} = \mathbf{0}$, and if $q = 0$, then \mathbf{d} is independent of the choice of moment point \mathbf{x}_0 . Furthermore, one can show that

$$\alpha \frac{1}{2} \delta_{ij} \frac{3(x_i - x_{0i})(x_j - x_{0j}) - |\mathbf{x} - \mathbf{x}_0|^2 \delta_{ij}}{|\mathbf{x} - \mathbf{x}_0|^5} = 0 \quad (3.16)$$

where α is an arbitrary constant. Choosing it to be

$$\alpha = \frac{1}{3} \int_{V'} d^3x' |\mathbf{x}' - \mathbf{x}_0|^2 \rho(\mathbf{x}') \quad (3.17)$$

we can transform Q_{ij} into

$$Q'_{ij} = Q_{ij} - \alpha \delta_{ij} = \int_{V'} d^3x' \left([(x'_i - x_{0i})(x'_j - x_{0j}) - \frac{1}{3} |\mathbf{x}' - \mathbf{x}_0|^2 \delta_{ij}] \rho(\mathbf{x}') \right) \quad (3.18)$$

or

$$\mathbf{Q}' = \mathbf{Q} - \mathbf{1}_3 \alpha = \int_{V'} d^3x' \left([(\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0) - \mathbf{1}_3 \frac{1}{3} |\mathbf{x}' - \mathbf{x}_0|^2] \rho(\mathbf{x}') \right) \quad (3.19)$$

where $\mathbf{1}_3 = \hat{\mathbf{x}}_i \hat{\mathbf{x}}_i$ is the unit tensor. It follows immediately that $Q'_{ii} = 0$ (Einstein summation), *i.e.*, that \mathbf{Q}' is *traceless*. Rotating the coordinate system, it is possible to diagonalise the tensors \mathbf{Q} and \mathbf{Q}' . For any spherical symmetric distribution of charge, all components of \mathbf{Q}' vanish if the moment point \mathbf{x}_0 is chosen as the symmetry centre of the distribution.

If the charge distribution $\rho(\mathbf{x})$ is made up of discrete point charges q_n with coordinates \mathbf{x}_n ,

the definitions above of q , \mathbf{d} , \mathbf{Q} and \mathbf{Q}' become

$$q = \sum_n q_n \quad (3.20a)$$

$$\mathbf{d} = \sum_n q_n (\mathbf{x}_n - \mathbf{x}_0) \quad (3.20b)$$

$$\mathbf{Q} = \sum_n q_n (\mathbf{x}_n - \mathbf{x}_0)(\mathbf{x}_n - \mathbf{x}_0) \quad (3.20c)$$

$$\mathbf{Q}' = \sum_n q_n [(\mathbf{x}_n - \mathbf{x}_0)(\mathbf{x}_n - \mathbf{x}_0) - \mathbf{1}_3 \frac{1}{3} |\mathbf{x}_n - \mathbf{x}_0|^2] \quad (3.20d)$$

—End of example 3.1◀

3.3 The electrodynamic potentials

Let us now generalise the static analysis above to the electrodynamic case, *i.e.*, the case with temporal and spatial dependent sources $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$, and corresponding fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$, as described the Maxwell-Lorentz equations (2.1) on page 20. In other words, let us study the *electrodynamic potentials* $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$.

According to the non-source Maxwell-Lorentz equation (2.1c), $\nabla \cdot \mathbf{B} = 0$ also in electrodynamics. Because of this divergence-free nature of the time- and space-dependent magnetic field, we can express it as the curl of an *electromagnetic vector potential*:

$$\mathbf{B}(t, \mathbf{x}) = \nabla \times \mathbf{A}(t, \mathbf{x}) \quad (3.21)$$

Inserting this expression into the other non-source Maxwell-Lorentz equation, *viz.* equation (2.1b) on page 20, we obtain

$$\nabla \times \mathbf{E}(t, \mathbf{x}) = -\frac{\partial}{\partial t} [\nabla \times \mathbf{A}(t, \mathbf{x})] = -\nabla \times \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \quad (3.22)$$

or, rearranging the terms,

$$\nabla \times \left(\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \right) = \mathbf{0} \quad (3.23)$$

As before we utilise the vanishing curl of a vector expression to write this vector expression as the gradient of a scalar function. If, in analogy with the electrostatic case, we introduce the *electromagnetic scalar potential* function $-\phi(t, \mathbf{x})$, equation (3.23) becomes equivalent to

$$\mathbf{E}(t, \mathbf{x}) + \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) = -\nabla \phi(t, \mathbf{x}) \quad (3.24)$$

This means that in electrodynamics, $\mathbf{E}(t, \mathbf{x})$ is calculated from the potentials according to the formula

$$\mathbf{E}(t, \mathbf{x}) = -\nabla\phi(t, \mathbf{x}) - \frac{\partial}{\partial t}\mathbf{A}(t, \mathbf{x}) \quad (3.25)$$

and $\mathbf{B}(t, \mathbf{x})$ from formula (3.21) on the previous page. Hence, it is a matter of convention (or taste) whether we want to express the laws of electrodynamics in terms of the potentials $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$, or in terms of the fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$. However, there is an important difference between the two approaches: in classical electrodynamics the only directly observable quantities are the fields themselves (and quantities derived from them) and not the potentials. On the other hand, the treatment becomes significantly simpler if we use the potentials in our calculations and then, at the final stage, use equation (3.21) on the preceding page and equation (3.25) above to calculate the fields or physical quantities expressed in the fields. This is the strategy we shall follow.

3.4 Gauge conditions

Inserting (3.25) and (3.21) on the preceding page into Maxwell's equations (2.1) on page 20, we obtain, after some simple algebra and the use of equation (1.13) on page 6 the equations

$$\nabla^2\phi = -\frac{\rho(t, \mathbf{x})}{\epsilon_0} - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \quad (3.26a)$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A}) = -\mu_0\mathbf{j}(t, \mathbf{x}) + \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} \quad (3.26b)$$

Subtracting $(1/c^2)\partial^2\phi/\partial t^2$ from both sides of the first equation and rearranging, these equations turn into the following symmetric *general inhomogeneous wave equations*

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \frac{\rho(t, \mathbf{x})}{\epsilon_0} + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \quad (3.27a)$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{j}(t, \mathbf{x}) - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) \quad (3.27b)$$

These two second order, coupled, partial differential equations, representing in all four scalar equations (one for ϕ and one each for the three components $A_i, i = 1, 2, 3$ of \mathbf{A}) are completely equivalent to the formulation of electrodynamics in terms of Maxwell's equations, which represent eight scalar first-order, coupled, partial differential equations.

As they stand, equations (3.26) and equations (3.27) on the preceding page look complicated and may seem to be of limited use. However, if we write equation (3.21) on page 35 in the form $\nabla \times \mathbf{A}(t, \mathbf{x}) = \mathbf{B}(t, \mathbf{x})$ we can consider this as a specification of $\nabla \times \mathbf{A}$. But we know from *Helmholtz' theorem* that in order to determine the (spatial) behaviour of \mathbf{A} completely, we must also specify $\nabla \cdot \mathbf{A}$. Since this divergence does not enter the derivation above, we are free to choose $\nabla \cdot \mathbf{A}$ in whatever way we like and still obtain the same physical results! This illustrates the power of the formulation of electrodynamics in terms of potentials.

3.4.1 Lorenz-Lorentz gauge

If we choose $\nabla \cdot \mathbf{A}$ to fulfil the so called *Lorenz-Lorentz gauge condition*¹

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \quad (3.28)$$

the coupled inhomogeneous wave equations (3.27) on page 36 simplify to the following set of *uncoupled inhomogeneous wave equations*:

$$\square^2 \phi \stackrel{\text{def}}{=} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho(t, \mathbf{x})}{\epsilon_0} \quad (3.29a)$$

$$\square^2 \mathbf{A} \stackrel{\text{def}}{=} \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}(t, \mathbf{x}) \quad (3.29b)$$

where \square^2 is the *d'Alembert operator*, discussed in example M.8 on page 223. Each of these four scalar equations is an *inhomogeneous wave equation* of the following generic form:

$$\square^2 \Psi(t, \mathbf{x}) = f(t, \mathbf{x}) \quad (3.30)$$

where Ψ denotes for either ϕ or one of the components A_i of the vector potential \mathbf{A} , and f is a shorthand for the pertinent generic source component, $\rho(t, \mathbf{x})/\epsilon_0$ or $\mu_0 j_i(t, \mathbf{x})$, respectively.

We assume that our sources are well-behaved enough in time t so that the *Fourier transform* pair for the generic source function f

$$f(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega f_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (3.31a)$$

$$f_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t, \mathbf{x}) e^{i\omega t} \quad (3.31b)$$

exists, and that the same is true for the generic potential component Ψ :

$$\Psi(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \Psi_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (3.32a)$$

$$\Psi_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \Psi(t, \mathbf{x}) e^{i\omega t} \quad (3.32b)$$

¹ In fact, the Dutch physicist HENDRIK ANTOON LORENTZ (1853-1928), who in 1903 demonstrated the covariance of Maxwell's equations, was not the original discoverer of the gauge condition (3.28). It had been discovered by the Danish physicist LUDVIG VALENTIN LORENZ (1829-1891) already in 1867. In the literature, this fact has sometimes been overlooked and the condition was earlier referred to as the *Lorentz gauge condition*.

Inserting the Fourier representations (3.31) and (3.32a) into equation (3.30) on the preceding page, and using the vacuum dispersion relation for electromagnetic waves relating the *angular frequency* ω , the speed of light c , and the *wave number* $k = (2\pi)/\lambda$ where λ is the *vacuum wavelength*,

$$\omega = ck \quad (3.33)$$

the generic 3D inhomogeneous wave equation (3.30) on the previous page, turns into

$$\nabla^2 \Psi_\omega(\mathbf{x}) + k^2 \Psi_\omega(\mathbf{x}) = -f_\omega(\mathbf{x}) \quad (3.34)$$

which is the 3D *inhomogeneous time-independent wave equation*, often called the 3D *inhomogeneous Helmholtz equation*.

As postulated by *Huygens's principle*, each point on a wave front acts as a point source for spherical wavelets of varying amplitude (weight). A new wave front is formed by a linear superposition of the individual weighted wavelets from each of the point sources on the old wave front. The solution of (3.34) can therefore be expressed as a weighted sum of solutions of an equation where the source term has been replaced by a single point source

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') + k^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \quad (3.35)$$

and the solution of equation (3.34) which corresponds to the frequency ω is given by the weighted superposition

$$\Psi_\omega(\mathbf{x}) = \int_{V'} d^3x' f_\omega(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \quad (3.36)$$

(plus boundary conditions) where $f_\omega(\mathbf{x}')$ is the wavelet amplitude at the source point \mathbf{x}' . The function $G(\mathbf{x}, \mathbf{x}')$ is called the *Green function* or the *propagator*.

Due to translational invariance in space, $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x} - \mathbf{x}')$. Furthermore, in equation (3.35) above, the Dirac generalised function $\delta(\mathbf{x} - \mathbf{x}')$, which represents the point source, depends only on $\mathbf{x} - \mathbf{x}'$ and there is no angular dependence in the equation. Hence, the solution can only be dependent on $r = |\mathbf{x} - \mathbf{x}'|$ and not on the direction of $\mathbf{x} - \mathbf{x}'$. If we interpret r as the radial coordinate in a spherically polar coordinate system, and recall the expression for the Laplace operator in such a coordinate system, equation (3.35) becomes

$$\frac{d^2}{dr^2}(rG) + k^2(rG) = -r\delta(r) \quad (3.37)$$

Away from $r = |\mathbf{x} - \mathbf{x}'| = 0$, *i.e.*, away from the source point \mathbf{x}' , this equation takes the form

$$\frac{d^2}{dr^2}(rG) + k^2(rG) = 0 \quad (3.38)$$

with the well-known general solution

$$G = C_+ \frac{e^{ikr}}{r} + C_- \frac{e^{-ikr}}{r} \equiv C_+ \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_- \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad (3.39)$$

where C_{\pm} are constants.

In order to evaluate the constants C_{\pm} , we insert the general solution, equation (3.39) above, into equation (3.35) on the facing page and integrate over a small volume around $r = |\mathbf{x} - \mathbf{x}'| = 0$. Since

$$G(|\mathbf{x} - \mathbf{x}'|) \sim C_+ \frac{1}{|\mathbf{x} - \mathbf{x}'|} + C_- \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \quad |\mathbf{x} - \mathbf{x}'| \rightarrow 0 \quad (3.40)$$

the volume integrated equation (3.35) on the preceding page can be approximated by

$$\begin{aligned} (C_+ + C_-) \int_{V'} d^3x' \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ + k^2 (C_+ + C_-) \int_{V'} d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} = - \int_{V'} d^3x' \delta(|\mathbf{x} - \mathbf{x}'|) \end{aligned} \quad (3.41)$$

In virtue of the fact that the volume element d^3x' in spherical polar coordinates is proportional to $r^2 = |\mathbf{x} - \mathbf{x}'|^2$ [see formula (F.52) on page 198], the second integral vanishes when $|\mathbf{x} - \mathbf{x}'| \rightarrow 0$. Furthermore, from equation (F.121) on page 201, we find that the integrand in the first integral can be written as $-4\pi\delta(|\mathbf{x} - \mathbf{x}'|)$ and, hence, that

$$C_+ + C_- = \frac{1}{4\pi} \quad (3.42)$$

Insertion of the general solution equation (3.39) above into equation (3.36) on the facing page gives

$$\Psi_{\omega}(\mathbf{x}) = C_+ \int_{V'} d^3x' f_{\omega}(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_- \int_{V'} d^3x' f_{\omega}(\mathbf{x}') \frac{e^{-ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad (3.43)$$

The inverse Fourier transform of this back to the t domain is obtained by inserting the above expression for $\Psi_{\omega}(\mathbf{x})$ into equation (3.32a) on page 37:

$$\begin{aligned} \Psi(t, \mathbf{x}) = C_+ \int_{V'} d^3x' \int_{-\infty}^{\infty} d\omega f_{\omega}(\mathbf{x}') \frac{\exp \left[-i\omega \left(t - \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega} \right) \right]}{|\mathbf{x} - \mathbf{x}'|} \\ + C_- \int_{V'} d^3x' \int_{-\infty}^{\infty} d\omega f_{\omega}(\mathbf{x}') \frac{\exp \left[-i\omega \left(t + \frac{k|\mathbf{x}-\mathbf{x}'|}{\omega} \right) \right]}{|\mathbf{x} - \mathbf{x}'|} \end{aligned} \quad (3.44)$$

If we introduce the *retarded time* t'_{ret} and the *advanced time* t'_{adv} in the following way [using the fact that $k/\omega = 1/c$ in free space, according to formula (3.33) on page 38]:

$$t'_{\text{ret}} = t'_{\text{ret}}(t, |\mathbf{x} - \mathbf{x}'|) = t - \frac{k |\mathbf{x} - \mathbf{x}'|}{\omega} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \quad (3.45a)$$

$$t'_{\text{adv}} = t'_{\text{adv}}(t, |\mathbf{x} - \mathbf{x}'|) = t + \frac{k |\mathbf{x} - \mathbf{x}'|}{\omega} = t + \frac{|\mathbf{x} - \mathbf{x}'|}{c} \quad (3.45b)$$

and use equation (3.31) on page 37, we obtain

$$\Psi(t, \mathbf{x}) = C_+ \int_{V'} d^3x' \frac{f(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + C_- \int_{V'} d^3x' \frac{f(t'_{\text{adv}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (3.46)$$

This is a solution to the generic inhomogeneous wave equation for the potential components equation (3.30) on page 37. We note that the solution at time t at the field point \mathbf{x} is dependent on the behaviour at other times t' of the source at \mathbf{x}' and that both retarded and advanced t' are mathematically acceptable solutions. However, if we assume that causality requires that the potential at (t, \mathbf{x}) is set up by the source at an earlier time, *i.e.*, at $(t'_{\text{ret}}, \mathbf{x}')$, we must in equation (3.46) above set $C_- = 0$ and therefore, according to equation (3.42) on the preceding page, $C_+ = 1/(4\pi)$.²

From the above discussion about the solution of the inhomogeneous wave equations in the Lorenz-Lorentz gauge we conclude that, if we discard the advanced potentials, the electrodynamic potentials in free space can be written

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}}))}{|\mathbf{x}(t) - \mathbf{x}'(t'_{\text{ret}})|} + \alpha \quad (3.47a)$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int_{V'} d^3x' \frac{\mathbf{j}(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}}))}{|\mathbf{x}(t) - \mathbf{x}'(t'_{\text{ret}})|} + \mathbf{a}(\mathbf{x}) \quad (3.47b)$$

where α is a scalar field dependent on time at most and therefore has a vanishing gradient, and \mathbf{a} is a time-independent vector field that has vanishing curl. These *retarded potentials* were obtained as solutions to the Lorenz-Lorentz equations (3.29) on page 37. They are therefore valid in the Lorenz-Lorentz gauge but may be gauge transformed according to the scheme described in subsection 3.5 on page 42. As they stand, we shall use them frequently in the sequel.

3.4.2 Coulomb gauge

In *Coulomb gauge*, often employed in *quantum electrodynamics*, one chooses $\nabla \cdot \mathbf{A} = 0$ so that equations (3.26) or equations (3.27) on page 36 become

$$\nabla^2 \phi = -\frac{\rho(t, \mathbf{x})}{\epsilon_0} \quad (3.48a)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}(t, \mathbf{x}) + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} \quad (3.48b)$$

² In fact, inspired by ideas put forward by PAUL ADRIEN MAURICE DIRAC (1902–1984), JOHN ARCHIBALD WHEELER (1911–2008) and RICHARD PHILLIPS FEYNMAN (1918–1988) derived, in 1945, a consistent electrodynamics based on both the retarded and the advanced potentials.

The first of these two is the *time-dependent Poisson's equation* which, in analogy with equation (3.3) on page 32, has the solution

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \alpha(t) \quad (3.49)$$

where α may vary in time but has a vanishing (spatial) gradient. We see that in the scalar potential expression, the charge density source is evaluated at time t . The retardation (and advancement) effects therefore occur only in the vector potential \mathbf{A} , which is the solution of the inhomogeneous wave equation (3.48b) on the preceding page.

In order to solve this equation, one splits up \mathbf{j} in a longitudinal (\parallel) and transverse (\perp) part, so that $\mathbf{j} \equiv \mathbf{j}_{\parallel} + \mathbf{j}_{\perp}$ where $\nabla \cdot \mathbf{j}_{\perp} = 0$ and $\nabla \times \mathbf{j}_{\parallel} = \mathbf{0}$ [cf. equations (2.48) on page 30], and note that the equation of continuity, equation (1.25) on page 10 becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_{\parallel} = \left[\frac{\partial}{\partial t} (-\epsilon_0 \nabla^2 \phi) + \nabla \cdot \mathbf{j}_{\parallel} \right] = \nabla \cdot \left[\left(-\epsilon_0 \nabla \frac{\partial \phi}{\partial t} \right) + \mathbf{j}_{\parallel} \right] = 0 \quad (3.50)$$

Furthermore, since $\nabla \times \nabla = \mathbf{0}$ and $\nabla \times \mathbf{j}_{\parallel} = \mathbf{0}$, one finds that

$$\nabla \times \left[\left(-\epsilon_0 \nabla \frac{\partial \phi}{\partial t} \right) + \mathbf{j}_{\parallel} \right] = \mathbf{0} \quad (3.51)$$

According to Helmholtz' theorem, this implies that

$$\epsilon_0 \nabla \frac{\partial \phi}{\partial t} = \mathbf{j}_{\parallel} \quad (3.52)$$

The inhomogeneous wave equation (3.48b) on the facing page thus becomes

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} = -\mu_0 \mathbf{j} + \mu_0 \mathbf{j}_{\parallel} = -\mu_0 \mathbf{j}_{\perp} \quad (3.53)$$

which shows that in Coulomb gauge the source of the vector potential \mathbf{A} is the transverse part of the current, \mathbf{j}_{\perp} . The longitudinal part of the current \mathbf{j}_{\parallel} does not contribute to the vector potential. The retarded solution is [cf. equation (3.47a) on the preceding page]:

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}_{\perp}(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{a}(\mathbf{x}) \quad (3.54)$$

where $\mathbf{a}(\mathbf{x})$ has vanishing curl. The Coulomb gauge condition is therefore also called the *transverse gauge* or *radiation gauge*.

3.4.3 Velocity gauge

If $\nabla \cdot \mathbf{A}$ fulfils the *velocity gauge condition*, sometimes referred to as the *complete α -Lorenz gauge*,

$$\nabla \cdot \mathbf{A} + \alpha \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0, \quad \alpha = \frac{c^2}{v^2} \quad (3.55)$$

we obtain the Lorenz-Lorentz gauge condition in the limit $v = c$, *i.e.*, $\alpha = 1$, and the Coulomb gauge condition in the limit $v \rightarrow \infty$, *i.e.*, $\alpha = 0$, respectively, where v is the propagation speed of the scalar potential. Hence, the velocity gauge is a generalisation of both these gauges.³ Inserting equation (3.55) into the coupled inhomogeneous wave equations (3.27) on page 36 they become

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho(t, \mathbf{x})}{\epsilon_0} - \frac{1-\alpha}{c^2} \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} \quad (3.56a)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}(t, \mathbf{x}) + \frac{1-\alpha}{c^2} \nabla \frac{\partial \phi}{\partial t} \quad (3.56b)$$

³ The value $\alpha = -1$, corresponding to an imaginary speed $v = ic$, gives the *Kirchhoff gauge*, published already in 1857 by GUSTAV ROBERT KIRCHHOFF (1824–1884).

3.4.4 Other gauges

Other useful gauges are

- The *Poincaré gauge* (or *radial gauge*) where

$$\phi(t, \mathbf{x}) = -\mathbf{x} \cdot \int_0^1 d\lambda \mathbf{E}(t, \lambda \mathbf{x}) \quad (3.57a)$$

$$\mathbf{A}(t, \mathbf{x}) = \int_0^1 d\lambda \mathbf{B}(t, \lambda \mathbf{x}) \times \lambda \mathbf{x} \quad (3.57b)$$

- The *Weyl gauge*, also known as the *temporal gauge* or *Hamilton gauge*, defined by $\phi = 0$.
- The *axial gauge*, defined by $A_3 = 0$.

The process of choosing a particular gauge condition is known as *gauge fixing*.

3.5 Gauge transformations

We saw in section 3.1 on page 31 and in section 3.2 on page 32 that in electrostatics and magnetostatics we have a certain *mathematical* degree of freedom, up to terms of vanishing gradients and curls, to pick suitable forms for the potentials and still get the same *physical* result. In fact, the way the electromagnetic scalar

potential $\phi(t, \mathbf{x})$ and the vector potential $\mathbf{A}(t, \mathbf{x})$ are related to the physical observables gives leeway for similar manipulation of them also in electrodynamics.

If we *simultaneously* transform both $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ into new ones $\phi'(t, \mathbf{x})$ and $\mathbf{A}'(t, \mathbf{x})$ according to the scheme

$$\phi(t, \mathbf{x}) \mapsto \phi'(t, \mathbf{x}) = \phi(t, \mathbf{x}) + \frac{\partial \Gamma(t, \mathbf{x})}{\partial t} \quad (3.58a)$$

$$\mathbf{A}(t, \mathbf{x}) \mapsto \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) - \nabla \Gamma(t, \mathbf{x}) \quad (3.58b)$$

where $\Gamma(t, \mathbf{x})$ is an arbitrary, sufficiently differentiable scalar function called the *gauge function*, and insert the transformed potentials into equation (3.25) on page 36 for the electric field and into equation (3.21) on page 35 for the magnetic field, we obtain the transformed fields

$$\mathbf{E}' = -\nabla \phi' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla \phi - \frac{\partial(\nabla \Gamma)}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial(\nabla \Gamma)}{\partial t} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \quad (3.59a)$$

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} - \nabla \times (\nabla \Gamma) = \nabla \times \mathbf{A} \quad (3.59b)$$

where, once again equation (F.107) on page 201 was used. We see that the fields are unaffected by the gauge transformation (3.58). A transformation of the potentials ϕ and \mathbf{A} which leaves the fields, and hence Maxwell's equations, invariant is called a *gauge transformation*. Any physical law that does not change under a gauge transformation is said to be *gauge invariant*. It is only those quantities (expressions) that are gauge invariant that are observable and therefore have experimental significance. Trivially, the electromagnetic fields and the Maxwell-Lorentz equations themselves are gauge invariant and electrodynamics is therefore a *gauge theory* and as such the prototype for all gauge theories.⁴

As just shown, the potentials $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ calculated from equations (3.26) on page 36, with an arbitrary choice of $\nabla \cdot \mathbf{A}$, can be gauge transformed according to (3.58). If, in particular, we choose $\nabla \cdot \mathbf{A}$ according to the Lorenz-Lorentz condition, equation (3.28) on page 37, and apply the gauge transformation (3.58) on the resulting Lorenz-Lorentz potential equations (3.29) on page 37, these equations will be transformed into

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2 \Gamma \right) = \frac{\rho(t, \mathbf{x})}{\epsilon_0} \quad (3.60a)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} - \nabla \left(\frac{1}{c^2} \frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2 \Gamma \right) = \mu_0 \mathbf{j}(t, \mathbf{x}) \quad (3.60b)$$

We notice that if we require that the gauge function $\Gamma(t, \mathbf{x})$ itself be restricted to fulfil the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \Gamma}{\partial t^2} - \nabla^2 \Gamma = 0 \quad (3.61)$$

⁴ A very important extension is the *Yang-Mills theory*, introduced in 1954. This theory has had a profound impact on modern physics.

these transformed Lorenz-Lorentz equations will keep their original form. The set of potentials which have been gauge transformed according to equation (3.58) on the previous page with a gauge function $\Gamma(t, \mathbf{x})$ restricted to fulfil equation (3.61) on the preceding page, or, in other words, those gauge transformed potentials for which the equations (3.29) on page 37 are invariant, comprise the *Lorenz-Lorentz gauge*.

EXAMPLE 3.2 ▷ Electromagnetodynamic potentials

In Dirac's symmetrised form of electrodynamics (electromagnetodynamics), Maxwell's equations are replaced by [see also equations (1.50) on page 16]:

$$\nabla \cdot \mathbf{E} = \frac{\rho^e}{\epsilon_0} \quad (3.62a)$$

$$\nabla \times \mathbf{E} = -\mu_0 \mathbf{j}^m - \frac{\partial \mathbf{B}}{\partial t} \quad (3.62b)$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^m \quad (3.62c)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^e + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (3.62d)$$

In this theory, one derives the inhomogeneous wave equations for the usual 'electric' scalar and vector potentials (ϕ^e, \mathbf{A}^e) and their 'magnetic' counterparts (ϕ^m, \mathbf{A}^m) by assuming that the potentials are related to the fields in the following symmetrised form:

$$\mathbf{E} = -\nabla \phi^e(t, \mathbf{x}) - \frac{\partial}{\partial t} \mathbf{A}^e(t, \mathbf{x}) - \nabla \times \mathbf{A}^m \quad (3.63a)$$

$$\mathbf{B} = -\frac{1}{c^2} \nabla \phi^m(t, \mathbf{x}) - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{A}^m(t, \mathbf{x}) + \nabla \times \mathbf{A}^e \quad (3.63b)$$

In the absence of magnetic charges, or, equivalently for $\phi^m \equiv 0$ and $\mathbf{A}^m \equiv \mathbf{0}$, these formulæ reduce to the usual Maxwell theory formula (3.25) on page 36 and formula (3.21) on page 35, respectively, as they should.

Inserting the symmetrised expressions (3.63) above into equations (3.62), one obtains [cf., equations (3.26) on page 36]

$$\nabla^2 \phi^e + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}^e) = -\frac{\rho^e(t, \mathbf{x})}{\epsilon_0} \quad (3.64a)$$

$$\nabla^2 \phi^m + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}^m) = -\frac{\rho^m(t, \mathbf{x})}{\epsilon_0} \quad (3.64b)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^e}{\partial t^2} - \nabla^2 \mathbf{A}^e + \nabla \left(\nabla \cdot \mathbf{A}^e + \frac{1}{c^2} \frac{\partial \phi^e}{\partial t} \right) = \mu_0 \mathbf{j}^e(t, \mathbf{x}) \quad (3.64c)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^m}{\partial t^2} - \nabla^2 \mathbf{A}^m + \nabla \left(\nabla \cdot \mathbf{A}^m + \frac{1}{c^2} \frac{\partial \phi^m}{\partial t} \right) = \mu_0 \mathbf{j}^m(t, \mathbf{x}) \quad (3.64d)$$

By choosing the conditions on the divergence of the vector potentials as the generalised

Lorenz-Lorentz condition [cf. equation (3.28) on page 37]

$$\nabla \cdot \mathbf{A}^e + \frac{1}{c^2} \frac{\partial}{\partial t} \phi^e = 0 \quad (3.65)$$

$$\nabla \cdot \mathbf{A}^m + \frac{1}{c^2} \frac{\partial}{\partial t} \phi^m = 0 \quad (3.66)$$

these coupled wave equations simplify to

$$\frac{1}{c^2} \frac{\partial^2 \phi^e}{\partial t^2} - \nabla^2 \phi^e = \frac{\rho^e(t, \mathbf{x})}{\epsilon_0} \quad (3.67a)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^e}{\partial t^2} - \nabla^2 \mathbf{A}^e = \mu_0 \mathbf{j}^e(t, \mathbf{x}) \quad (3.67b)$$

$$\frac{1}{c^2} \frac{\partial^2 \phi^m}{\partial t^2} - \nabla^2 \phi^m = \frac{\rho^m(t, \mathbf{x})}{\epsilon_0} \quad (3.67c)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}^m}{\partial t^2} - \nabla^2 \mathbf{A}^m = \mu_0 \mathbf{j}^m(t, \mathbf{x}) \quad (3.67d)$$

exhibiting, once again, the striking properties of Dirac's symmetrised Maxwell theory.

—End of example 3.2◁

▷Gauge transformations and quantum mechanics

EXAMPLE 3.3

As discussed in section 2.2 on page 20 quantum theory requires that we take the magnitude rather than the real part of our mathematical variables in order to turn them into physical observables. In non-relativistic quantum mechanics, the physical observable *probability density* is $\psi \psi^* = |\psi|^2$, where the wave function $\psi \in \mathbb{C}$ solves the *Schrödinger equation*

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi \quad (3.68)$$

and \hat{H} is the *Hamilton operator*.

The non-relativistic *Hamiltonian* for a classical particle with charge q in an electromagnetic field, described by the scalar potential ϕ and vector potential \mathbf{A} , is

$$H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2 + q\phi \quad (3.69)$$

where \mathbf{p} is the linear momentum. The corresponding quantal Hamilton operator is obtained by replacing \mathbf{p} by the operator $\hat{\mathbf{p}} = -i\hbar \nabla$, which is called *minimal coupling*. This gives the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 \psi + q\phi \psi \quad (3.70)$$

The idea is to perform a gauge transformation from the potentials $\phi(t, \mathbf{x})$ and $\mathbf{A}(t, \mathbf{x})$ to new potentials

$$\phi \rightarrow \phi'(t, \mathbf{x}) = \phi(t, \mathbf{x}) + \frac{\partial \Gamma(t, \mathbf{x})}{\partial t} \quad (3.71)$$

$$\mathbf{A} \rightarrow \mathbf{A}'(t, \mathbf{x}) = \mathbf{A}(t, \mathbf{x}) - \nabla \Gamma(t, \mathbf{x}) \quad (3.72)$$

and then find a $\theta(t, \mathbf{x})$, expressed in the gauge function $\Gamma(t, \mathbf{x})$, so that the transformed

Schrödinger equation can be written

$$i\hbar \frac{\partial e^{i\theta} \psi}{\partial t} = \frac{1}{2m} (-i\hbar \nabla - q\mathbf{A})^2 e^{i\theta} \psi + q\phi e^{i\theta} \psi \quad (3.73)$$

Under the gauge transformation, the Schrödinger equation (3.70) transforms into

$$i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m} [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)]^2 \psi' + q\phi \psi' + q \frac{\partial \Gamma}{\partial t} \psi' \quad (3.74)$$

Now, setting

$$\psi'(t, \mathbf{x}) = e^{i\theta(t, \mathbf{x})} \psi(t, \mathbf{x}) \quad (3.75)$$

we see that

$$\begin{aligned} & [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)]^2 \psi' \\ &= [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)] [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)] e^{i\theta} \psi \\ &= [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)] \\ &\quad \times \left[-i\hbar e^{i\theta} (\nabla\psi) - i\hbar e^{i\theta} (\nabla\theta)\psi - q\mathbf{A} e^{i\theta} \psi + (q\nabla\Gamma) e^{i\theta} \psi \right] \\ &= [-i\hbar \nabla - q\mathbf{A} + (q\nabla\Gamma)] e^{i\theta} [-i\hbar \nabla - i\hbar(\nabla\theta) - q\mathbf{A} + (q\nabla\Gamma)] \psi \\ &= \left[-i\hbar(\nabla\theta) e^{i\theta} - i\hbar e^{i\theta} \nabla - q\mathbf{A} e^{i\theta} + (q\nabla\Gamma) e^{i\theta} \right] \\ &\quad \times [-i\hbar \nabla - i\hbar(\nabla\theta) - q\mathbf{A} + (q\nabla\Gamma)] \psi \\ &= e^{i\theta} [-i\hbar \nabla - i\hbar(\nabla\theta) - q\mathbf{A} + (q\nabla\Gamma)]^2 \psi \end{aligned} \quad (3.76)$$

Clearly, the gauge transformed Hamilton operator is unchanged iff $\hbar(\nabla\theta) = -q(\nabla\Gamma)$, or, equivalently, iff $\theta(t, \mathbf{x}) = -q\Gamma(t, \mathbf{x})/\hbar$. This has as a consequence that

$$\begin{aligned} i\hbar \frac{\partial \psi'}{\partial t} - q \frac{\partial \Gamma}{\partial t} \psi' &= i\hbar \frac{\partial \psi'}{\partial t} + \hbar \frac{\partial \theta}{\partial t} \psi' = i\hbar \frac{\partial}{\partial t} (e^{i\theta} \psi) + \hbar \frac{\partial \theta}{\partial t} e^{i\theta} \psi \\ &= i^2 \hbar \frac{\partial \theta}{\partial t} e^{i\theta} \psi + i\hbar e^{i\theta} \frac{\partial \psi}{\partial t} + \hbar \frac{\partial \theta}{\partial t} e^{i\theta} \psi = e^{i\theta} i\hbar \frac{\partial \psi}{\partial t} \end{aligned} \quad (3.77)$$

Inserting this into the transformed Schrödinger equation (3.74) we recover the untransformed Schrödinger equation (3.70).

We conclude that for a gauge transformation of the potentials ϕ and \mathbf{A} and using the minimal coupling as in equation (3.70) on the preceding page, the Schrödinger equation is invariant, but that the wave function changes from ψ to $e^{i\theta} \psi$ where the phase angle is real-valued. This means that the non-relativistic quantum physical observable $|\psi|^2$ is unaffected by a gauge transformation of the classical potentials ϕ and \mathbf{A} that appear in the Hamilton operator. The fact that $\theta(t, \mathbf{x})$ is coordinate dependent, means that we are dealing with a *local gauge transformation*.

For the gauge transformation given by formulæ (3.71) on the previous page WOLFGANG PAULI introduced the notation *gauge transformation of the second kind* whereas he called a wavefunction phase change, expression (3.75) a *gauge transformation of the first kind*.

—End of example 3.3◀

▷Longitudinal and transverse components in gauge transformations——— EXAMPLE 3.4

If we represent the vector potential $\mathbf{A}(t, \mathbf{x})$ in the reciprocal \mathbf{k} space as described at the end of subsection 2.3.1, the gauge transformation equation (3.58b) on page 43 becomes

$$\mathbf{A}_{\mathbf{k}}(t) \mapsto \mathbf{A}'_{\mathbf{k}}(t) - i\mathbf{k}\Gamma_{\mathbf{k}}(t) \quad (3.78)$$

we can separate it into its longitudinal and transverse components

$$\mathbf{A}'_{\parallel} = \mathbf{A}_{\parallel} - \nabla\Gamma \quad (3.79a)$$

$$\mathbf{A}'_{\perp} = \mathbf{A}_{\perp} \quad (3.79b)$$

Hence, a law (expression) that depends on \mathbf{A} only through its transverse component \mathbf{A}_{\perp} is gauge invariant, whereas a law that depends on the longitudinal component \mathbf{A}_{\parallel} is in general not gauge invariant and therefore does not represent a physical observable.

For the electric and magnetic field the following then applies:

$$\mathbf{E}_{\parallel} = -\nabla\Phi - \frac{\partial\mathbf{A}_{\parallel}}{\partial t} \quad (3.80a)$$

$$\mathbf{E}_{\perp} = -\frac{\partial\mathbf{A}_{\perp}}{\partial t} \quad (3.80b)$$

$$\mathbf{B} = \nabla \times \mathbf{A}_{\perp} \quad (3.80c)$$

—End of example 3.4◁

3.6 Bibliography

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DRAFT

4

FUNDAMENTAL PROPERTIES OF THE ELECTROMAGNETIC FIELD

In this chapter we will explore a number of fundamental properties of the electromagnetic fields and their sources, as well as of the physical observables constructed from them. Of particular interest are symmetries since they have a striking predictive power and are essential ingredients of the physics. This includes both discrete and continuous geometric symmetries (reflection, translation, rotation) and intrinsic symmetries (duality, reciprocity). Intimately related to symmetries are conserved quantities (constants of motion) of which our primary interest will be the electromagnetic energy, linear momentum, angular momentum, and centre of energy. These conserved quantities can carry information over large distances and are all more or less straightforwardly related to their counterparts in classical mechanics (indeed in all field theories). But we will also consider other conserved quantities, where this relation is perhaps less straightforward.

To derive useful mathematical expressions for the physical observables that we want to study, we once again take the microscopic Maxwell-Lorentz equations (2.1) on page 20 as our axiomatic starting point.

4.1 Charge, space, and time inversion symmetries

Let us first investigate how the charge density ρ , the current density \mathbf{j} , and the fields \mathbf{E} and \mathbf{B} behave under *charge conjugation*, *i.e.*, a change of sign of charge $q \rightarrow q' = -q$, called *C* symmetry; under *space inversion*, *i.e.*, a change of sign

of the space coordinates $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$, called *parity transformation* and *P* symmetry; and under *time reversal*, *i.e.*, a change of sign of the time coordinate $t \rightarrow t' = -t$, called *T* symmetry. Recalling that

$$\rho \propto q \quad (4.1a)$$

and

$$\mathbf{j} = \rho \mathbf{v} = \rho \frac{d\mathbf{x}}{dt} \quad (4.1b)$$

the transformation properties follow directly from the Maxwell-Lorentz equations (2.1) on page 20. Let us study them one by one.

C SYMMETRY, CHARGE CONJUGATION $q \rightarrow q' = -q$:

$$\rho \rightarrow \rho' = -\rho \quad (4.2a)$$

$$\mathbf{j} \rightarrow \mathbf{j}' = \rho' \frac{d\mathbf{x}'}{dt} = -\rho \frac{d\mathbf{x}'}{dt} = -\rho \frac{d\mathbf{x}}{dt} = -\mathbf{j} \quad (4.2b)$$

$$\nabla \rightarrow \nabla' = \nabla \quad (4.2c)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \quad (4.2d)$$

When inserted into the two first of the Maxwell-Lorentz equations (2.1) on page 20 we find that they become

$$\nabla \cdot \mathbf{E}'(t, \mathbf{x}) = \frac{\rho'}{\epsilon_0} = -\frac{\rho}{\epsilon_0} = -\nabla \cdot \mathbf{E}(t, \mathbf{x})$$

and

$$\nabla \times \mathbf{E}'(t, \mathbf{x}) = \nabla \times [-\mathbf{E}(t, \mathbf{x})] = \frac{\partial}{\partial t} [-\mathbf{B}'(t, \mathbf{x})] = \frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x})$$

respectively, implying that

$$\mathbf{E}'(t, \mathbf{x}) = -\mathbf{E}(t, \mathbf{x}) \quad (4.3a)$$

$$\mathbf{B}'(t, \mathbf{x}) = -\mathbf{B}(t, \mathbf{x}) \quad (4.3b)$$

Consequently, the Maxwell-Lorentz equations are invariant under charge conjugation. No violation of this invariance has yet been found experimentally.

P SYMMETRY, SPACE INVERSION $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$:

$$\rho \rightarrow \rho' = \rho \quad (4.4a)$$

$$\mathbf{j} \rightarrow \mathbf{j}' = \rho' \frac{d\mathbf{x}'}{dt} = \rho \frac{d(-\mathbf{x}')}{dt} = -\rho \frac{d\mathbf{x}}{dt} = -\mathbf{j} \quad (4.4b)$$

$$\nabla \rightarrow \nabla' = -\nabla \quad (4.4c)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} \quad (4.4d)$$

implying that

$$\mathbf{E}'(t, \mathbf{x}) \equiv \mathbf{E}(t, -\mathbf{x}) = -\mathbf{E}(t, \mathbf{x}) \quad (4.5a)$$

$$\mathbf{B}'(t, \mathbf{x}) \equiv \mathbf{B}(t, -\mathbf{x}) = \mathbf{B}(t, \mathbf{x}) \quad (4.5b)$$

Since, by assumption, t , ρ and ε_0 are ordinary scalars and that, consequently, ρ/ε_0 is an ordinary scalar, and that the position vector \mathbf{x} is the prototype of all ordinary (polar) vectors, we note that \mathbf{j} and \mathbf{E} are ordinary vectors, whereas \mathbf{B} is a *pseudovector* (axial vector) as described in subsection M.1.3.2 on page 217.

T SYMMETRY, TIME REVERSAL $t \rightarrow t' = -t$:

$$\rho \rightarrow \rho' = \rho \quad (4.6a)$$

$$\mathbf{j} \rightarrow \mathbf{j}' = -\mathbf{j} \quad (4.6b)$$

$$\nabla \rightarrow \nabla' = \nabla \quad (4.6c)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} = -\frac{\partial}{\partial t} \quad (4.6d)$$

implying that

$$\mathbf{E}'(t, \mathbf{x}) \equiv \mathbf{E}(-t, \mathbf{x}) = \mathbf{E}(t, \mathbf{x}) \quad (4.7a)$$

$$\mathbf{B}'(t, \mathbf{x}) \equiv \mathbf{B}(-t, \mathbf{x}) = -\mathbf{B}(t, \mathbf{x}) \quad (4.7b)$$

We see that \mathbf{E} is even and \mathbf{B} odd under time reversal. The Universe as a whole is asymmetric under time reversal. On quantum scales this is exhibited by the *uncertainty principle* and on classical scales by the *arrow of time* which is related to the increase of *thermodynamic entropy* in a closed system.

The *CPT theorem* states that the combined CPT symmetry must hold for all physical phenomena. No violation of this law has been observed so far.

4.2 Electromagnetic duality

We notice that Dirac's symmetrised version of the Maxwell equations (1.50) on page 16 exhibit the following symmetry (recall that $\varepsilon_0\mu_0 = 1/c^2$):

$$\mathbf{E} \rightarrow c\mathbf{B} \quad (4.8a)$$

$$c\mathbf{B} \rightarrow -\mathbf{E} \quad (4.8b)$$

$$c\rho^e \rightarrow \rho^m \quad (4.8c)$$

$$\rho^m \rightarrow -c\rho^e \quad (4.8d)$$

$$c\mathbf{j}^e \rightarrow \mathbf{j}^m \quad (4.8e)$$

$$\mathbf{j}^m \rightarrow -c\mathbf{j}^e \quad (4.8f)$$

which is a particular case ($\theta = \pi/2$) of the general *duality transformation*, also known as the *Heaviside-Larmor-Rainich transformation* (indicated by the *Hodge star operator* \star in the upper left-hand corner of the symbol in question)

$$\star \mathbf{E} = \mathbf{E} \cos \theta + c \mathbf{B} \sin \theta \quad (4.9a)$$

$$c \star \mathbf{B} = -\mathbf{E} \sin \theta + c \mathbf{B} \cos \theta \quad (4.9b)$$

$$c \star \rho^e = c \rho^e \cos \theta + \rho^m \sin \theta \quad (4.9c)$$

$$\star \rho^m = -c \rho^e \sin \theta + \rho^m \cos \theta \quad (4.9d)$$

$$c \star \mathbf{j}^e = c \mathbf{j}^e \cos \theta + \mathbf{j}^m \sin \theta \quad (4.9e)$$

$$\star \mathbf{j}^m = -c \mathbf{j}^e \sin \theta + \mathbf{j}^m \cos \theta \quad (4.9f)$$

This transformation leaves the symmetrised Maxwell equations, and hence the physics they describe (often referred to as *electromagnetodynamics*), invariant. Since \mathbf{E} and \mathbf{j}^e are true (polar) vectors, \mathbf{B} a pseudovector (axial vector), ρ^e a (true) scalar, we conclude that the magnetic charge density ρ^m and θ , which behaves as a *mixing angle* in a two-dimensional ‘charge space’, must be pseudo-scalars¹ and \mathbf{j}^m a pseudovector.

¹ Recall that the Taylor expansion for $\cos \theta$ contains only even powers of θ whereas the expansion for $\sin \theta$ contains only odd.

The invariance of Dirac’s symmetrised Maxwell equations under the duality transformation (4.9) means that the amount of magnetic monopole density ρ^m is irrelevant for the physics as long as the ratio $\rho^m/\rho^e = \tan \theta$ is kept constant. So whether we assume that the particles are only electrically charged or have also a magnetic charge with a given, fixed ratio between the two types of charges is a matter of convention, as long as we assume that this fraction is *the same for all particles*. Such particles are referred to as *dyons*. By varying the mixing angle θ we can change the fraction of magnetic monopoles at will without changing the laws of electrodynamics. For $\theta = 0$ we recover the usual Maxwell electrodynamics.

EXAMPLE 4.1 ▷Duality of the electromagnetodynamic equations

Show that the symmetric, electromagnetodynamic Maxwell-Lorentz equations (Dirac’s symmetrised Maxwell equations), equations (1.50) on page 16, are invariant under the duality transformation (4.9).

Explicit application of the transformation yields

$$\begin{aligned} \nabla \cdot \star \mathbf{E} &= \nabla \cdot (\mathbf{E} \cos \theta + c \mathbf{B} \sin \theta) = \frac{\rho^e}{\varepsilon_0} \cos \theta + c \mu_0 \rho^m \sin \theta \\ &= \frac{1}{\varepsilon_0} \left(\rho^e \cos \theta + \frac{1}{c} \rho^m \sin \theta \right) = \frac{\star \rho^e}{\varepsilon_0} \end{aligned} \quad (4.10)$$

$$\begin{aligned}\nabla \times \star \mathbf{E} + \frac{\partial \star \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{E} \cos \theta + c \mathbf{B} \sin \theta) + \frac{\partial}{\partial t} \left(-\frac{1}{c} \mathbf{E} \sin \theta + \mathbf{B} \cos \theta \right) \\ &= -\mu_0 \mathbf{j}^m \cos \theta - \frac{\partial \mathbf{B}}{\partial t} \cos \theta + c \mu_0 \mathbf{j}^e \sin \theta + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \sin \theta \\ &\quad - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \sin \theta + \frac{\partial \mathbf{B}}{\partial t} \cos \theta = -\mu_0 \mathbf{j}^m \cos \theta + c \mu_0 \mathbf{j}^e \sin \theta \\ &= -\mu_0 (-c \mathbf{j}^e \sin \theta + \mathbf{j}^m \cos \theta) = -\mu_0 \star \mathbf{j}^m\end{aligned}\quad (4.11)$$

$$\begin{aligned}\nabla \cdot \star \mathbf{B} &= \nabla \cdot \left(-\frac{1}{c} \mathbf{E} \sin \theta + \mathbf{B} \cos \theta \right) = -\frac{\rho^e}{c \epsilon_0} \sin \theta + \mu_0 \rho^m \cos \theta \\ &= \mu_0 (-c \rho^e \sin \theta + \rho^m \cos \theta) = \mu_0 \star \rho^m\end{aligned}\quad (4.12)$$

$$\begin{aligned}\nabla \times \star \mathbf{B} - \frac{1}{c^2} \frac{\partial \star \mathbf{E}}{\partial t} &= \nabla \times \left(-\frac{1}{c} \mathbf{E} \sin \theta + \mathbf{B} \cos \theta \right) - \frac{1}{c^2} \frac{\partial}{\partial t} (\mathbf{E} \cos \theta + c \mathbf{B} \sin \theta) \\ &= \frac{1}{c} \mu_0 \mathbf{j}^m \sin \theta + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \cos \theta + \mu_0 \mathbf{j}^e \cos \theta + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \cos \theta \\ &\quad - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \cos \theta - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \sin \theta \\ &= \mu_0 \left(\frac{1}{c} \mathbf{j}^m \sin \theta + \mathbf{j}^e \cos \theta \right) = \mu_0 \star \mathbf{j}^e\end{aligned}\quad (4.13)$$

QED ■

End of example 4.1 ◁

▷ Duality expressed in Riemman-Silberstein formalism

EXAMPLE 4.2

Expressed in the Riemann-Silberstein complex field vector, introduced in equation (2.6) on page 22, the duality transformation equations (4.9) on the preceding page become

$$\begin{aligned}\star \mathbf{G} &= \star \mathbf{E} + ic \star \mathbf{B} = \mathbf{E} \cos \theta + c \mathbf{B} \sin \theta - i \mathbf{E} \sin \theta + ic \mathbf{B} \cos \theta \\ &= \mathbf{E}(\cos \theta - i \sin \theta) + ic \mathbf{B}(\cos \theta - i \sin \theta) = (\mathbf{E} + ic \mathbf{B})e^{-i\theta} = \mathbf{G}e^{-i\theta}\end{aligned}\quad (4.14)$$

from which it is easy to see that

$$\star \mathbf{G} \cdot \star \mathbf{G} = |\star \mathbf{G}|^2 = \mathbf{G}e^{-i\theta} \cdot \mathbf{G}e^{i\theta} = |\mathbf{G}|^2\quad (4.15)$$

whereas

$$\star \mathbf{G} \cdot \star \mathbf{G} = \mathbf{G} \cdot \mathbf{G}e^{-2i\theta}\quad (4.16)$$

Furthermore, assuming that $\theta = \theta(t, \mathbf{x})$, we see that spatial and temporal differentiation of $\star \mathbf{G}$ leads to

$$\partial_t \star \mathbf{G} \equiv \frac{\partial \star \mathbf{G}}{\partial t} = -i(\partial_t \theta)e^{-i\theta} \mathbf{G} + e^{-i\theta} \partial_t \mathbf{G}\quad (4.17a)$$

$$\partial \cdot \star \mathbf{G} \equiv \nabla \cdot \star \mathbf{G} = -ie^{-i\theta} \nabla \theta \cdot \mathbf{G} + e^{-i\theta} \nabla \cdot \mathbf{G}\quad (4.17b)$$

$$\partial \times \star \mathbf{G} \equiv \nabla \times \star \mathbf{G} = -ie^{-i\theta} \nabla \theta \times \mathbf{G} + e^{-i\theta} \nabla \times \mathbf{G}\quad (4.17c)$$

which means that $\partial_t \star \mathbf{G}$ transforms as $\star \mathbf{G}$ itself only if θ is time-independent, and that $\nabla \cdot \star \mathbf{G}$ and $\nabla \times \star \mathbf{G}$ transform as $\star \mathbf{G}$ itself only if θ is space-independent.

End of example 4.2 ◁

4.3 Conservation laws

It is well established that a deeper understanding of a physical system can be obtained by studying the system's *conserved quantities* (*constants of motion*), *i.e.*, those observables that do not change with time. In other words, the task is to find the *conservation laws* for the physical system under study, which, according to *Noether's theorem*,² is closely related to finding the symmetries of the system.

² AMALIE EMMY NOETHER (1882–1935) made important contributions to mathematics and theoretical physics. Her (first) theorem, which states that any differentiable symmetry of the action of a physical system has a corresponding conservation law, is considered to be a fundamental tool of theoretical physics.

Consider a certain physical property or ‘substance’, *e.g.* electric charge, that flows, in a time-dependent way, in 3D space. Let us denote the volume density of this ‘substance’ by $\varrho(t, \mathbf{x})$ and its flow velocity by $\mathbf{v}(t, \mathbf{x})$. Let us also introduce a closed volume V , fixed in space and enclosed by a perfectly permeable surface S with a infinitesimally small directed area element $d^2x \hat{\mathbf{n}} = dx_i dx_j \hat{\mathbf{n}}$, where dx_i and dx_j are two coordinates that span the local tangent plane of the surface and $\hat{\mathbf{n}}$ is a unit vector, orthogonal to the tangent plane and pointing outward. In the special case that the volume V is spherical, it is usually referred to as the *control sphere*.

The property or ‘substance’ flows into the volume V at the rate $\varrho \mathbf{v}$ per unit area. Clearly, this inward flow will increase the amount of ‘substance’ in V . We also allow for an increase due to the production of the ‘substance’ inside V . This increase is quantified by the source density s . Recalling that the normal vector $\hat{\mathbf{n}}$ points outward, the following balance equation must hold:

$$\underbrace{\frac{d}{dt} \int_V d^3x \varrho}_{\text{Total change within } V} = - \underbrace{\oint_S d^2x \hat{\mathbf{n}} \cdot \varrho \mathbf{v}}_{\text{Flow into } V} + \underbrace{\int_V d^3x s}_{\text{Production inside } V} \quad (4.18)$$

With the help of Gauss's theorem, formula (F.126) on page 202, this balance equation can be rewritten

$$\int_V d^3x \left(\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) - s \right) = 0 \quad (4.19)$$

Since this equation must hold for an arbitrary volume V , the integrand must vanish and we obtain the *continuity equation*

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) = s \quad (4.20)$$

which expresses, in differential form, the balance between the explicit temporal change of the density of the ‘substance’, the flow of it across the surface of a closed volume, and the production of the ‘substance’ in the volume enclosed by the surface. If there is no production of the ‘substance’ in V , *i.e.*, if $s = 0$, the equation describes the constancy of the amount of ‘substance’.

In fact, since $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ for any arbitrary, differentiable vector field \mathbf{a} , we can generalise equation (4.20) on the preceding page to

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v} + \nabla \times \mathbf{a}) = s \quad (4.21)$$

Furthermore, since the time derivative d/dt operating on a scalar, vector or tensor field, dependent on t and $\mathbf{x}(t)$, is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (4.22)$$

according to the chain rule [cf. equation (1.36) on page 13], and since

$$\nabla \cdot (\varrho \mathbf{v}) = \varrho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \varrho \quad (4.23)$$

we can rewrite the equation of continuity as

$$\frac{d\varrho}{dt} + \varrho \nabla \cdot (\mathbf{v} + \nabla \times \mathbf{a}) = s \quad (4.24)$$

where, again, \mathbf{a} is an arbitrary, differentiable pseudovector field with dimension m^2s^{-1} , e.g., the *moment of velocity* with respect to \mathbf{x}_0 , i.e., $(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}$.

4.3.1 Conservation of electric charge

When the ‘substance’ density ϱ in equation (4.20) on the preceding page is the electric charge density ρ , we exclude any charge-generating mechanisms, i.e., we assume that $s = 0$, and interpret $\rho \mathbf{v}$ as the electric current density \mathbf{j} . Then the continuity equation takes the form

$$\frac{\partial \rho(t, \mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}(t, \mathbf{x}) = 0 \quad (4.25)$$

This is our first *conservation law* derived from the microscopic Maxwell equations (2.1) on page 20. This result is, of course, consistent *a posteriori* with the fact that these axiomatic laws were formulated with the *a priori* assumption that the continuity equation (1.25) on page 10 was valid. The continuity equation (4.25) describes the fact that electric charge is conserved, i.e., indestructible.³

³ This was postulated in 1747 by BENJAMIN FRANKLIN (1706–1770), printer, scientist, inventor, philosopher, statesman, and one of the founding fathers of the United States of America.

4.3.2 Conservation of total electric current

If we use the notation \mathbf{j}^c for the electric current associated with electric charge transport and \mathbf{j}^d for the displacement current $\partial(\epsilon_0 \mathbf{E})/\partial t$, equation (2.1d) on page 20 can be rewritten

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{j}^c + \mathbf{j}^d \quad (4.26)$$

Differentiating this with respect to time t and using the Maxwell equation (2.1b) on page 20 and formula (F.110) on page 201, we obtain the following *conservation law for the total current*

$$\frac{\partial \mathbf{j}^c}{\partial t} + \frac{\partial \mathbf{j}^d}{\partial t} + \nabla \cdot \left(\mathbf{1}_3 \times \frac{1}{\mu_0} (\nabla \times \mathbf{E}) \right) = \mathbf{0} \quad (4.27)$$

This is a rather unusual way of writing the *wave equation* for the electric field vector \mathbf{E} . Usually it is written as in equation (2.28a) on page 25.

4.3.3 Conservation of energy

The continuity equation (4.25) on the previous page contains the divergence of the first-order vector quantity \mathbf{j} . As one example of the divergence of a second order vector quantity, let us study the divergence of $\mathbf{E} \times \mathbf{B}$. Using the Maxwell equations (2.1) on page 20 we obtain the balance equation

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \\ &= -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{E} \cdot \mathbf{j} - \varepsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\ &= -\mu_0 \left(\frac{\partial}{\partial t} \frac{\varepsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) + \mathbf{j} \cdot \mathbf{E} \right) \end{aligned} \quad (4.28)$$

where formula (F.95) on page 200 was used.

Let us introduce the *electromagnetic field energy density*

$$u^{\text{em}} = \frac{\varepsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \quad (\text{Jm}^{-3}) \quad (4.29)$$

and the *electromagnetic energy flux* or the *Poynting vector*

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B} \quad (\text{Wm}^{-2}) \quad (4.30)$$

that can also be viewed as the *electromagnetic energy current density*.

Employing formula (3.21) on page 35, the Poynting vector \mathbf{S} can be expressed in terms of the vector potential \mathbf{A} vector as

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times (\nabla \times \mathbf{A}) \quad (4.31a)$$

$$= \frac{1}{\mu_0} ((\nabla \mathbf{A}) \cdot \mathbf{E} - \mathbf{E} \cdot (\nabla \mathbf{A})) \quad (4.31b)$$

$$= \frac{1}{\mu_0} ((\nabla \mathbf{A}) \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \mathbf{A}) + (\nabla \cdot \mathbf{E}) \mathbf{A}) \quad (4.31c)$$

$$= \frac{1}{\mu_0} (\rho \mathbf{A} + (\nabla \mathbf{A}) \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \mathbf{A})) \quad (4.31d)$$

where we used formula (F.100) on page 201 in the first step, formula (F.96) on page 200 in the second step and the Maxwell-Lorentz equation (2.1a) on page 20. For each spatial Fourier component, we can use the results in subsection 2.3.1 to resolve \mathbf{A} into its longitudinal and transverse components \mathbf{A}_{\parallel} and \mathbf{A}_{\perp} , respectively, with the result that

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times (\nabla \times \mathbf{A}_{\perp}) \quad (4.32a)$$

$$= \frac{1}{\mu_0} ((\nabla \mathbf{A}_{\perp}) \cdot \mathbf{E} - \mathbf{E} \cdot (\nabla \mathbf{A}_{\perp})) \quad (4.32b)$$

$$= \frac{1}{\mu_0} ((\nabla \mathbf{A}_{\perp}) \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \mathbf{A}_{\perp}) + (\nabla \cdot \mathbf{E}) \mathbf{A}_{\perp}) \quad (4.32c)$$

$$= \frac{1}{\mu_0} (\rho \mathbf{A}_{\perp} + (\nabla \mathbf{A}_{\perp}) \cdot \mathbf{E} - \nabla \cdot (\mathbf{E} \mathbf{A}_{\perp})) \quad (4.32d)$$

With the help of these definitions we can write the balance equation (4.28) on the facing page as

$$\frac{\partial u^{\text{em}}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \quad (4.33)$$

i.e., as a continuity equation with a source density

$$s = -\mathbf{j} \cdot \mathbf{E} = -\rho \mathbf{E} \cdot \mathbf{v} \quad (4.34)$$

From the definition of \mathbf{E} , we conclude that $\rho \mathbf{E}$ represents force per unit volume (Nm^{-3}) and that $\rho \mathbf{E} \cdot \mathbf{v}$ therefore represents work per unit volume or, in other words, power density (Wm^{-3}). It is known as the *Lorentz power density* and is equivalent to the time rate of change of the *mechanical energy density* (Jm^{-3}) of the current carrying particles

$$\frac{\partial u^{\text{mech}}}{\partial t} = \mathbf{j} \cdot \mathbf{E} \quad (4.35)$$

Equation (4.33) above can therefore be written

$$\frac{\partial u^{\text{mech}}}{\partial t} + \frac{\partial u^{\text{em}}}{\partial t} + \nabla \cdot \mathbf{S} = 0 \quad (4.36)$$

which is the *energy density balance equation* in differential form.

Expressing the *Lorentz power*, $\int_V d^3x \mathbf{j} \cdot \mathbf{E}$, as the time rate of change of the *mechanical energy*:

$$\frac{dU^{\text{mech}}}{dt} = \int_V d^3x \mathbf{j} \cdot \mathbf{E} \quad (4.37)$$

and introducing the *electromagnetic field energy* $U^{\text{em}} = U^{\text{e}} + U^{\text{m}}$, where

$$U^{\text{e}} = \frac{\epsilon_0}{2} \int_V d^3x \mathbf{E} \cdot \mathbf{E} \quad (4.38)$$

is the *electric field energy* and

$$U^m = \frac{1}{2\mu_0} \int_V d^3x \mathbf{B} \cdot \mathbf{B} = \frac{\varepsilon_0}{2} \int_V d^3x c^2 \mathbf{B} \cdot \mathbf{B} \quad (4.39)$$

is the *magnetic field energy*, we can write the integral version of the balance equation (4.36) on the preceding page as

$$\frac{dU^{\text{mech}}}{dt} + \frac{dU^{\text{em}}}{dt} + \oint_S d^2x \hat{\mathbf{n}} \cdot \mathbf{S} = 0 \quad (4.40)$$

where

This is the *energy theorem in Maxwell's theory*, also known as *Poynting's theorem*.

Allowing for an EMF and assuming that Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{E}^{\text{emf}}) \quad (4.41)$$

is a valid approximation, or, equivalently, that

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{E}^{\text{emf}} \quad (4.42)$$

we obtain the relation

$$\int_V d^3x \mathbf{j} \cdot \mathbf{E} = \int_V d^3x \frac{j^2}{\sigma} - \int_V d^3x \mathbf{j} \cdot \mathbf{E}^{\text{emf}} \quad (4.43)$$

which, when inserted into equation (4.40) above and use is made of equation (4.37) on the preceding page, yields

$$\underbrace{\int_V d^3x \mathbf{j} \cdot \mathbf{E}^{\text{emf}}}_{\text{Supplied electric power}} = \underbrace{\frac{dU^{\text{em}}}{dt}}_{\text{Field power}} + \underbrace{\oint_S d^2x \hat{\mathbf{n}} \cdot \mathbf{S}}_{\text{Radiated power}} \underbrace{\int_V d^3x \mathbf{j} \cdot \mathbf{E}^{\text{emf}}}_{\text{Supplied electric power}} \quad (4.44)$$

This shows how the supplied power (left-hand side, LHS) is expelled in the form of a time rate change of *electric and magnetic field energy*, *radiated electromagnetic power*, and *Joule heat power*, i.e., *Ohmic losses* in the system (right-hand side, RHS).

EXAMPLE 4.3 ▷Conservation of energy

Show, by explicit calculation, that the total energy H of a closed system comprising charged particles of mass m_i and speed v_i and pertinent electromagnetic fields \mathbf{E} and \mathbf{B}

$$H = \frac{1}{2} m_i v_i^2 + \frac{\varepsilon_0}{2} \int_V d^3x (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \quad (4.45)$$

is conserved.

—End of example 4.3◁

4.3.4 Conservation of linear momentum

The derivation of the energy conservation formula (4.33) on page 57 introduced the Poynting (energy flux) vector $\mathbf{S}(t, \mathbf{x})$. We now seek a balance equation involving the time derivative of \mathbf{S} and find, using the Maxwell equations (2.1) on page 20, that

$$\begin{aligned}\frac{\partial \mathbf{S}}{\partial t} &= \varepsilon_0 c^2 \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\varepsilon_0 c^2 (\mathbf{E} \times (\nabla \times \mathbf{E}) + c^2 \mathbf{B} \times (\nabla \times \mathbf{B})) - c^2 \mathbf{j} \times \mathbf{B}\end{aligned}\quad (4.46)$$

A combination of formulæ (F.100), (F.91), and (F.96) on page 200, and formula (F.110) on page 201, using the Maxwell-Lorentz equations (2.1) on page 20 once more, allows us to write

$$\begin{aligned}\mathbf{E} \times (\nabla \times \mathbf{E}) &= \nabla \mathbf{E} \cdot \mathbf{E} - \mathbf{E} \cdot \nabla \mathbf{E} = \frac{1}{2} \nabla (\mathbf{E} \cdot \mathbf{E}) - \mathbf{E} \cdot \nabla \mathbf{E} \\ &= -\nabla \cdot (\mathbf{E} \mathbf{E}) + \frac{1}{2} \nabla (\mathbf{E} \cdot \mathbf{E}) + (\nabla \cdot \mathbf{E}) \mathbf{E} \\ &= -\nabla \cdot \left(\mathbf{E} \mathbf{E} - \frac{1}{2} \mathbf{1}_3 (\mathbf{E} \cdot \mathbf{E}) \right) + \rho \mathbf{E}\end{aligned}\quad (4.47)$$

and similarly for \mathbf{B} . Dividing out c^2 , and rearranging terms, we can write equation (4.46) as

$$\frac{1}{c^2} \frac{\partial \mathbf{S}}{\partial t} + \varepsilon_0 \nabla \cdot \left[\frac{1}{2} \mathbf{1}_3 (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) - (\mathbf{E} \mathbf{E} + c^2 \mathbf{B} \mathbf{B}) \right] + \mathbf{f} = \mathbf{0} \quad (4.48)$$

where

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \quad (4.49)$$

The polar vector \mathbf{f} has the dimension Nm^{-3} and we therefore identify it as a force density and call it the *Lorentz force density*. According to classical mechanics (*Newton's second law*, *Euler's first law*), this mechanical force density must equal the time rate change of the *mechanical linear momentum density*

$$\mathbf{g}^{\text{mech}}(t, \mathbf{x}) = \varrho_m(t, \mathbf{x}) \mathbf{v} \quad (4.50)$$

where ϱ_m is the (volumetric) *mass density* (kgm^{-3}).

In the same vein, we introduce the *electromagnetic linear momentum density* \mathbf{g}^{em} by making the identification

$$\mathbf{g}^{\text{em}}(t, \mathbf{x}) = \varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{\mathbf{S}(t, \mathbf{x})}{c^2} \quad (4.51)$$

or, alternatively,

$$\mathbf{S} = \mathbf{g}^{\text{em}} c^2 \quad (4.52)$$

Since the LHS of this equation is (energy density) \times (velocity) and the RHS is (mass density) \times (velocity) $\times c^2$, we see that this relation forebodes the relativistic relation $E = mc^2$, where E is energy and m is mass.

We are now able to write down the *linear momentum density* balance equation (4.46) on the previous page in differential form. It is

$$\frac{\partial \mathbf{g}^{\text{mech}}}{\partial t} + \frac{\partial \mathbf{g}^{\text{em}}}{\partial t} + \nabla \cdot \mathbf{T} = \mathbf{0} \quad (4.53)$$

where

$$\mathbf{T} = \mathbf{1}_3 \frac{\epsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) - \epsilon_0 (\mathbf{E} \mathbf{E} + c^2 \mathbf{B} \mathbf{B}) \quad (4.54)$$

is the *electromagnetic linear momentum flux tensor* (the negative of the *Maxwell stress tensor*), measured in Pa, i.e., Nm^{-2} . The tensor \mathbf{T} is the *electromagnetic linear momentum current density* of the electromagnetic field. In tensor component form, it can be written

$$T_{ij} = \delta_{ij} u^{\text{em}} - \epsilon_0 E_i E_j - \frac{1}{\mu_0} B_i B_j \quad (4.55)$$

where the electromagnetic energy density u^{em} is defined in formula (4.29) on page 56. The component T_{ij} is the electromagnetic linear momentum in the i th direction that passes a surface element in the j th direction per unit time, per unit area.

Integration of equation (4.53) above over the entire finite volume V , enclosed by the surface S , yields the *conservation law for linear momentum*

$$\frac{d\mathbf{p}^{\text{mech}}}{dt} + \frac{d\mathbf{p}^{\text{em}}}{dt} + \oint_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{T} = \mathbf{0} \quad (4.56)$$

where

$$\mathbf{p}^{\text{mech}} = \int_V d^3x \, \mathbf{g}^{\text{mech}} = \int_V d^3x \, \varrho_m \mathbf{v} \quad (4.57a)$$

and

$$\mathbf{p}^{\text{em}} = \int_V d^3x \, \mathbf{g}^{\text{em}} = \int_V d^3x \, \epsilon_0 (\mathbf{E} \times \mathbf{B}) \quad (4.57b)$$

or

$$\underbrace{\int_V d^3x \, \mathbf{f}}_{\text{Force on the matter}} + \frac{d}{dt} \underbrace{\int_V d^3x \, \epsilon_0 (\mathbf{E} \times \mathbf{B})}_{\text{EM field momentum}} + \underbrace{\oint_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{T}}_{\text{Linear momentum flow}} = \mathbf{0} \quad (4.58)$$

This is the *linear momentum theorem in Maxwell's theory* which describes how the electromagnetic field carries linear momentum.

If we assume that we have a single localised charge q , *i.e.*, that the charge density is given in terms of a Dirac distribution as in equation (1.8) on page 5, where the sum is over one particle only, the evaluation of the first integral in equation (4.58) on the preceding page shows that the force on this single charged particle is

$$\mathbf{F} = \int_V d^3x \mathbf{f} = \int_V d^3x (\rho \mathbf{E} + \mathbf{j} \times \mathbf{B}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (4.59)$$

which is the *Lorentz force*; see also equation (1.46) on page 14. Note that equation (4.59) above follows directly from a conservation law, and therefore is a consequence of a symmetry of the microscopic Maxwell equations (2.1) on page 20. Hence, the Lorentz force does not have to be presupposed.

Using formula (4.31d) on page 56 in formula (4.57b) on the facing page, the Maxwell-Lorentz equation (2.1a) on page 20, and applying the divergence theorem, we can write the electromagnetic linear momentum of the field as

$$\mathbf{p}^{\text{em}} = \int_V d^3x \rho \mathbf{A} + \varepsilon_0 \int_V d^3x (\nabla \mathbf{A}) \cdot \mathbf{E} - \varepsilon_0 \oint_S d^2x \hat{\mathbf{n}} \cdot \mathbf{E} \mathbf{A} \quad (4.60)$$

For each spatial Fourier component we use equation (4.32d) on page 57 to find the manifestly gauge-independent expression

$$\mathbf{p}^{\text{em}} = \int_V d^3x \rho \mathbf{A}_\perp + \varepsilon_0 \int_V d^3x (\nabla \mathbf{A})_\perp \cdot \mathbf{E} - \varepsilon_0 \oint_S d^2x \hat{\mathbf{n}} \cdot \mathbf{E} \mathbf{A}_\perp \quad (4.61)$$

If we restrict ourselves to consider a single temporal Fourier component of the transverse component and use the results in example 3.4 on page 47 we find that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = i\omega \mathbf{A} \quad (4.62)$$

which allows us to replace \mathbf{A} by $-i\mathbf{E}/\omega$, yielding

$$\mathbf{p}_{\text{trans}}^{\text{em}} = \text{Re} \left\{ \frac{-i\varepsilon_0}{2\omega} \int_V d^3x (\nabla \mathbf{E}) \cdot \mathbf{E}^* \right\} + \text{Re} \left\{ \frac{i\varepsilon_0}{\omega} \oint_S d^2x \hat{\mathbf{n}} \cdot \mathbf{E} \mathbf{E}^* \right\} \quad (4.63)$$

If $\mathbf{E} \mathbf{E}^*$ falls off sufficiently rapidly at large distances, we can discard the surface integral term and find that the cycle averaged linear momentum of the transverse field is

$$\mathbf{p}_{\text{trans}}^{\text{em}} = \text{Re} \left\{ \frac{-i\varepsilon_0}{2\omega} \int_V d^3x (\nabla \mathbf{E}) \cdot \mathbf{E}^* \right\} \quad (4.64)$$

where we used formula (2.5) on page 22. In complex tensor notation with Einstein's summation convention applied, this can be written

$$\mathbf{p}_{\text{trans}}^{\text{em}} = \frac{-i\varepsilon_0}{2\omega} \int_V d^3x E_i^* \nabla E_i \quad (4.65)$$

If we recall that the quantal *linear momentum operator* is

$$\hat{\mathbf{p}} = -i\hbar\nabla \quad (4.66)$$

we see that we can write the expression for the linear momentum of the electromagnetic field in terms of this operator as

$$\mathbf{p}_{\text{trans}}^{\text{em}} = \frac{\varepsilon_0}{2\hbar\omega} \int_V d^3x E_i^* \hat{\mathbf{p}} E_i \quad (4.67)$$

i.e., as a quantum expectation value where the components of the electric field vector behave as wavefunctions.

EXAMPLE 4.4 ▷Conservation of linear momentum◁

Show, by explicit calculation, that the linear momentum \mathbf{p} of a closed system comprising charged particles of mass m_i and velocity \mathbf{v}_i and pertinent electromagnetic fields \mathbf{E} and \mathbf{B}

$$\mathbf{p} = m_i \mathbf{v}_i + \varepsilon_0 \int_V d^3x (\mathbf{E} \times \mathbf{B}) \quad (4.68)$$

is conserved.

◁End of example 4.4▷

4.3.5 Conservation of angular momentum

Already 1776 it was shown by LEONHARD EULER that the most general dynamical state of a mechanical system is the sum of its translational motion, described by the linear momentum *mechanical linear momentum* \mathbf{p}^{mech} , and its rotational motion, described by the system's *mechanical moment of momentum*, or *mechanical angular momentum* $\mathbf{J}^{\text{mech}}(\mathbf{x}_0)$ about the moment point \mathbf{x}_0 , and that these two momenta are in general independent of each other⁴.

In the special case of a classical rigid body for which the contribution from internal angular momenta cancel the angular momentum around \mathbf{x}_0 is given by the pseudovector $\mathbf{J}^{\text{mech}}(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{p}^{\text{mech}}$. For a closed system of rotating and orbiting bodies, *e.g.*, a spinning planet orbiting a (non-rotating) star, the *total mechanical angular momentum* of the system is the vectorial sum of two contributions.

$$\mathbf{J}^{\text{mech}}(\mathbf{x}_0) = \boldsymbol{\Sigma}^{\text{mech}} + \mathbf{L}^{\text{mech}}(\mathbf{x}_0) \quad (4.69)$$

where $\boldsymbol{\Sigma}^{\text{mech}}$ is the intrinsic *mechanical spin angular momentum*, describing the spin of the planet around its own axis, and \mathbf{L}^{mech} is the extrinsic *mechanical orbital angular momentum*, describing the motion of the planet in an orbit around

⁴ Rational mechanist CLIFFORD AMBROSE TRUESDELL III (1919–2000) wrote several excellent accounts on how this result was arrived at by Euler in the latter half of the 18th century.

the star. As is well known from mechanics, \mathbf{J}^{mech} of a closed mechanical system is conserved and to change it one has to apply a *mechanical torque*

$$\boldsymbol{\tau}(\mathbf{x}_0) = \frac{d\mathbf{J}^{\text{mech}}(\mathbf{x}_0)}{dt} \quad (4.70)$$

Starting from the definition of the mechanical linear momentum density \mathbf{g}^{mech} , formula (4.50) on page 59, and suppressing the obvious dependence of \mathbf{x} , we define the *mechanical angular momentum density* about a fixed moment point \mathbf{x}_0 as

$$\mathbf{h}^{\text{mech}}(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{mech}} \quad (4.71)$$

Analogously, we define the *electromagnetic moment of momentum density* or *electromagnetic angular momentum density* about a moment point \mathbf{x}_0 as the pseudovector

$$\mathbf{h}^{\text{em}}(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{em}} \quad (4.72)$$

where, according to equation (4.52) on page 59, $\mathbf{g}^{\text{em}} = \varepsilon_0 \mathbf{E} \times \mathbf{B} = \mathbf{S}/c^2$.

Now, if \mathbf{f} is the Lorentz force density as defined in equation (4.49) on page 59, then

$$\begin{aligned} \frac{\partial \mathbf{h}^{\text{mech}}(\mathbf{x}_0)}{\partial t} &= \frac{\partial}{\partial t} [(\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{mech}}] = \frac{d\mathbf{x}}{dt} \times \mathbf{g}^{\text{mech}} \\ &\quad + (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial \mathbf{g}^{\text{mech}}}{\partial t} = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f} \end{aligned} \quad (4.73)$$

is the *Lorentz torque density*. Furthermore,

$$\begin{aligned} \frac{\partial \mathbf{h}^{\text{em}}(\mathbf{x}_0)}{\partial t} &= \frac{\partial}{\partial t} [(\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{em}}] = \frac{d\mathbf{x}}{dt} \times \mathbf{g}^{\text{em}} + (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial \mathbf{g}^{\text{em}}}{\partial t} \\ &= \varepsilon_0 (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) = \frac{1}{c^2} (\mathbf{x} - \mathbf{x}_0) \times \frac{\partial \mathbf{S}}{\partial t} \end{aligned} \quad (4.74)$$

By introducing the *electromagnetic angular momentum flux tensor*

$$\mathbf{K}(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0) \times \mathbf{T} \quad (4.75)$$

where \mathbf{T} is the electromagnetic linear momentum flux tensor given by expression (4.54) on page 60, the differential form of the balance equation for angular momentum density can be written

$$\frac{\partial \mathbf{h}^{\text{mech}}(\mathbf{x}_0)}{\partial t} + \frac{\partial \mathbf{h}^{\text{em}}(\mathbf{x}_0)}{\partial t} + \nabla \cdot \mathbf{K}(\mathbf{x}_0) = \mathbf{0} \quad (4.76)$$

where the pseudotensor $\mathbf{K}(\mathbf{x}_0)$ represents the *electromagnetic angular momentum current density* around \mathbf{x}_0 .

Integration over the entire volume V , enclosed by the surface S , yields the *conservation law for angular momentum*

$$\frac{d\mathbf{J}^{\text{mech}}(\mathbf{x}_0)}{dt} + \frac{d\mathbf{J}^{\text{em}}(\mathbf{x}_0)}{dt} + \oint_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{K}(\mathbf{x}_0) = \mathbf{0} \quad (4.77)$$

where the mechanical and electromagnetic angular momentum pseudovectors are

$$\mathbf{J}^{\text{mech}}(\mathbf{x}_0) = \int_V d^3x \, \mathbf{h}^{\text{mech}}(\mathbf{x}_0) \quad (4.78a)$$

and

$$\begin{aligned} \mathbf{J}^{\text{em}}(\mathbf{x}_0) &= \int_V d^3x \, \mathbf{h}^{\text{em}}(\mathbf{x}_0) = \int_V d^3x \, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{g}^{\text{em}} \\ &= \varepsilon_0 \int_V d^3x \, (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{E} \times \mathbf{B}) \end{aligned} \quad (4.78b)$$

respectively. We can formulate—and interpret—this conservation law in the following way:

$$\begin{aligned} \underbrace{\int_V d^3x \, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f}}_{\text{Torque on the matter}} + \underbrace{\frac{d}{dt} \varepsilon_0 \int_V d^3x \, (\mathbf{x} - \mathbf{x}_0) \times (\mathbf{E} \times \mathbf{B})}_{\text{Field angular momentum}} \\ + \underbrace{\oint_S d^2x \, \hat{\mathbf{n}} \cdot \mathbf{K}(\mathbf{x}_0)}_{\text{Angular momentum flow}} = \mathbf{0} \end{aligned} \quad (4.79)$$

This *angular momentum theorem* is the angular analogue of the linear momentum theorem, equation (4.58) on page 60. It shows that the electromagnetic field, like any physical field, can carry angular momentum.

For a single localised charge q , *i.e.*, for a charge density given by equation (1.8) on page 5 with summation over one particle only, the evaluation of the first integral in equation (4.79) above shows that the mechanical torque on this single charged particle is

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{x}_0) &= \int_V d^3x \, (\mathbf{x} - \mathbf{x}_0) \times \mathbf{f} = \int_V d^3x \, ((\mathbf{x} - \mathbf{x}_0) \times (\rho \mathbf{E} + \mathbf{j} \times \mathbf{B})) \\ &= (\mathbf{x} - \mathbf{x}_0) \times q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \end{aligned} \quad (4.80)$$

We call it the *Lorentz torque*.

If we study formula (4.78b) a little closer, we see that it can be written

$$\mathbf{J}^{\text{em}}(\mathbf{x}_0) = \mathbf{J}^{\text{em}}(\mathbf{0}) - \mathbf{x}_0 \times \varepsilon_0 \int_V d^3x \, \mathbf{E} \times \mathbf{B} = \mathbf{J}^{\text{em}}(\mathbf{0}) - \mathbf{x}_0 \times \mathbf{p}^{\text{em}} \quad (4.81)$$

where use was made of equation (4.57b) on page 60. With the help of equation (4.31) on page 56 we can write

$$\mathbf{J}^{\text{em}}(\mathbf{0}) = \varepsilon_0 \int_V d^3x \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) = \frac{1}{c^2} \int_V d^3x \mathbf{x} \times \mathbf{S} \quad (4.82)$$

$$= \varepsilon_0 \int_V d^3x \mathbf{x} \times [(\nabla \mathbf{A}) \cdot \mathbf{E}] - \varepsilon_0 \int_V d^3x [\mathbf{x} \times \mathbf{E} \cdot (\nabla \mathbf{A})] \quad (4.83)$$

Partial integration of this yields

$$\begin{aligned} \mathbf{J}^{\text{em}}(\mathbf{0}) &= \varepsilon_0 \int_V d^3x \mathbf{E} \times \mathbf{A} + \varepsilon_0 \int_V d^3x \mathbf{x} \times [(\nabla \mathbf{A}) \cdot \mathbf{E}] \\ &\quad - \varepsilon_0 \int_V d^3x \nabla \cdot (\mathbf{E} \mathbf{x} \times \mathbf{A}) + \varepsilon_0 \int_V d^3x (\mathbf{x} \times \mathbf{A})(\nabla \cdot \mathbf{E}) \end{aligned} \quad (4.84)$$

By introducing

$$\Sigma^{\text{em}} = \varepsilon_0 \int_V d^3x \mathbf{E} \times \mathbf{A} \quad (4.85a)$$

and

$$\mathbf{L}^{\text{em}}(\mathbf{0}) = \varepsilon_0 \int_V d^3x \mathbf{x} \times [(\nabla \mathbf{A}) \cdot \mathbf{E}] + \int_V d^3x \mathbf{x} \times \rho \mathbf{A} \quad (4.85b)$$

applying Gauss's theorem, formula (F.126) on page 202, and using the Maxwell-Lorentz equation (2.1a) on page 20, we find that expression (4.84) above can be written

$$\mathbf{J}^{\text{em}}(\mathbf{0}) = \Sigma^{\text{em}} + \mathbf{L}^{\text{em}}(\mathbf{0}) - \varepsilon_0 \oint_S d^2x \hat{\mathbf{n}} \cdot (\mathbf{E} \mathbf{x} \times \mathbf{A}) \quad (4.86)$$

If there is no net electric charge density ρ in the integration volume, the second integral in the RHS of equation (4.85b) vanishes, and if $\mathbf{E} \mathbf{x} \times \mathbf{A} \equiv \mathbf{E}(\mathbf{x} \times \mathbf{A})$ falls off sufficiently fast with $|\mathbf{x}|$, the contribution from the surface integral in equation (4.86) above can be neglected. Furthermore, for a single Fourier component and in the complex notation, we obtain the expressions

$$\Sigma^{\text{em}} = -i \frac{\varepsilon_0}{2\omega} \int_V d^3x (\mathbf{E}^* \times \mathbf{E}) \quad (4.87a)$$

$$\mathbf{L}^{\text{em}}(\mathbf{0}) = -i \frac{\varepsilon_0}{2\omega} \int_V d^3x E_i^* (\mathbf{x} \times \nabla) E_i \quad (4.87b)$$

Recalling that in quantum mechanics the *spin angular momentum operator* is

$$\hat{\Sigma}_{jk} = -i\hbar \epsilon_{ijk} \hat{x}_i \quad (4.88)$$

which, with the help of the matrix vector expression (M.27) on page 210 can be written

$$\hat{\Sigma} = -\hbar \mathbf{S} \quad (4.89)$$

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and the *orbital angular momentum operator* is

$$\hat{\mathbf{L}} = -i\hbar \mathbf{x} \times \nabla = -i\hbar \epsilon_{ijk} x_j \partial_k \hat{\mathbf{x}}_i \quad (4.90)$$

we can write

$$\Sigma^{\text{em}} = \frac{\epsilon_0}{2\hbar\omega} \int_V d^3x E_j^* \hat{\Sigma}_{jk} E_k \quad (4.91a)$$

$$\mathbf{L}^{\text{em}}(\mathbf{0}) = \frac{\epsilon_0}{2\hbar\omega} \int_V d^3x E_i^* \hat{\mathbf{L}} E_i \quad (4.91b)$$

Hence, under the assumptions made above, we can interpret Σ^{em} as the *electromagnetic spin angular momentum* and \mathbf{L}^{em} as the *electromagnetic orbital angular momentum*.

EXAMPLE 4.5 ▷Conservation of angular momentum

Show, by explicit calculation, that the total angular momentum \mathbf{J} of a closed system comprising charged particles located at \mathbf{x}_i , of mass m_i and velocity \mathbf{v}_i and pertinent electromagnetic fields \mathbf{E} and \mathbf{B}

$$\mathbf{J} = \mathbf{x}_i \times m_i \mathbf{v}_i + \epsilon_0 \int_V d^3x [\mathbf{x} \times (\mathbf{E} \times \mathbf{B})] \quad (4.92)$$

is conserved.

End of example 4.5◁

EXAMPLE 4.6 ▷Spin angular momentum and wave polarisation

Consider a generic temporal Fourier mode of the electric field vector \mathbf{E} of a circularly polarised wave with (angular) frequency ω . According to equation (2.34) on page 27 it can be written

$$\mathbf{E}(t, \mathbf{x}) = E(t, \mathbf{x}) \hat{\mathbf{h}}_{\pm} \quad (4.93)$$

where

$$E(t, \mathbf{x}) = \sqrt{2} E_0 e^{i(kx_3 - \omega t + \delta_1)} \quad (4.94)$$

and

$$\hat{\mathbf{h}}_{\pm} = \frac{1}{\sqrt{2}} (\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \quad (4.95)$$

As before, we use the convention that $\hat{\mathbf{h}}_+$ represents *left-hand circular polarisation* and $\hat{\mathbf{h}}_-$ *right-hand circular polarisation*. Noting that

$$(\hat{\mathbf{h}}_{\pm})^* \times \hat{\mathbf{h}}_{\pm} = \pm i\hat{\mathbf{z}} \quad (4.96)$$

we see that

$$\mathbf{E}^* \times \mathbf{E} = \pm i |E|^2 \hat{\mathbf{z}} = \pm i E^2 \hat{\mathbf{z}} \quad (4.97)$$

When this is inserted into equation (4.87a) on the preceding page one finds that the spin

angular momentum of a circularly polarised wave is

$$\mathbf{\Sigma}^{\text{em}} = \pm \frac{\epsilon_0}{2\omega} \int_V d^3x E^2 \hat{\mathbf{z}} = \pm \frac{U^{\text{em}}}{\omega} \hat{\mathbf{z}} \quad (4.98)$$

where U^{em} is the field energy. Considering the wave consisting of N photons, then $U^{\text{em}} = N\hbar\omega$ which means that the spin of the wave is

$$\mathbf{\Sigma}^{\text{em}} = \pm \frac{N\hbar\omega}{\omega} \hat{\mathbf{z}} = \pm N\hbar \hat{\mathbf{z}} \quad (4.99)$$

Hence, each photon of a right-hand or a left-hand circular polarised wave carries a spin angular momentum of \hbar or $-\hbar$, respectively.

—End of example 4.6◁

▷Orbital angular momentum

EXAMPLE 4.7

The Cartesian components of the quantal *orbital angular momentum operator* (OAM) $\hat{\mathbf{L}}$ as given by expression (4.90) on the preceding page are

$$\hat{L}_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \quad (4.100a)$$

$$\hat{L}_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \quad (4.100b)$$

$$\hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \quad (4.100c)$$

In cylindrical coordinates (ρ, φ, z) they are

$$\hat{L}_x = -i\hbar \left[\sin \varphi \left(z \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial z} \right) + \frac{z}{\rho} \cos \varphi \frac{\partial}{\partial \varphi} \right] \quad (4.101a)$$

$$\hat{L}_y = -i\hbar \left[\cos \varphi \left(z \frac{\partial}{\partial \rho} - \rho \frac{\partial}{\partial z} \right) - \frac{z}{\rho} \sin \varphi \frac{\partial}{\partial \varphi} \right] \quad (4.101b)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \quad (4.101c)$$

and in spherical coordinates (r, φ, θ) they are

$$\hat{L}_x = -i\hbar \left(\sin \varphi \frac{\partial}{\partial \theta} + \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} \right) \quad (4.102a)$$

$$\hat{L}_y = -i\hbar \left(\cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} \right) \quad (4.102b)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \quad (4.102c)$$

For an electric field \mathbf{E} that depends on the azimuthal angle φ in such a way that

$$\mathbf{E} = \mathbf{E}_0(t, \mathbf{x}) \Phi(\varphi) \quad (4.103)$$

we find that

$$\hat{L}_z \mathbf{E}(t, \mathbf{x}) = -i\hbar \left(\frac{\partial \mathbf{E}_0(t, \mathbf{x})}{\partial \varphi} \Phi(\varphi) - \mathbf{E}_0(t, \mathbf{x}) \frac{\partial \Phi(\varphi)}{\partial \varphi} \right) \quad (4.104)$$

If $\mathbf{E}_0(t, \mathbf{x})$ is rotationally symmetric around the the z axis, so that $\mathbf{E}_0 = \mathbf{E}_0(\rho, z)$ in cylindrical coordinates, $\mathbf{E}_0 = \mathbf{E}_0(r, \theta)$ in spherical (polar) coordinates *etc.*, and if the azi-

muthal part is expressed in a Fourier series

$$\Phi(\varphi) = \sum_{m=-\infty}^{\infty} c_m e^{im\varphi} \quad (4.105)$$

we see that

$$\hat{L}_z \mathbf{E}(t, \mathbf{x}) = \sum_{m=-\infty}^{\infty} c_m m \hbar \mathbf{E}_0 e^{im\varphi} = \sum_{m=-\infty}^{\infty} c_m m \hbar \mathbf{E}_m \quad (4.106)$$

i.e., that \mathbf{E} is a weighted superposition of OAM eigenstates $\mathbf{E}_m = \mathbf{E}_0 e^{im\varphi}$. Furthermore,

$$\hat{L}_z \mathbf{E}_m = m \hbar \mathbf{E}_m \quad (4.107)$$

which means that for a rotationally symmetric beam with an *azimuthal phase* dependence given by $\exp(im\varphi)$ one deduces, according to formula (4.87b) on page 65, that the z component of the orbital angular momentum of each photon in this beam is $m\hbar$.

—End of example 4.7◁

4.3.6 Conservation of centre of energy

In analogy with the existence of a centre of mass in a classical mechanical system, there exists a *centre of energy* in electrodynamics. It can be shown that this vectoral quantity is also conserved.

EXAMPLE 4.8 ▷Examples of other conservation laws

In addition to the conservation laws for energy, linear momentum and angular momentum, a large number of other electromagnetic conservation laws can be derived. For instance, as can be shown by direct calculation based on the Maxwell-Lorentz the following conservation law holds in free space (vacuum):

$$\frac{\partial \chi}{\partial t} + \nabla \cdot \mathbf{X} = 0 \quad (4.108)$$

where the pseudoscalar

$$\chi = \mathbf{E} \cdot (\nabla \times \mathbf{E}) + c^2 \mathbf{B} \cdot (\nabla \times \mathbf{B}) \quad (4.109)$$

is the *chirality density*, and the pseudovector

$$\mathbf{X} = \mathbf{E} \times \frac{\partial \mathbf{E}}{\partial t} + c^2 \mathbf{B} \times \frac{\partial \mathbf{B}}{\partial t} \quad (4.110)$$

is the *chirality flow*.

—End of example 4.8◁

4.3.7 Electromagnetic virial theorem

If instead of vector multiplying the linear momentum densities \mathbf{g}^{mech} and \mathbf{g}^{em} by the position vector relative to a fix point \mathbf{x}_0 as we did in subsection 4.3.5 on page 62, we scalar multiply, the following differential balance equation is obtained:

$$\frac{\partial((\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{g}^{\text{mech}})}{\partial t} + \frac{\partial((\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{g}^{\text{em}})}{\partial t} + \nabla \cdot ((\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{T}) = u^{\text{em}} \quad (4.111)$$

This is the *electromagnetic virial theorem*, analogous to the virial theorem of Clausius in mechanics. The quantity $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{g}^{\text{em}}$ is the *electromagnetic virial density*. When integrated over space and time averaged, this theorem is a statement of the partitioning of energy in electrodynamics and finds use in, *e.g.*, *plasma physics*.

4.4 Bibliography

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DRAFT

5

FIELDS FROM ARBITRARY CHARGE AND CURRENT DISTRIBUTIONS

While, in principle, the electric and magnetic fields can be calculated from the Maxwell equations in chapter 1, or even from the wave equations in chapter 2, it is often physically more lucid to calculate them from the electromagnetic potentials derived in chapter 3. In this chapter we will derive the electric and magnetic fields from the potentials.

We recall that in order to find the solution (3.46) for the generic inhomogeneous wave equation (3.30) on page 37 we presupposed the existence of a Fourier transform pair (3.31) on page 37 for the generic source term

$$f(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega f_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (5.1a)$$

$$f_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t, \mathbf{x}) e^{i\omega t} \quad (5.1b)$$

That such transform pairs exist is true for most well-behaved physical variables which are neither strictly monotonically increasing nor strictly monotonically decreasing with time.¹ For charge and current densities varying in time we can therefore, without loss of generality, work with individual Fourier components $\rho_{\omega}(\mathbf{x})$ and $\mathbf{j}_{\omega}(\mathbf{x})$, respectively. Strictly speaking, the existence of a single Fourier component assumes a *monochromatic* source (*i.e.*, a source containing only one single frequency component), which in turn requires that the electric and magnetic fields exist for infinitely long times. However, by taking the proper limits, we may still use this approach even for sources and fields of finite duration.

¹ An example of a physical quantity that cannot be represented in Fourier form is *white noise*.

This is the method we shall utilise in this chapter in order to derive the electric and magnetic fields in vacuum from arbitrary given charge densities $\rho(t, \mathbf{x})$ and current densities $\mathbf{j}(t, \mathbf{x})$, defined by the temporal Fourier transform pairs

$$\rho(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \rho_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (5.2a)$$

$$\rho_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \rho(t, \mathbf{x}) e^{i\omega t} \quad (5.2b)$$

and

$$\mathbf{j}(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \mathbf{j}_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (5.3a)$$

$$\mathbf{j}_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{j}(t, \mathbf{x}) e^{i\omega t} \quad (5.3b)$$

under the assumption that only *retarded* potentials produce physically acceptable solutions.

The temporal Fourier transform pair for the retarded scalar potential can then be written

$$\phi(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \phi_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (5.4a)$$

$$\phi_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \phi(t, \mathbf{x}) e^{i\omega t} = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \rho_{\omega}(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad (5.4b)$$

where in the last step, we made use of the explicit expression for the temporal Fourier transform of the generic potential component $\Psi_{\omega}(\mathbf{x})$, equation (3.43) on page 39. Similarly, the following Fourier transform pair for the vector potential must exist:

$$\mathbf{A}(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \mathbf{A}_{\omega}(\mathbf{x}) e^{-i\omega t} \quad (5.5a)$$

$$\mathbf{A}_{\omega}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{A}(t, \mathbf{x}) e^{i\omega t} = \frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}_{\omega}(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad (5.5b)$$

Similar transform pairs exist for the fields themselves.

In the limit that the sources can be considered monochromatic containing only one single frequency ω_0 , we have the much simpler expressions

$$\rho(t, \mathbf{x}) = \rho_0(\mathbf{x}) e^{-i\omega_0 t} \quad (5.6a)$$

$$\mathbf{j}(t, \mathbf{x}) = \mathbf{j}_0(\mathbf{x}) e^{-i\omega_0 t} \quad (5.6b)$$

$$\phi(t, \mathbf{x}) = \phi_0(\mathbf{x}) e^{-i\omega_0 t} \quad (5.6c)$$

$$\mathbf{A}(t, \mathbf{x}) = \mathbf{A}_0(\mathbf{x}) e^{-i\omega_0 t} \quad (5.6d)$$

where again the real-valuedness of all these quantities is implied. As discussed above, we can safely assume that all formulae derived for a general temporal

Fourier representation of the source (general distribution of frequencies in the source) are valid for these simple limiting cases. We note that in this context, we can make the formal identification $\rho_\omega = \rho_0 \delta(\omega - \omega_0)$, $\mathbf{j}_\omega = \mathbf{j}_0 \delta(\omega - \omega_0)$ etc., and that we therefore, without any loss of stringency, let ρ_0 mean the same as the Fourier amplitude ρ_ω and so on.

5.1 The magnetic field

Let us now compute the magnetic field from the vector potential, defined by equation (5.5a) and equation (5.5b) on the preceding page, and formula (3.21) on page 35:

$$\mathbf{B}(t, \mathbf{x}) = \nabla \times \mathbf{A}(t, \mathbf{x}) \quad (5.7)$$

The calculations are much simplified if we work in ω space and, at the final stage, inverse Fourier transform back to ordinary t space. We are working in the Lorenz-Lorentz gauge and note that in ω space the Lorenz-Lorentz condition, equation (3.28) on page 37, takes the form

$$\nabla \cdot \mathbf{A}_\omega - i \frac{k}{c} \phi_\omega = 0 \quad (5.8)$$

which provides a relation between (the Fourier transforms of) the vector and scalar potentials.

Using the Fourier transformed version of equation (5.7) and equation (5.5b) on the preceding page, we obtain

$$\mathbf{B}_\omega(\mathbf{x}) = \nabla \times \mathbf{A}_\omega(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad (5.9)$$

Utilising formula (F.97) on page 200 and recalling that $\mathbf{j}_\omega(\mathbf{x}')$ does not depend on \mathbf{x} , we can rewrite this as

$$\begin{aligned}\mathbf{B}_\omega(\mathbf{x}) &= -\frac{\mu_0}{4\pi} \int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \times \left[\nabla \left(\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \right) \right] \\ &= -\frac{\mu_0}{4\pi} \left[\int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \times \left(-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \right) e^{ik|\mathbf{x}-\mathbf{x}'|} \right. \\ &\quad \left. + \int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \times \left(ik \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} e^{ik|\mathbf{x}-\mathbf{x}'|} \right) \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right] \quad (5.10) \\ &= \frac{\mu_0}{4\pi} \left[\int_{V'} d^3x' \frac{\mathbf{j}_\omega(\mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|} \mathbf{x} (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \right. \\ &\quad \left. + \int_{V'} d^3x' \frac{(-ik) \mathbf{j}_\omega(\mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|} \mathbf{x} (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \right]\end{aligned}$$

From this expression for the magnetic field in the frequency (ω) domain, we obtain the total magnetic field in the temporal (t) domain by taking the inverse Fourier transform (using the identity $-ik = -i\omega/c$):

$$\begin{aligned}\mathbf{B}(t, \mathbf{x}) &= \int_{-\infty}^{\infty} d\omega \mathbf{B}_\omega(\mathbf{x}) e^{-i\omega t} \\ &= \frac{\mu_0}{4\pi} \left\{ \int_{V'} d^3x' \frac{\left[\int_{-\infty}^{\infty} d\omega \mathbf{j}_\omega(\mathbf{x}') e^{-i(\omega t - k|\mathbf{x}-\mathbf{x}'|)} \right] \mathbf{x} (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \right. \\ &\quad \left. + \frac{1}{c} \int_{V'} d^3x' \frac{\left[\int_{-\infty}^{\infty} d\omega (-i\omega) \mathbf{j}_\omega(\mathbf{x}') e^{-i(\omega t - k|\mathbf{x}-\mathbf{x}'|)} \right] \mathbf{x} (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} \right\} \\ &= \underbrace{\frac{\mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}}_{\text{Induction field}} + \underbrace{\frac{\mu_0}{4\pi c} \int_{V'} d^3x' \frac{\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}}_{\text{Radiation field}} \quad (5.11)\end{aligned}$$

where

$$\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \stackrel{\text{def}}{=} \left(\frac{\partial \mathbf{j}}{\partial t} \right)_{t=t'_{\text{ret}}} \quad (5.12)$$

and t'_{ret} is given in equation (3.45) on page 40. The first term, the *induction field*, dominates near the current source but falls off rapidly with distance from it, is the electrodynamic version of the Biot-Savart law in electrostatics, formula (1.17) on page 8. The second term, the *radiation field* or the *far field*, dominates at large distances and represents energy that is transported out to infinity. Note how the spatial derivatives (∇) gave rise to a time derivative ($\dot{}$)!

5.2 The electric field

In order to calculate the electric field, we use the temporally Fourier transformed version of formula (3.25) on page 36, inserting equations (5.4b) and (5.5b) as the explicit expressions for the Fourier transforms of ϕ and \mathbf{A} :

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) &= -\nabla\phi_\omega(\mathbf{x}) + i\omega\mathbf{A}_\omega(\mathbf{x}) \\ &= -\frac{1}{4\pi\epsilon_0}\nabla\int_{V'}d^3x'\rho_\omega(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + \frac{i\mu_0\omega}{4\pi}\int_{V'}d^3x'\mathbf{j}_\omega(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{1}{4\pi\epsilon_0}\left[\int_{V'}d^3x'\frac{\rho_\omega(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\ &\quad \left.-ik\int_{V'}d^3x'\left(\frac{\rho_\omega(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{\mathbf{j}_\omega(\mathbf{x}')}{c}\right)\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right]\end{aligned}\quad (5.13)$$

Using the Fourier transform of the continuity equation (1.25) on page 10

$$\nabla' \cdot \mathbf{j}_\omega(\mathbf{x}') - i\omega\rho_\omega(\mathbf{x}') = 0 \quad (5.14)$$

we see that we can express ρ_ω in terms of \mathbf{j}_ω as follows

$$\rho_\omega(\mathbf{x}') = -\frac{i}{\omega}\nabla' \cdot \mathbf{j}_\omega(\mathbf{x}') \quad (5.15)$$

Doing so in the last term of equation (5.13) above, and also using the fact that $k = \omega/c$, we can rewrite this equation as

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\left[\int_{V'}d^3x'\frac{\rho_\omega(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right. \\ &\quad \left.-\frac{1}{c}\int_{V'}d^3x'\underbrace{\left(\frac{[\nabla' \cdot \mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - ik\mathbf{j}_\omega(\mathbf{x}')\right)}_{\mathbf{I}_\omega}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right]\end{aligned}\quad (5.16)$$

The last vector-valued integral can be further rewritten in the following way:

$$\begin{aligned}\mathbf{I}_\omega &= \int_{V'}d^3x'\left(\frac{[\nabla' \cdot \mathbf{j}_\omega(\mathbf{x}')](\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - ik\mathbf{j}_\omega(\mathbf{x}')\right)\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\ &= \int_{V'}d^3x'\left(\frac{\partial j_{\omega m}}{\partial x'_m}\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|} - ikj_{\omega l}(\mathbf{x}')\right)\hat{\mathbf{x}}_l\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\end{aligned}\quad (5.17)$$

But, since

$$\begin{aligned}\frac{\partial}{\partial x'_m}\left(j_{\omega m}\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{ik|\mathbf{x}-\mathbf{x}'|}\right) &= \left(\frac{\partial j_{\omega m}}{\partial x'_m}\right)\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{ik|\mathbf{x}-\mathbf{x}'|} \\ &\quad + j_{\omega m}\frac{\partial}{\partial x'_m}\left(\frac{x_l - x'_l}{|\mathbf{x}-\mathbf{x}'|^2}e^{ik|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned}\quad (5.18)$$

we can rewrite \mathbf{I}_ω as

$$\begin{aligned}\mathbf{I}_\omega = & - \int_{V'} d^3x' \left[j_{\omega m} \frac{\partial}{\partial x'_m} \left(\frac{x_l - x'_l}{|\mathbf{x} - \mathbf{x}'|^2} \hat{\mathbf{x}}_l e^{ik|\mathbf{x} - \mathbf{x}'|} \right) + ik \mathbf{j}_\omega \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \right] \\ & + \int_{V'} d^3x' \frac{\partial}{\partial x'_m} \left(j_{\omega m} \frac{x_l - x'_l}{|\mathbf{x} - \mathbf{x}'|^2} \hat{\mathbf{x}}_l e^{ik|\mathbf{x} - \mathbf{x}'|} \right)\end{aligned}\quad (5.19)$$

where, according to Gauss's theorem, the last term vanishes if \mathbf{j}_ω is assumed to be limited and tends to zero at large distances. Further evaluation of the derivative in the first term makes it possible to write

$$\begin{aligned}\mathbf{I}_\omega = & - \int_{V'} d^3x' \left(-\mathbf{j}_\omega \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|^2} + \frac{2}{|\mathbf{x} - \mathbf{x}'|^4} [\mathbf{j}_\omega \cdot (\mathbf{x} - \mathbf{x}')] (\mathbf{x} - \mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|} \right) \\ & - ik \int_{V'} d^3x' \left(-\frac{[\mathbf{j}_\omega \cdot (\mathbf{x} - \mathbf{x}')] (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} e^{ik|\mathbf{x} - \mathbf{x}'|} + \mathbf{j}_\omega \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \right)\end{aligned}\quad (5.20)$$

Using the triple product 'bac-cab' formula (F.73) on page 200 backwards, and inserting the resulting expression for \mathbf{I}_ω into equation (5.16) on the previous page, we arrive at the following final expression for the Fourier transform of the total \mathbf{E} field:

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) = & -\frac{1}{4\pi\epsilon_0} \nabla \int_{V'} d^3x' \rho_\omega(\mathbf{x}') \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} + \frac{i\mu_0\omega}{4\pi} \int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \\ = & \frac{1}{4\pi\epsilon_0} \left[\int_{V'} d^3x' \frac{\rho_\omega(\mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|} (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right. \\ & + \frac{1}{c} \int_{V'} d^3x' \frac{[\mathbf{j}_\omega(\mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|} \cdot (\mathbf{x} - \mathbf{x}')] (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^4} \\ & + \frac{1}{c} \int_{V'} d^3x' \frac{[\mathbf{j}_\omega(\mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|} \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^4} \\ & \left. - \frac{ik}{c} \int_{V'} d^3x' \frac{[\mathbf{j}_\omega(\mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|} \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \right]\end{aligned}\quad (5.21)$$

Taking the inverse Fourier transform of equation (5.21), once again using the vacuum relation $\omega = kc$, we find, at last, the expression in time domain for the

total electric field:

$$\begin{aligned}
 \mathbf{E}(t, \mathbf{x}) &= \int_{-\infty}^{\infty} d\omega \mathbf{E}_{\omega}(\mathbf{x}) e^{-i\omega t} \\
 &= \underbrace{\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(t'_{\text{ret}}, \mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}}_{\text{Retarded Coulomb field}} \\
 &\quad + \underbrace{\frac{1}{4\pi\epsilon_0 c} \int_{V'} d^3x' \frac{[\mathbf{j}(t'_{\text{ret}}, \mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')](\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^4}}_{\text{Intermediate field}} \\
 &\quad + \underbrace{\frac{1}{4\pi\epsilon_0 c} \int_{V'} d^3x' \frac{[\mathbf{j}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^4}}_{\text{Intermediate field}} \\
 &\quad + \underbrace{\frac{1}{4\pi\epsilon_0 c^2} \int_{V'} d^3x' \frac{[\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}}_{\text{Radiation field}}
 \end{aligned} \tag{5.22}$$

Here, the first term represents the *retarded Coulomb field* and the last term represents the *radiation field* which carries energy over very large distances. The other two terms represent an *intermediate field* which contributes only in the *near zone* and must be taken into account there.

With this we have achieved our goal of finding closed-form analytic expressions for the electric and magnetic fields when the sources of the fields are completely arbitrary, prescribed distributions of charges and currents. The only assumption made is that the advanced potentials have been discarded; recall the discussion following equation (3.46) on page 40 in chapter 3.

5.3 The radiation fields

In this section we study electromagnetic radiation, *i.e.*, those parts of the electric and magnetic fields, calculated above, which are capable of carrying energy and momentum over large distances. We shall therefore make the assumption that the observer is located in the *far zone*, *i.e.*, very far away from the source region(s). The fields which are dominating in this zone are by definition the *radiation fields*.

From equation (5.11) on page 74 and equation (5.22) above, which give the

total electric and magnetic fields, we obtain

$$\mathbf{B}^{\text{far}}(t, \mathbf{x}) = \int_{-\infty}^{\infty} d\omega \mathbf{B}_{\omega}^{\text{far}}(\mathbf{x}) e^{-i\omega t} = \frac{\mu_0}{4\pi c} \int_{V'} d^3x' \frac{\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} \quad (5.23a)$$

$$\begin{aligned} \mathbf{E}^{\text{far}}(t, \mathbf{x}) &= \int_{-\infty}^{\infty} d\omega \mathbf{E}_{\omega}^{\text{far}}(\mathbf{x}) e^{-i\omega t} \\ &= \frac{1}{4\pi \epsilon_0 c^2} \int_{V'} d^3x' \frac{[\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \end{aligned} \quad (5.23b)$$

where

$$\dot{\mathbf{j}}(t'_{\text{ret}}, \mathbf{x}') \stackrel{\text{def}}{=} \left(\frac{\partial \mathbf{j}}{\partial t} \right)_{t=t'_{\text{ret}}} \quad (5.24)$$

Instead of studying the fields in the time domain, we can often make a spectrum analysis into the frequency domain and study each Fourier component separately. A superposition of all these components and a transformation back to the time domain will then yield the complete solution.

The Fourier representation of the radiation fields equation (5.23a) and equation (5.23b) above were included in equation (5.10) on page 74 and equation (5.21) on page 76, respectively and are explicitly given by

$$\begin{aligned} \mathbf{B}_{\omega}^{\text{far}}(\mathbf{x}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{B}^{\text{far}}(t, \mathbf{x}) e^{i\omega t} \\ &= -i \frac{k \mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}_{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} e^{ik|\mathbf{x} - \mathbf{x}'|} \end{aligned} \quad (5.25a)$$

$$\begin{aligned} &= -i \frac{\mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}}{|\mathbf{x} - \mathbf{x}'|} e^{ik|\mathbf{x} - \mathbf{x}'|} \\ \mathbf{E}_{\omega}^{\text{far}}(\mathbf{x}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{E}^{\text{far}}(t, \mathbf{x}) e^{i\omega t} \\ &= -i \frac{k}{4\pi \epsilon_0 c} \int_{V'} d^3x' \frac{[\mathbf{j}_{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} e^{ik|\mathbf{x} - \mathbf{x}'|} \\ &= -i \frac{1}{4\pi \epsilon_0 c} \int_{V'} d^3x' \frac{[\mathbf{j}_{\omega}(\mathbf{x}') \times \mathbf{k}] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} e^{ik|\mathbf{x} - \mathbf{x}'|} \end{aligned} \quad (5.25b)$$

where we used the fact that $\mathbf{k} = k \hat{\mathbf{k}} = k(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$.

If the source is located near a point \mathbf{x}_0 inside a volume V' and has such a limited spatial extent that $\sup |\mathbf{x}' - \mathbf{x}_0| \ll |\mathbf{x} - \mathbf{x}'|$, and the integration surface S , centred on \mathbf{x}_0 , has a large enough radius $|\mathbf{x} - \mathbf{x}_0| \gg \sup |\mathbf{x}' - \mathbf{x}_0|$, we see

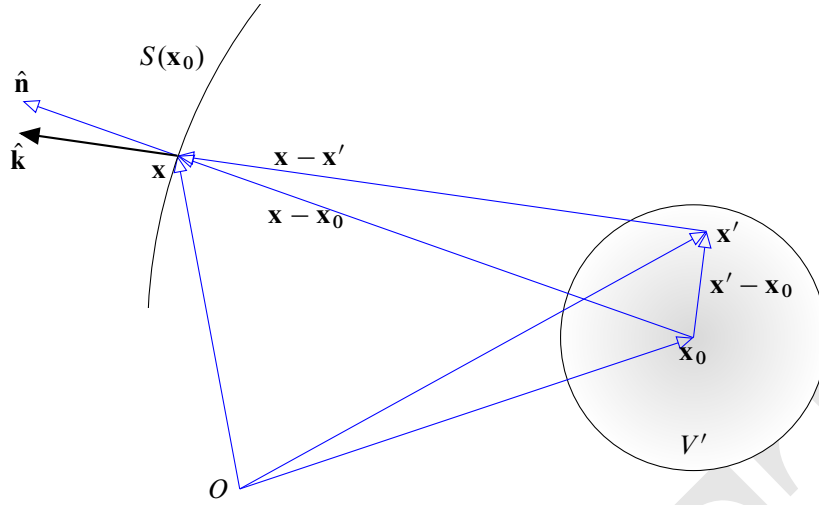


Figure 5.1: Relation between the surface normal and the \mathbf{k} vector for radiation generated at source points \mathbf{x}' near the point \mathbf{x}_0 in the source volume V' . At distances much larger than the extent of V' , the unit vector $\hat{\mathbf{n}}$, normal to the surface S which has its centre at \mathbf{x}_0 , and the unit vector $\hat{\mathbf{k}}$ of the radiation \mathbf{k} vector from \mathbf{x}' are nearly coincident.

from figure 5.1 that we can approximate

$$\begin{aligned} k |\mathbf{x} - \mathbf{x}'| &\equiv \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') \equiv \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_0) - \mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0) \\ &\approx k |\mathbf{x} - \mathbf{x}_0| - \mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0) \end{aligned} \quad (5.26)$$

Recalling from Formula (F.50) and formula (F.51) on page 198 that

$$dS = |\mathbf{x} - \mathbf{x}_0|^2 d\Omega = |\mathbf{x} - \mathbf{x}_0|^2 \sin \theta d\theta d\varphi$$

and noting from figure 5.1 that $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ are nearly parallel, we see that we can approximate

$$\frac{\hat{\mathbf{k}} \cdot d\mathbf{S}}{|\mathbf{x} - \mathbf{x}_0|^2} \equiv \frac{d^2x}{|\mathbf{x} - \mathbf{x}_0|^2} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \approx d\Omega \quad (5.27)$$

Both these approximations will be used in the following.

Within approximation (5.26) the expressions (5.25a) and (5.25b) for the radiation fields can be approximated as

$$\begin{aligned} \mathbf{B}_\omega^{\text{far}}(\mathbf{x}) &\approx -i \frac{\mu_0}{4\pi} e^{ik|\mathbf{x}-\mathbf{x}_0|} \int_{V'} d^3x' \frac{\mathbf{j}_\omega(\mathbf{x}') \times \mathbf{k}}{|\mathbf{x} - \mathbf{x}'|} e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \\ &\approx -i \frac{\mu_0}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} \int_{V'} d^3x' [\mathbf{j}_\omega(\mathbf{x}') \times \mathbf{k}] e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \end{aligned} \quad (5.28a)$$

$$\begin{aligned} \mathbf{E}_\omega^{\text{far}}(\mathbf{x}) &\approx -i \frac{1}{4\pi\epsilon_0 c} e^{ik|\mathbf{x}-\mathbf{x}_0|} \int_{V'} d^3x' \frac{[\mathbf{j}_\omega(\mathbf{x}') \times \mathbf{k}] \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \\ &\approx i \frac{1}{4\pi\epsilon_0 c} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} \frac{(\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} \times \int_{V'} d^3x' [\mathbf{j}_\omega(\mathbf{x}') \times \mathbf{k}] e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \end{aligned} \quad (5.28b)$$

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I.e., if $\sup |\mathbf{x}' - \mathbf{x}_0| \ll |\mathbf{x} - \mathbf{x}'|$, then the fields can be approximated as *spherical* waves multiplied by dimensional and angular factors, with integrals over points in the source volume only.

DRAFT

6

RADIATION AND RADIATING SYSTEMS

In chapter 3 we were able to derive general expressions for the scalar and vector potentials from which we later (in chapter 5) derived exact analytic expressions for the total electric and magnetic fields generated by completely arbitrary distributions of charge and current sources that are located in certain regions. The only limitation in the calculation of the fields was that the advanced potentials were discarded on—admittedly not totally convincing—physical grounds.

In chapter 4 we showed that the electromagnetic energy, linear momentum, and angular momentum are all conserved quantities and in this chapter we will show that these quantities can be radiated all the way to infinity and therefore be used in wireless communications over long distances and for observing very remote objects in Nature, including electromagnetic radiation sources in the Universe. Radiation processes are irreversible in that the radiation does not return to the radiator but is lost from it forever.¹ However, the radiation can, of course, be sensed by other charges and currents that are located in free space, possibly very far away from the sources. This is precisely what makes it possible for our eyes to observe light, and even more so radio signals, from extremely distant stars and galaxies. This consequence of Maxwell's equations, with the displacement current included, was verified experimentally by HEINRICH RUDOLF HERTZ about twenty years after Maxwell had made his theoretical predictions. Hertz's experimental and theoretical studies paved the way for radio and TV broadcasting, radar, wireless communications, radio astronomy and a host of other applications and technologies.

¹ This is referred to as *time arrow of radiation*.

Thus, one can, at least in principle, calculate the radiated fields, flux of energy, linear momentum and angular momentum, as well as other electromagnetic observables at any time at any point in space for an arbitrary charge and current density of the source. However, in practice, it is often difficult to evaluate the source integrals unless the charge and current densities have a simple distribution in space. In the general case, one has to resort to approximations. We shall consider both these situations in this chapter.

6.1 Radiation of linear momentum and energy

We saw in chapter 5 that at large distances r from the source, the leading order contributions to the \mathbf{E} and \mathbf{B} fields are purely transverse and fall off as $1/r$. Hence, at large distances r , the dominating component of the Poynting vector, \mathbf{S}^{far} , falls off as $1/r^2$ and is purely radial there. Consequently, when \mathbf{S}^{far} is integrated over a large spherical shell (centred on the source) for which the directed area element is $d^2x \, \hat{\mathbf{n}} = r^2 d\Omega \, \hat{\mathbf{r}} = r^2 \sin \theta \, d\theta \, d\varphi \, \hat{\mathbf{r}}$ [cf. formula (F.51) on page 198], the total integrated power, as given by the surface integral in equation (4.40) on page 58, tends to a constant at infinity, showing that energy U^{em} and electromagnetic linear momentum \mathbf{p}^{em} is carried all the way to infinity and is irreversibly lost there. This is the physical foundation of the well-known fact that \mathbf{p}^{em} and U^{em} can be transmitted over extremely long distances. The force action (time rate of change of \mathbf{p}^{em}) on charges in one region in space can therefore cause a force action on charges in another region in space. Today's wireless communication technology, be it classical or quantal, is based almost exclusively on the utilisation of this *translational degree of freedom* of the charges (currents) and the fields.

Let us consider the linear momentum and energy that is carried in the far fields \mathbf{B}^{far} , equation (5.25a), and \mathbf{E}^{far} , equation (5.25b) on page 78. Signals with limited lifetime and hence finite frequency bandwidth have to be analysed differently from monochromatic signals.

6.1.1 Monochromatic signals

If the source is strictly monochromatic, we can obtain the temporal average of the radiated power P directly, simply by averaging over one period so that

$$\begin{aligned}\langle \mathbf{S} \rangle &= \frac{1}{\mu_0} \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{1}{2\mu_0} \text{Re} \{ \mathbf{E} \times \mathbf{B}^* \} = \frac{1}{2\mu_0} \text{Re} \{ \mathbf{E}_\omega e^{-i\omega t} \times (\mathbf{B}_\omega e^{-i\omega t})^* \} \\ &= \frac{1}{2\mu_0} \text{Re} \{ \mathbf{E}_\omega \times \mathbf{B}_\omega^* e^{-i\omega t} e^{i\omega t} \} = \frac{1}{2\mu_0} \text{Re} \{ \mathbf{E}_\omega \times \mathbf{B}_\omega^* \}\end{aligned}\quad (6.1)$$

From formula (F.50) on page 198 and formula (F.51) on page 198 we see that

$$d^2x = |\mathbf{x} - \mathbf{x}_0|^2 d\Omega = |\mathbf{x} - \mathbf{x}_0|^2 \sin \theta d\theta d\varphi$$

We also note from figure 5.1 on page 79 that $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ are nearly parallel. Hence, we can approximate

$$\frac{\hat{\mathbf{k}} \cdot d^2x \hat{\mathbf{n}}}{|\mathbf{x} - \mathbf{x}_0|^2} \equiv \frac{d^2x}{|\mathbf{x} - \mathbf{x}_0|^2} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \equiv d\Omega \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \approx d\Omega \quad (6.2)$$

Using the far-field approximations (5.28a) and (5.28b) for the fields and the fact that $1/c = \sqrt{\varepsilon_0 \mu_0}$, and also introducing the *characteristic impedance of vacuum*

$$R_0 \stackrel{\text{def}}{=} \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 376.7 \, \Omega \quad (6.3)$$

for the vacuum resistance R_0 , we obtain

$$\langle \mathbf{S}^{\text{far}} \rangle = \frac{1}{32\pi^2} R_0 \frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} \left| \int_{V'} d^3x' (\mathbf{j}_\omega \times \mathbf{k}) e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \quad (6.4)$$

Consequently, the amount of power per unit solid angle $d\Omega$ that flows across an infinitesimal surface element $r^2 d\Omega = |\mathbf{x} - \mathbf{x}_0|^2 d\Omega$ of a large spherical shell with its origin at \mathbf{x}_0 and enclosing all sources, is

$$\frac{dP}{d\Omega} = \frac{1}{32\pi^2} R_0 \left| \int_{V'} d^3x' (\mathbf{j}_\omega \times \mathbf{k}) e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 \quad (6.5)$$

This formula is valid far away from the sources and shows that the radiated power is given by an expression which is resistance (R_0) times the square of the supplied current (the integrated current density \mathbf{j}_ω), as expected. We note that the emitted power is independent of distance and is therefore carried all the way to infinity. The possibility to transmit electromagnetic power over large distances, even in vacuum, is the physical foundation for the extremely important wireless communications technology. Besides determining the strength of the radiated power, the integral in formula (6.5) above also determines the angular distribution.

6.1.2 Finite bandwidth signals

A signal with finite pulse width in time (t) domain has a certain spread in frequency (ω) domain. To calculate the total radiated energy we need to integrate over the whole bandwidth. The total energy transmitted through a unit area is the time integral of the Poynting vector:

$$\begin{aligned}\int_{-\infty}^{\infty} dt \mathbf{S}(t) &= \frac{1}{\mu_0} \int_{-\infty}^{\infty} dt (\mathbf{E} \times \mathbf{B}) \\ &= \frac{1}{\mu_0} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} dt (\mathbf{E}_\omega \times \mathbf{B}_{\omega'}) e^{-i(\omega+\omega')t}\end{aligned}\quad (6.6)$$

If we carry out the temporal integration first and use the fact that

$$\int_{-\infty}^{\infty} dt e^{-i(\omega+\omega')t} = 2\pi \delta(\omega + \omega') \quad (6.7)$$

equation (6.6) can be written

$$\begin{aligned}\int_{-\infty}^{\infty} dt \mathbf{S}(t) &= \frac{2\pi}{\mu_0} \int_{-\infty}^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) \\ &= \frac{2\pi}{\mu_0} \left(\int_0^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) + \int_{-\infty}^0 d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) \right) \\ &= \frac{2\pi}{\mu_0} \left(\int_0^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) - \int_0^{-\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) \right) \\ &= \frac{2\pi}{\mu_0} \left(\int_0^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega}) + \int_0^{\infty} d\omega (\mathbf{E}_{-\omega} \times \mathbf{B}_\omega) \right) \\ &= \frac{2\pi}{\mu_0} \int_0^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_{-\omega} + \mathbf{E}_{-\omega} \times \mathbf{B}_\omega) \\ &= \frac{2\pi}{\mu_0} \int_0^{\infty} d\omega (\mathbf{E}_\omega \times \mathbf{B}_\omega^* + \mathbf{E}_\omega^* \times \mathbf{B}_\omega)\end{aligned}\quad (6.8)$$

where the last step follows from physical requirement of real-valuedness of \mathbf{E}_ω and \mathbf{B}_ω . We insert the Fourier transforms of the field components which dominate at large distances, *i.e.*, the far fields (5.25a) and (5.25b). The result, after integration over the area S of a large sphere which encloses the source volume V' , is

$$U = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \oint_S d^2x \hat{\mathbf{n}} \cdot \int_0^{\infty} d\omega \left| \int_{V'} d^3x' \frac{\mathbf{j}_\omega \times \mathbf{k}}{|\mathbf{x} - \mathbf{x}'|} e^{ik|\mathbf{x} - \mathbf{x}'|} \right|^2 \hat{\mathbf{k}} \quad (6.9)$$

Inserting the approximations (5.26) and (6.2) into equation (6.9) above, introducing the *spectral energy density* U_ω via the definition

$$U \stackrel{\text{def}}{=} \int_0^{\infty} d\omega U_\omega \quad (6.10)$$

and recalling the definition (6.3) on page 83, we obtain

$$\frac{dU_\omega}{d\Omega} d\omega \approx \frac{1}{4\pi} R_0 \left| \int_{V'} d^3x' (\mathbf{j}_\omega \times \mathbf{k}) e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 d\omega \quad (6.11)$$

which, at large distances, is a good approximation to the energy that is radiated per unit solid angle $d\Omega$ in a frequency band $d\omega$. It is important to notice that Formula (6.11) includes only source coordinates. This means that the amount of energy that is being radiated is independent on the distance to the source (as long as it is large).

6.2 Radiation of angular momentum

Not only electromagnetic linear momentum (Poynting vector) can be radiated from a source and transmitted over very long distances, but the same is also true for electromagnetic angular momentum \mathbf{J}^{em} . Then torque action (the time rate of change of \mathbf{J}^{em}) in one region causes torque action on charges. The use of this *rotational degree of freedom* of the fields has only recently been put to practical use even if it has been known for more than a century.

After lengthy calculations, based on the results obtained in chapter 5, one finds that the complete cycle averaged far-zone expression for a frequency component ω of the electromagnetic angular momentum density generated by arbitrary charge and current sources can be approximated by

$$\langle \mathbf{h}^{\text{em}}(\mathbf{x}_0) \rangle = \frac{1}{32\pi^2 \epsilon_0 c^3} \left(\frac{\hat{\mathbf{n}} \times \text{Re} \{ (cq + I_n) \dot{\mathbf{I}}^* \}}{c |\mathbf{x} - \mathbf{x}_0|^2} + \frac{\hat{\mathbf{n}} \times \text{Re} \{ (cq + I_n) \mathbf{I}^* \}}{|\mathbf{x} - \mathbf{x}_0|^3} \right) \quad (6.12)$$

where, in complex notation,

$$\mathbf{I}(t') \approx \int_{V'} d^3x' \mathbf{j}(t', \mathbf{x}') \quad (6.13)$$

and

$$\dot{\mathbf{I}}(t') \approx \int_{V'} d^3x' \dot{\mathbf{j}}(t', \mathbf{x}') \quad (6.14)$$

We see that at very large distances r , the angular momentum density \mathbf{h}^{em} falls off as $1/r^2$, *i.e.*, it has precisely the same behaviour in the far zone as the linear momentum density and can therefore also transfer information wirelessly over large distances. The only difference is that while the direction of the linear momentum (Poynting vector) becomes purely radial at infinity, the angular momentum becomes perpendicular to the linear momentum, *i.e.* purely transverse, there.

6.3 Radiation from a localised source volume at rest

In the general case, and when we are interested in evaluating the radiation far from a source at rest and which is localised in a small volume, we can introduce an approximation which leads to a *multipole expansion* where individual terms can be evaluated analytically. Here we use *Hertz's method*, which focuses on the physics rather than the mathematics, to obtain this expansion.

6.3.1 Electric multipole moments

Let us assume that the charge distribution ρ determining the potential in equation (3.47a) on page 40 has such a small extent that all the source points \mathbf{x}' can be assumed to be located very near a point \mathbf{x}_0 . At a large distance $|\mathbf{x} - \mathbf{x}_0|$, one can then, to a good approximation, approximate the potential by the Taylor expansion [*cf.* example 3.1 on page 33]

$$\begin{aligned} \phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q(t)}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} d_i(t, \mathbf{x}_0) \frac{(\mathbf{x} - \mathbf{x}_0)_i}{|\mathbf{x} - \mathbf{x}_0|} \right. \\ \left. + \frac{1}{|\mathbf{x} - \mathbf{x}_0|^3} Q_{ij}(t, \mathbf{x}_0) \left(\frac{3}{2} \frac{(\mathbf{x} - \mathbf{x}_0)_i}{|\mathbf{x} - \mathbf{x}_0|} \frac{(\mathbf{x} - \mathbf{x}_0)_j}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{2} \delta_{ij} \right) + \dots \right] \end{aligned} \quad (6.15)$$

where

$$q(t) = \int_{V'} d^3x' \rho(t'_{\text{ret}}, \mathbf{x}') \quad (6.16a)$$

is the *total charge* or *electric monopole moment*,

$$\mathbf{d}(t, \mathbf{x}_0) = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \rho(t'_{\text{ret}}, \mathbf{x}') \quad (6.16b)$$

with components d_i , $i = 1, 2, 3$ is the *electric dipole moment vector*, and

$$\mathbf{Q}(t, \mathbf{x}_0) = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0) \rho(t'_{\text{ret}}, \mathbf{x}') \quad (6.16c)$$

with components Q_{ij} , $i, j = 1, 2, 3$ (Einstein's summation convention over i and j is implied), is the *electric quadrupole moment tensor*.

The source volume is so small that internal retardation effects can be neglected, *i.e.*, that we can set $t'_{\text{ret}} \approx t - |\mathbf{x} - \mathbf{x}_0|/c$. Then

$$t = t(t'_{\text{ret}}) \approx t'_{\text{ret}} + \text{Const} \quad (6.17)$$

where

$$Const = \frac{|\mathbf{x} - \mathbf{x}_0|}{c} \quad (6.18)$$

Hence the transformation between t and t'_{ret} is a trivial. In the subsequent analysis in this Subsection we shall use t' to denote this approximate t'_{ret} .

For a normal medium, the major contributions to the electrostatic interactions come from the net charge and the lowest order electric multipole moments induced by the polarisation due to an applied electric field. Particularly important is the dipole moment. Let \mathbf{P} denote the electric dipole moment density (electric dipole moment per unit volume; unit: C m^{-2}), also known as the *electric polarisation*, in some medium. In analogy with the second term in the expansion equation (6.15) on the preceding page, the electric potential from this volume distribution $\mathbf{P}(t, \mathbf{x}')$ of electric dipole moments \mathbf{d} at the source point \mathbf{x}' can be written

$$\begin{aligned} \phi_{\mathbf{d}}(t, \mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \mathbf{P}(t', \mathbf{x}') \cdot \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\ &= -\frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \mathbf{P}(t', \mathbf{x}') \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \mathbf{P}(t', \mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \end{aligned} \quad (6.19)$$

Using expression (M.113) on page 224 and applying the divergence theorem, we can rewrite this expression for the potential as follows:

$$\begin{aligned} \phi_{\mathbf{d}}(t, \mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \left[\int_{V'} d^3x' \nabla' \cdot \left(\frac{\mathbf{P}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) - \int_{V'} d^3x' \frac{\nabla' \cdot \mathbf{P}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] \\ &= \frac{1}{4\pi\epsilon_0} \left[\oint_{S'} d^2x' \hat{\mathbf{n}}' \cdot \frac{\mathbf{P}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \int_{V'} d^3x' \frac{\nabla' \cdot \mathbf{P}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right] \end{aligned} \quad (6.20)$$

where the first term, which describes the effects of the induced, non-cancelling dipole moment on the surface of the volume, can be neglected, unless there is a discontinuity in $\hat{\mathbf{n}} \cdot \mathbf{P}$ at the surface. Doing so, we find that the contribution from the electric dipole moments to the potential is given by

$$\phi_{\mathbf{d}}(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{-\nabla' \cdot \mathbf{P}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (6.21)$$

Comparing this expression with expression equation (3.47a) on page 40 for the potential from a charge distribution $\rho(t, \mathbf{x})$, we see that $-\nabla' \cdot \mathbf{P}(t, \mathbf{x})$ has the characteristics of a charge density and that, to the lowest order, the effective charge density becomes $\rho(t, \mathbf{x}) - \nabla' \cdot \mathbf{P}(t, \mathbf{x})$, in which the second term is a polarisation term that we call $\rho^{\text{pol}}(t, \mathbf{x})$.

6.3.2 The Hertz potential

In section 6.3.1 we introduced the electric polarisation $\mathbf{P}(t, \mathbf{x})$ such that the polarisation charge density

$$\rho^{\text{pol}} = -\nabla \cdot \mathbf{P} \quad (6.22)$$

If we adopt the same idea for the ‘true’ charge density due to free charges and introduce a vector field $\boldsymbol{\pi}(t, \mathbf{x})$, analogous to $\mathbf{P}(t, \mathbf{x})$, but such that

$$\rho^{\text{true}} \stackrel{\text{def}}{=} -\nabla \cdot \boldsymbol{\pi} \quad (6.23a)$$

which means that the associated ‘polarisation current’ now is the true current:

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = \mathbf{j}^{\text{true}} \quad (6.23b)$$

As a consequence, the equation of continuity for ‘true’ charges and currents [cf. expression (1.25) on page 10] is satisfied:

$$\frac{\partial \rho^{\text{true}}(t, \mathbf{x})}{\partial t} + \nabla \cdot \mathbf{j}^{\text{true}}(t, \mathbf{x}) = -\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{\pi} + \nabla \cdot \frac{\partial \boldsymbol{\pi}}{\partial t} = 0 \quad (6.24)$$

The vector $\boldsymbol{\pi}$ is called the *polarisation vector* because, formally, it treats also the ‘true’ (free) charges as polarisation charges. Since in the microscopic Maxwell-Lorentz equation (2.1a) on page 20, the charge density ρ must include all charges, we can write this equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{\rho^{\text{true}} + \rho^{\text{pol}}}{\epsilon_0} = \frac{-\nabla \cdot \boldsymbol{\pi} - \nabla \cdot \mathbf{P}}{\epsilon_0} \quad (6.25)$$

i.e., in a form where all the charges are considered to be polarisation charges.

We now introduce a further potential $\boldsymbol{\Pi}^e$ with the following property

$$\nabla \cdot \boldsymbol{\Pi}^e = -\phi \quad (6.26a)$$

$$\frac{1}{c^2} \frac{\partial \boldsymbol{\Pi}^e}{\partial t} = \mathbf{A} \quad (6.26b)$$

where ϕ and \mathbf{A} are the electromagnetic scalar and vector potentials, respectively. As we see, $\boldsymbol{\Pi}^e$ acts as a ‘*super-potential*’ in the sense that it is a potential from which we can obtain other potentials. It is called the *Hertz vector* or *polarisation potential*. Requiring that the scalar and vector potentials ϕ and \mathbf{A} , respectively, satisfy their inhomogeneous wave equations, equations (3.27) on page 36, one finds, using (6.23) and (6.26), that the Hertz vector must satisfy the inhomogeneous wave equation

$$\square^2 \boldsymbol{\Pi}^e = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{\Pi}^e - \nabla^2 \boldsymbol{\Pi}^e = \frac{\boldsymbol{\pi}}{\epsilon_0} \quad (6.27)$$

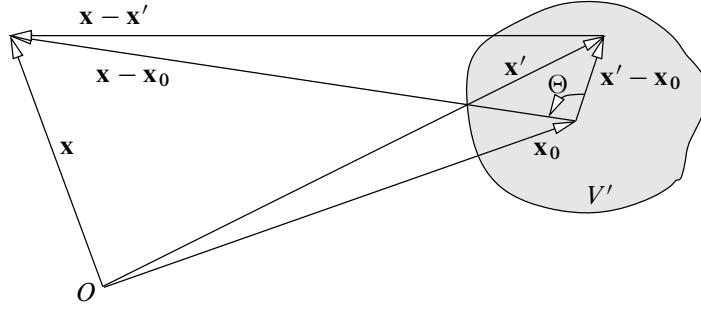


Figure 6.1: Geometry of a typical multipole radiation problem where the field point \mathbf{x} is located some distance away from the finite source volume V' centred around \mathbf{x}_0 . If $k |\mathbf{x}' - \mathbf{x}_0| \ll 1 \ll k |\mathbf{x} - \mathbf{x}_0|$, then the radiation at \mathbf{x} is well approximated by a few terms in the multipole expansion.

Assume that the source region is a limited volume around some central point \mathbf{x}_0 far away from the field (observation) point \mathbf{x} illustrated in figure 6.1. Under these assumptions, we can expand the Hertz vector, expression (6.32) on the next page, due to the presence of non-vanishing $\pi(t'_{\text{ret}}, \mathbf{x}')$ in the vicinity of \mathbf{x}_0 , in a formal series. For this purpose we recall from *potential theory* that

$$\begin{aligned} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &\equiv \frac{e^{ik|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|}}{|(\mathbf{x}-\mathbf{x}_0)-(\mathbf{x}'-\mathbf{x}_0)|} \\ &= ik \sum_{n=0}^{\infty} (2n+1) P_n(\cos \Theta) j_n(k|\mathbf{x}'-\mathbf{x}_0|) h_n^{(1)}(k|\mathbf{x}-\mathbf{x}_0|) \end{aligned} \quad (6.28)$$

where (see figure 6.1)

$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ is a *Green function* or *propagator*

Θ is the angle between $\mathbf{x}' - \mathbf{x}_0$ and $\mathbf{x} - \mathbf{x}_0$

$P_n(\cos \Theta)$ is the *Legendre polynomial* of order n

$j_n(k|\mathbf{x}' - \mathbf{x}_0|)$ is the *spherical Bessel function of the first kind* of order n

$h_n^{(1)}(k|\mathbf{x} - \mathbf{x}_0|)$ is the *spherical Hankel function of the first kind* of order n

According to the addition theorem for Legendre polynomials

$$P_n(\cos \Theta) = \sum_{m=-n}^n (-1)^m P_n^m(\cos \theta) P_n^{-m}(\cos \theta') e^{im(\varphi - \varphi')} \quad (6.29)$$

where P_n^m is an *associated Legendre polynomial of the first kind*, related to the *spherical harmonic* Y_n^m as

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos \theta) e^{im\varphi}$$

and, in spherical polar coordinates,

$$\mathbf{x}' - \mathbf{x}_0 = (|\mathbf{x}' - \mathbf{x}_0|, \theta', \varphi') \quad (6.30a)$$

$$\mathbf{x} - \mathbf{x}_0 = (|\mathbf{x} - \mathbf{x}_0|, \theta, \varphi) \quad (6.30b)$$

This equation is of the same type as equation (3.30) on page 37, and has therefore the retarded solution

$$\mathbf{\Pi}^e(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\boldsymbol{\pi}(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (6.31)$$

with Fourier components

$$\mathbf{\Pi}_\omega^e(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\boldsymbol{\pi}_\omega(\mathbf{x}') e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \quad (6.32)$$

If we introduce the *help* vector \mathbf{C} such that

$$\mathbf{C} = \nabla \times \mathbf{\Pi}^e \quad (6.33)$$

we see that we can calculate the magnetic and electric fields, respectively, as follows

$$\mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{C}}{\partial t} \quad (6.34a)$$

$$\mathbf{E} = \nabla \times \mathbf{C} \quad (6.34b)$$

Clearly, the last equation is valid only if $\nabla \cdot \mathbf{E} = 0$ *i.e.*, if we are outside the source volume. Since we are mainly interested in the fields in the so called *far zone*, a long distance from the source region, this is no essential limitation.

Inserting equation (6.28) on the preceding page, together with formula (6.29) on the previous page, into equation (6.32), we can in a formally exact way expand the Fourier component of the Hertz vector as

$$\begin{aligned} \mathbf{\Pi}_\omega^e &= \frac{ik}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \sum_{m=-n}^n (2n+1)(-1)^m h_n^{(1)}(k|\mathbf{x} - \mathbf{x}_0|) P_n^m(\cos\theta) e^{im\varphi} \\ &\times \int_{V'} d^3x' \boldsymbol{\pi}_\omega(\mathbf{x}') j_n(k|\mathbf{x}' - \mathbf{x}_0|) P_n^{-m}(\cos\theta') e^{-im\varphi'} \end{aligned} \quad (6.35)$$

We notice that there is no dependence on $\mathbf{x} - \mathbf{x}_0$ inside the integral; the integrand is only dependent on the relative source vector $\mathbf{x}' - \mathbf{x}_0$.

We are interested in the case where the field point is many wavelengths away from the well-localised sources, *i.e.*, when the following inequalities

$$k|\mathbf{x}' - \mathbf{x}_0| \ll 1 \ll k|\mathbf{x} - \mathbf{x}_0| \quad (6.36)$$

hold. Then we may to a good approximation replace $h_n^{(1)}$ with the first term in its asymptotic expansion:

$$h_n^{(1)}(k|\mathbf{x} - \mathbf{x}_0|) \approx (-i)^{n+1} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{k|\mathbf{x} - \mathbf{x}_0|} \quad (6.37)$$

and replace j_n with the first term in its power series expansion:

$$j_n(k|\mathbf{x}' - \mathbf{x}_0|) \approx \frac{2^n n!}{(2n+1)!} (k|\mathbf{x}' - \mathbf{x}_0|)^n \quad (6.38)$$

Inserting these expansions into equation (6.35) on the preceding page, we obtain the multipole expansion of the Fourier component of the Hertz vector

$$\Pi_\omega^e \approx \sum_{n=0}^{\infty} \Pi_\omega^{e(n)} \quad (6.39a)$$

where

$$\begin{aligned} \Pi_\omega^{e(n)} = & (-i)^n \frac{1}{4\pi\epsilon_0} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} \frac{2^n n!}{(2n)!} \\ & \times \int_{V'} d^3x' \pi_\omega(\mathbf{x}') (k|\mathbf{x}' - \mathbf{x}_0|)^n P_n(\cos \Theta) \end{aligned} \quad (6.39b)$$

This expression is approximately correct only if certain care is exercised; if many $\Pi_\omega^{e(n)}$ terms are needed for an accurate result, the expansions of the spherical Hankel and Bessel functions used above may not be consistent and must be replaced by more accurate expressions. Furthermore, asymptotic expansions as the one used in equation (6.37) on page 91 are not unique.

Taking the inverse Fourier transform of Π_ω^e will yield the Hertz vector in time domain, which inserted into equation (6.33) on the facing page will yield **C**. The resulting expression can then in turn be inserted into equations (6.34) on the preceding page in order to obtain the radiation fields.

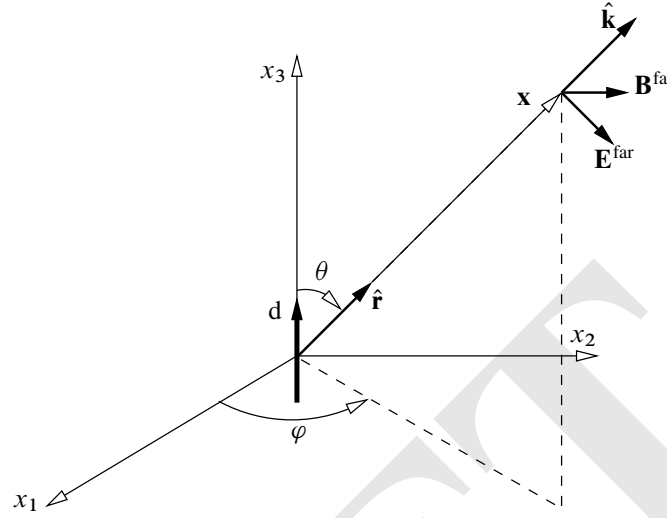
For a linear source distribution along the polar axis, $\Theta = \theta$ in expression (6.39b), and $P_n(\cos \theta)$ gives the angular distribution of the radiation. In the general case, however, the angular distribution must be computed with the help of formula (6.29) on page 89. Let us now study the lowest order contributions to the expansion of the Hertz vector.

6.3.3 Electric dipole radiation

Choosing $n = 0$ in expression (6.39b), we obtain

$$\Pi_\omega^{e(0)} = \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{4\pi\epsilon_0|\mathbf{x} - \mathbf{x}_0|} \int_{V'} d^3x' \pi_\omega(\mathbf{x}') = \frac{1}{4\pi\epsilon_0} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} \mathbf{d}_\omega \quad (6.40)$$

Figure 6.2: If a spherical polar coordinate system (r, θ, φ) is chosen such that the electric dipole moment \mathbf{d} (and thus its Fourier transform \mathbf{d}_ω) is located at the origin and directed along the polar axis, the calculations are simplified.



Since π represents a dipole moment density for the ‘true’ charges (in the same vein as \mathbf{P} does so for the polarised charges), $\mathbf{d}_\omega = \int_{V'} d^3x' \pi_\omega(\mathbf{x}')$ is, by definition, the Fourier component of the *electric dipole moment*

$$\mathbf{d}(t, \mathbf{x}_0) = \int_{V'} d^3x' \pi(t', \mathbf{x}') = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \rho(t', \mathbf{x}') \quad (6.41)$$

[cf. equation (6.16b) on page 86]. If a spherical coordinate system is chosen with its polar axis along \mathbf{d}_ω as in figure 6.2, the components of $\Pi_\omega^{e(0)}$ are

$$\Pi_r^e \stackrel{\text{def}}{=} \Pi_\omega^{e(0)} \cdot \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \cos \theta \quad (6.42a)$$

$$\Pi_\theta^e \stackrel{\text{def}}{=} \Pi_\omega^{e(0)} \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{4\pi\epsilon_0} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \sin \theta \quad (6.42b)$$

$$\Pi_\varphi^e \stackrel{\text{def}}{=} \Pi_\omega^{e(0)} \cdot \hat{\boldsymbol{\varphi}} = 0 \quad (6.42c)$$

Evaluating formula (6.33) on page 90 for the help vector \mathbf{C} , with the spherically polar components (6.42) of $\Pi_\omega^{e(0)}$ inserted, we obtain

$$\mathbf{C}_\omega = C_{\omega,\varphi}^{(0)} \hat{\boldsymbol{\varphi}} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|} - ik \right) \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} d_\omega \sin \theta \hat{\boldsymbol{\varphi}} \quad (6.43)$$

Applying this to equations (6.34) on page 90, we obtain directly the Fourier

components of the fields

$$\mathbf{B}_\omega = -i \frac{\omega \mu_0}{4\pi} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} - ik \right) \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} d_\omega \sin \theta \hat{\boldsymbol{\phi}} \quad (6.44a)$$

$$\begin{aligned} \mathbf{E}_\omega = \frac{1}{4\pi \varepsilon_0} \left[2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} - \frac{ik}{|\mathbf{x} - \mathbf{x}_0|} \right) \cos \theta \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \right. \\ \left. + \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} - \frac{ik}{|\mathbf{x} - \mathbf{x}_0|} - k^2 \right) \sin \theta \hat{\boldsymbol{\theta}} \right] \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} d_\omega \end{aligned} \quad (6.44b)$$

Keeping only those parts of the fields which dominate at large distances (the radiation fields) and recalling that the wave vector $\mathbf{k} = k(\mathbf{x} - \mathbf{x}_0)/|\mathbf{x} - \mathbf{x}_0|$ where $k = \omega/c$, we can now write down the Fourier components of the radiation parts of the magnetic and electric fields from the dipole:

$$\mathbf{B}_\omega^{\text{far}} = -\frac{\omega \mu_0}{4\pi} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} d_\omega k \sin \theta \hat{\boldsymbol{\phi}} = -\frac{\omega \mu_0}{4\pi} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} (\mathbf{d}_\omega \times \mathbf{k}) \quad (6.45a)$$

$$\mathbf{E}_\omega^{\text{far}} = -\frac{1}{4\pi \varepsilon_0} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} d_\omega k^2 \sin \theta \hat{\boldsymbol{\theta}} = -\frac{1}{4\pi \varepsilon_0} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|} [(\mathbf{d}_\omega \times \mathbf{k}) \times \mathbf{k}] \quad (6.45b)$$

These fields constitute the *electric dipole radiation*, also known as *E1 radiation*.

▷ Linear and angular momenta radiated from an electric dipole in vacuum ————— EXAMPLE 6.1

The Fourier amplitudes of the fields generated by an electric dipole, \mathbf{d}_ω , oscillating at the angular frequency ω , are given by formulæ (6.44) above. Inverse Fourier transforming to the time domain, and using a spherical coordinate system (r, θ, φ) , the physically observable fields are found to be

$$\mathbf{B}(t, \mathbf{x}) = \frac{\omega \mu_0}{4\pi} d_\omega \sin \theta \left(\frac{1}{r^2} \sin(kr - \omega t') - \frac{k}{r} \cos(kr - \omega t') \right) \hat{\boldsymbol{\phi}} \quad (6.46a)$$

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) = \frac{1}{4\pi \varepsilon_0} d_\omega \sin \theta \left(\frac{1}{r^3} \cos(kr - \omega t') + \frac{k}{r^2} \sin(kr - \omega t') - \frac{k^2}{r} \cos(kr - \omega t') \right) \hat{\boldsymbol{\theta}} \\ + \frac{1}{2\pi \varepsilon_0} d_\omega \cos \theta \left(\frac{1}{r^3} \cos(kr - \omega t') + \frac{k}{r^2} \sin(kr - \omega t') \right) \hat{\mathbf{r}} \end{aligned} \quad (6.46b)$$

where $t' = t - r/c$ is the retarded time.

Applying formula (4.51) on page 59 for the *electromagnetic linear momentum density* to the fields from a pure electric dipole, equations (6.46) above, one obtains

$$\begin{aligned}
\mathbf{g}^{\text{em}} = \varepsilon_0 \mathbf{E} \times \mathbf{B} = & -\frac{\omega\mu_0}{8\pi^2} d_\omega^2 \sin\theta \cos\theta \left\{ \frac{1}{r^5} \sin(kr - \omega t') \cos(kr - \omega t') \right. \\
& + \frac{k}{r^4} [\sin^2(kr - \omega t') - \cos^2(kr - \omega t')] - \frac{k^2}{r^3} \sin(kr - \omega t') \cos(kr - \omega t') \left. \right\} \hat{\boldsymbol{\theta}} \\
& + \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \sin^2\theta \left\{ \frac{1}{r^5} \sin(kr - \omega t') \cos(kr - \omega t') \right. \\
& + \frac{k}{r^4} [\sin^2(kr - \omega t') - \cos^2(kr - \omega t')] - 2\frac{k^2}{r^3} \sin(kr - \omega t') \cos(kr - \omega t') \\
& + \frac{k^3}{r^2} \cos^2(kr - \omega t') \left. \right\} \hat{\mathbf{r}}
\end{aligned} \tag{6.47}$$

Using well-known trigonometric relations, this can be put in the form

$$\begin{aligned}
\mathbf{g}^{\text{em}} = & -\frac{\omega\mu_0}{16\pi^2} d_\omega^2 \sin\theta \cos\theta \left[\frac{1}{r^5} \sin[2(kr - \omega t')] - 2\frac{k}{r^4} \cos[2(kr - \omega t')] \right. \\
& \left. - \frac{k^2}{r^3} \sin[2(kr - \omega t')] \right] \hat{\boldsymbol{\theta}} \\
& + \frac{\omega\mu_0}{32\pi^2} d_\omega^2 \sin^2\theta \left[\frac{1}{r^5} \sin[2(kr - \omega t')] - 2\frac{k}{r^4} \cos[2(kr - \omega t')] \right. \\
& \left. - 2\frac{k^2}{r^3} \sin[2(kr - \omega t')] + \frac{k^3}{r^2} (1 + \cos[2(kr - \omega t')]) \right] \hat{\mathbf{r}}
\end{aligned} \tag{6.48}$$

which shows that the linear momentum density, and hence the Poynting vector, is strictly radial only at infinity.

Applying formula (4.72) on page 63 for the *electromagnetic angular momentum density* with the momentum point chosen as $\mathbf{x}_0 = \mathbf{0}$, i.e.,

$$\mathbf{h}^{\text{em}} = \mathbf{x} \times \mathbf{g}^{\text{em}} \tag{6.49}$$

and using equation (6.48) above, we find that for a pure electric dipole

$$\begin{aligned}
\mathbf{h}^{\text{em}} = & -\frac{\omega\mu_0}{8\pi^2} d_\omega^2 \sin\theta \cos\theta \left[\frac{1}{r^4} \sin(kr - \omega t') \cos(kr - \omega t') \right. \\
& + \frac{k}{r^3} [\sin^2(kr - \omega t') - \cos^2(kr - \omega t')] - \frac{k^2}{r^2} \sin(kr - \omega t') \cos(kr - \omega t') \left. \right] \hat{\boldsymbol{\phi}}
\end{aligned} \tag{6.50}$$

or

$$\begin{aligned}
\mathbf{h}^{\text{em}} = & -\frac{\omega\mu_0}{16\pi^2} d_\omega^2 \sin\theta \cos\theta \left[\frac{1}{r^4} \sin[2(kr - \omega t')] \right. \\
& \left. - 2\frac{k}{r^3} \cos[2(kr - \omega t')] - \frac{k^2}{r^2} \sin[2(kr - \omega t')] \right] \hat{\boldsymbol{\phi}}
\end{aligned} \tag{6.51}$$

The *total electromagnetic linear momentum* is (cf. formula (4.57b) on page 60)

$$\mathbf{p}^{\text{em}} = \int_{V'} d^3x' \mathbf{g}^{\text{em}}(t', \mathbf{x}') \tag{6.52}$$

and the *total electromagnetic angular momentum* is (cf. formula (4.78b) on page 64)

$$\mathbf{J}^{\text{em}} = \int_{V'} d^3x' \mathbf{h}^{\text{em}}(t', \mathbf{x}') \tag{6.53}$$

In order to get a total net \mathbf{J}^{em} , it is convenient to superimpose several individual dipoles of (possibly) different strengths and relative phases. Perhaps the most common configuration yielding a total net \mathbf{J}^{em} is two orthogonal co-located dipoles with $\pi/2$ phase shift between them.

We note that in the far zone the linear and angular momentum densities tend to

$$\begin{aligned}\mathbf{g}^{\text{em, far}}(t, \mathbf{x}) &\approx \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \frac{k^3}{r^2} \sin^2 \theta \cos^2(kr - \omega t') \hat{\mathbf{r}} \\ &= \frac{\omega\mu_0}{32\pi^2} d_\omega^2 \frac{k^3}{r^2} \sin^2 \theta (1 + \cos[2(kr - \omega t')]) \hat{\mathbf{r}}\end{aligned}\quad (6.54)$$

and

$$\begin{aligned}\mathbf{h}^{\text{em, far}}(t, \mathbf{x}) &\approx \frac{\omega\mu_0}{8\pi^2} d_\omega^2 \frac{k^2}{r^2} \sin \theta \cos \theta \sin(kr - \omega t') \cos(kr - \omega t') \hat{\boldsymbol{\phi}} \\ &= \frac{\omega\mu_0}{16\pi^2} d_\omega^2 \frac{k^2}{r^2} \sin \theta \cos \theta \sin[2(kr - \omega t')] \hat{\boldsymbol{\phi}}\end{aligned}\quad (6.55)$$

respectively. *I.e.*, to leading order, both the linear momentum density \mathbf{g}^{em} and the angular momentum density \mathbf{h}^{em} fall off as $\sim 1/r^2$ far away from the source region. This means that when they are integrated over a spherical surface $\propto r^2$ located at a large distance from the source [*cf.* the last term in the LHS of formula (4.40) on page 58], there is a net flux so that the integrated momenta do not fall off with distance and can therefore be transported all the way to infinity.

—End of example 6.1◁

6.3.4 Magnetic dipole radiation

The next term in the expression (6.39b) on page 91 for the expansion of the Fourier transform of the Hertz vector is for $n = 1$:

$$\begin{aligned}\Pi_\omega^{e(1)} &= -i \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{4\pi\epsilon_0 |\mathbf{x}-\mathbf{x}_0|} \int_{V'} d^3x' k |\mathbf{x}' - \mathbf{x}_0| \pi_\omega(\mathbf{x}') \cos \Theta \\ &= -ik \frac{1}{4\pi\epsilon_0 |\mathbf{x}-\mathbf{x}_0|^2} \int_{V'} d^3x' [(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x}' - \mathbf{x}_0)] \pi_\omega(\mathbf{x}')\end{aligned}\quad (6.56)$$

Here, the term $[(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x}' - \mathbf{x}_0)] \pi_\omega(\mathbf{x}')$ can be rewritten

$$[(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x}' - \mathbf{x}_0)] \pi_\omega(\mathbf{x}') = (x_i - x_{0,i})(x'_i - x_{0,i}) \pi_\omega(\mathbf{x}') \quad (6.57)$$

and introducing

$$\eta_i = x_i - x_{0,i} \quad (6.58a)$$

$$\eta'_i = x'_i - x_{0,i} \quad (6.58b)$$

the j th component of the integrand in $\Pi_\omega^{e(1)}$ can be broken up into

$$\begin{aligned} \{[(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x}' - \mathbf{x}_0)] \pi_\omega(\mathbf{x}')\}_j &= \frac{1}{2} \eta_i (\pi_{\omega,j} \eta'_i + \pi_{\omega,i} \eta'_j) \\ &+ \frac{1}{2} \eta_i (\pi_{\omega,j} \eta'_i - \pi_{\omega,i} \eta'_j) \end{aligned} \quad (6.59)$$

i.e., as the sum of two parts, the first being symmetric and the second antisymmetric in the indices i, j . We note that the antisymmetric part can be written as

$$\begin{aligned} \frac{1}{2} \eta_i (\pi_{\omega,j} \eta'_i - \pi_{\omega,i} \eta'_j) &= \frac{1}{2} [\pi_{\omega,j} (\eta_i \eta'_i) - \eta'_j (\eta_i \pi_{\omega,i})] \\ &= \frac{1}{2} [\pi_\omega (\boldsymbol{\eta} \cdot \boldsymbol{\eta}') - \boldsymbol{\eta}' (\boldsymbol{\eta} \cdot \boldsymbol{\pi}_\omega)]_j \\ &= \frac{1}{2} \{(\mathbf{x} - \mathbf{x}_0) \times [\boldsymbol{\pi}_\omega \times (\mathbf{x}' - \mathbf{x}_0)]\}_j \end{aligned} \quad (6.60)$$

The utilisation of equations (6.23) on page 88, and the fact that we are considering a single Fourier component,

$$\boldsymbol{\pi}(t, \mathbf{x}) = \boldsymbol{\pi}_\omega e^{-i\omega t} \quad (6.61)$$

allow us to express $\boldsymbol{\pi}_\omega$ in \mathbf{j}_ω as

$$\boldsymbol{\pi}_\omega = i \frac{\mathbf{j}_\omega}{\omega} \quad (6.62)$$

Hence, we can write the antisymmetric part of the integral in formula (6.56) on the previous page as

$$\begin{aligned} \frac{1}{2} (\mathbf{x} - \mathbf{x}_0) \times \int_{V'} d^3x' \boldsymbol{\pi}_\omega(\mathbf{x}') \times (\mathbf{x}' - \mathbf{x}_0) \\ = i \frac{1}{2\omega} (\mathbf{x} - \mathbf{x}_0) \times \int_{V'} d^3x' \mathbf{j}_\omega(\mathbf{x}') \times (\mathbf{x}' - \mathbf{x}_0) \\ = -i \frac{1}{\omega} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{m}_\omega \end{aligned} \quad (6.63)$$

where we introduced the Fourier transform of the *magnetic dipole moment*

$$\mathbf{m}_\omega = \frac{1}{2} \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \times \mathbf{j}_\omega(\mathbf{x}') \quad (6.64)$$

The final result is that the antisymmetric, magnetic dipole, part of $\Pi_\omega^{e(1)}$ can be written

$$\Pi_\omega^{e, \text{antisym}(1)} = -\frac{k}{4\pi \epsilon_0 \omega} \frac{e^{ik|\mathbf{x} - \mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|^2} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{m}_\omega \quad (6.65)$$

In analogy with the electric dipole case, we insert this expression into equation (6.33) on page 90 to evaluate \mathbf{C} , with which equations (6.34) on page 90 then gives the \mathbf{B} and \mathbf{E} fields. Discarding, as before, all terms belonging to the near fields and transition fields and keeping only the terms that dominate at large distances, we obtain

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} (\mathbf{m}_\omega \times \mathbf{k}) \times \mathbf{k} \quad (6.66a)$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{k}{4\pi\epsilon_0 c} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} \mathbf{m}_\omega \times \mathbf{k} \quad (6.66b)$$

which are the fields of the *magnetic dipole radiation* (*M1 radiation*).

6.3.5 Electric quadrupole radiation

The symmetric part $\Pi_\omega^{e, \text{sym}(1)}$ of the $n = 1$ contribution in the equation (6.39b) on page 91 for the expansion of the Hertz vector can be expressed in terms of the *electric quadrupole tensor*, which is defined in accordance with equation (6.16c) on page 86:

$$\mathbf{Q}(t, \mathbf{x}_0) = \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0)(\mathbf{x}' - \mathbf{x}_0) \rho(t'_{\text{ret}}, \mathbf{x}') \quad (6.67)$$

Again we use this expression in equation (6.33) on page 90 to calculate the fields via equations (6.34) on page 90. Tedious, but fairly straightforward algebra (which we will not present here), yields the resulting fields. The components of the fields that dominate in the far field zone (wave zone) are given by

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = \frac{i\mu_0\omega}{8\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} (\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k} \quad (6.68a)$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{i}{8\pi\epsilon_0} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x}-\mathbf{x}_0|} [(\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k}] \times \mathbf{k} \quad (6.68b)$$

This type of radiation is called *electric quadrupole radiation* or *E2 radiation*.

6.4 Radiation from an extended source volume at rest

Certain radiating systems have a symmetric geometry or are in any other way simple enough that a direct (semi-)analytic calculation of the radiated fields and energy is possible. This is for instance the case when the radiating current flows in a finite, conducting medium of simple geometry at rest such as in a stationary *antenna*.

6.4.1 Radiation from a one-dimensional current distribution

Let us apply equation (6.5) on page 83 to calculate the radiated EM power from a one-dimensional, time-varying current. Such a current can be set up by feeding the EMF of a generator (*e.g.*, a transmitter) onto a stationary, linear, straight, thin, conducting wire across a very short gap at its centre. Due to the applied EMF, the charges in this thin wire of finite length L are set into linear motion to produce a time-varying *antenna current* which is the source of the EM radiation. Linear antennas of this type are called *dipole antennas*. For simplicity, we assume that the conductor resistance and the energy loss due to the electromagnetic radiation are negligible.

Choosing our coordinate system such that the x_3 axis is along the antenna axis, the antenna current density can be represented, in complex notation, by $\mathbf{j}(t', \mathbf{x}') = \delta(x'_1)\delta(x'_2)J(t', x'_3)\hat{\mathbf{x}}_3$ (measured in Am^{-2}) where $J(t', x'_3)$ is the current (measured in A) along the antenna wire. Since we can assume that the antenna wire is infinitely thin, the antenna current must vanish at the endpoints $-L/2$ and $L/2$. At the midpoint, where the antenna is fed across a very short gap in the conducting wire, the antenna current is, of course, equal to the supplied current.

For each Fourier frequency component ω_0 , the antenna current $J(t', x'_3)$ can be written as $I(x'_3)\exp(-i\omega_0 t')$ so that the antenna current density can be represented as $\mathbf{j}(t', \mathbf{x}') = \mathbf{j}_0(\mathbf{x}')\exp(-i\omega_0 t')$ [*cf.* equations (5.6) on page 72] where

$$\mathbf{j}_0(\mathbf{x}') = \delta(x'_1)\delta(x'_2)I(x'_3)\hat{\mathbf{x}}_3 \quad (6.69)$$

and where the spatially varying Fourier amplitude $I(x'_3)$ of the antenna current fulfils the *time-independent wave equation* (*Helmholtz equation*)

$$\frac{d^2 I}{dx_3'^2} + k^2 I(x'_3) = 0, \quad I(-L/2) = I(L/2) = 0, \quad I(0) = I_0 \quad (6.70)$$

This second-order ordinary differential equation with constant coefficients has the well-known solution

$$I(x'_3) = I_0 \frac{\sin[k(L/2 - |x'_3|)]}{\sin(kL/2)} \quad (6.71)$$

where I_0 is the amplitude of the antenna current (measured in A), assumed to be constant and supplied by the generator/transmitter at the *antenna feed point* (in our case the midpoint of the antenna wire) and $1/\sin(kL/2)$ is a normalisation factor. The antenna current forms a *standing wave* as indicated in figure 6.3 on the facing page.

When the antenna is short we can approximate the current distribution formula (6.71) above by the first term in its Taylor expansion, *i.e.*, by $I_0(1 -$

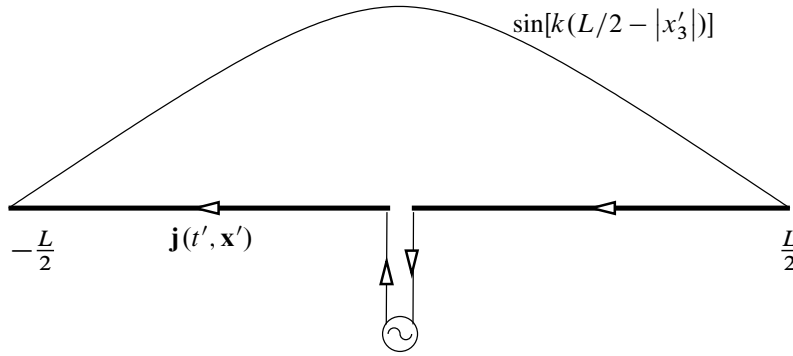


Figure 6.3: A linear antenna used for transmission. The current in the feeder and the antenna wire is set up by the EMF of the generator (the transmitter). At the ends of the wire, the current is reflected back with a 180° phase shift to produce a antenna current in the form of a standing wave.

$2|x'_3|/L$). For a half-wave antenna ($L = \lambda/2 \Leftrightarrow kL = \pi$) formula (6.71) on the facing page simplifies to $I_0 \cos(kx'_3)$. Hence, in the most general case of a straight, infinitely thin antenna of finite, arbitrary length L directed along the x'_3 axis, the Fourier amplitude of the antenna current density is

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x'_1) \delta(x'_2) \frac{\sin[k(L/2 - |x'_3|)]}{\sin(kL/2)} \hat{\mathbf{x}}_3 \quad (6.72)$$

For a half-wave dipole antenna ($L = \lambda/2$), the antenna current density is simply

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x'_1) \delta(x'_2) \cos(kx'_3) \quad (6.73)$$

while for a short antenna ($L \ll \lambda$) it can be approximated by

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x'_1) \delta(x'_2) (1 - 2|x'_3|/L) \quad (6.74)$$

In the case of a travelling wave antenna, in which one end of the antenna is connected to ground via a resistance so that the current at this end does not vanish, the Fourier amplitude of the antenna current density is

$$\mathbf{j}_0(\mathbf{x}') = I_0 \delta(x'_1) \delta(x'_2) \exp(kx'_3) \quad (6.75)$$

In order to evaluate formula (6.5) on page 83 with the explicit monochromatic current (6.72) inserted, we use a spherical polar coordinate system as in

Figure 6.4: We choose a spherical polar coordinate system ($\mathbf{r} = |\mathbf{x}|, \theta, \varphi$) and arrange it so that the linear electric dipole antenna axis (and thus the antenna current density \mathbf{j}_ω) is along the polar axis with the feed point at the origin.

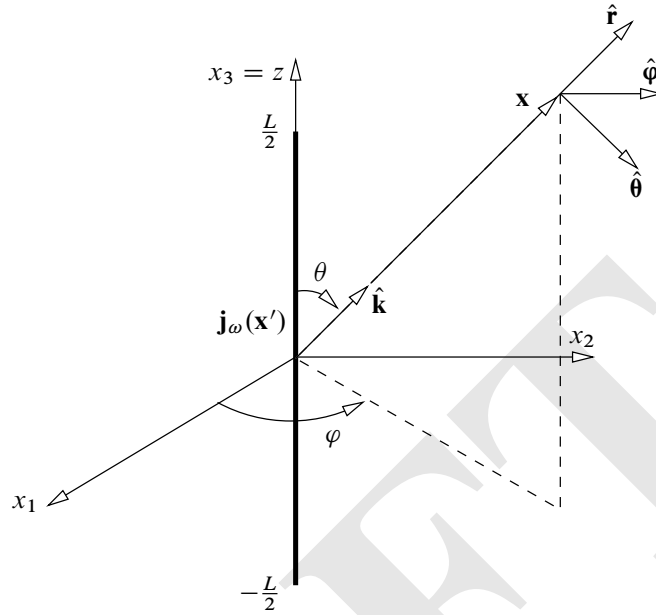


figure 6.4 to evaluate the source integral

$$\begin{aligned}
 \mathcal{I} &= \left| \int_{V'} d^3x' \mathbf{j}_0 \cdot \mathbf{x} \mathbf{k} e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}_0)} \right|^2 \\
 &= \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} dx'_3 I_0 \frac{\sin[k(L/2 - |x'_3|)]}{\sin(kL/2)} k \sin \theta e^{-ikx'_3 \cos \theta} e^{ikx_0 \cos \theta} \right|^2 \\
 &= I_0^2 \frac{k^2 \sin^2 \theta}{\sin^2(kL/2)} \left| e^{ikx_0 \cos \theta} \right|^2 \left| 2 \int_0^{\frac{L}{2}} dx'_3 \sin(kL/2 - kx'_3) \cos(kx'_3 \cos \theta) \right|^2 \\
 &= 4I_0^2 \left(\frac{\cos[(kL/2) \cos \theta] - \cos(kL/2)}{\sin \theta \sin(kL/2)} \right)^2
 \end{aligned} \tag{6.76}$$

Inserting this expression and $d\Omega = 2\pi \sin \theta d\theta$ into formula (6.5) on page 83 and integrating over θ , we find that the total radiated power from the antenna is

$$P(L) = R_0 I_0^2 \frac{1}{4\pi} \int_0^\pi d\theta \left(\frac{\cos[(kL/2) \cos \theta] - \cos(kL/2)}{\sin \theta \sin(kL/2)} \right)^2 \sin \theta \tag{6.77}$$

One can show that

$$\lim_{kL \rightarrow 0} P(L) = \frac{\pi}{12} \left(\frac{L}{\lambda} \right)^2 R_0 I_0^2 \tag{6.78}$$

where λ is the vacuum wavelength.

The quantity

$$R^{\text{rad}}(L) = \frac{P(L)}{I_{\text{eff}}^2} = \frac{P(L)}{\frac{1}{2}I_0^2} = R_0 \frac{\pi}{6} \left(\frac{L}{\lambda} \right)^2 \approx 197 \left(\frac{L}{\lambda} \right)^2 \Omega \quad (6.79)$$

is called the *radiation resistance*. For the technologically important case of a half-wave antenna, *i.e.*, for $L = \lambda/2$ or $kL = \pi$, formula (6.77) on the facing page reduces to

$$P(\lambda/2) = R_0 I_0^2 \frac{1}{4\pi} \int_0^\pi d\theta \frac{\cos^2(\frac{\pi}{2} \cos \theta)}{\sin \theta} \quad (6.80)$$

The integral in (6.80) can always be evaluated numerically. But, it can in fact also be evaluated analytically as follows:

$$\begin{aligned} \int_0^\pi \frac{\cos^2(\frac{\pi}{2} \cos \theta)}{\sin \theta} d\theta &= [\cos \theta \rightarrow u] = \int_{-1}^1 \frac{\cos^2(\frac{\pi}{2} u)}{1 - u^2} du = \\ &= \left[\cos^2\left(\frac{\pi}{2} u\right) = \frac{1 + \cos(\pi u)}{2} \right] \\ &= \frac{1}{2} \int_{-1}^1 \frac{1 + \cos(\pi u)}{(1 + u)(1 - u)} du \\ &= \frac{1}{4} \int_{-1}^1 \frac{1 + \cos(\pi u)}{(1 + u)} du + \frac{1}{4} \int_{-1}^1 \frac{1 + \cos(\pi u)}{(1 - u)} du \\ &= \frac{1}{2} \int_{-1}^1 \frac{1 + \cos(\pi u)}{(1 + u)} du = \left[1 + u \rightarrow \frac{v}{\pi} \right] \\ &= \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos v}{v} dv = \frac{1}{2} [\gamma + \ln 2\pi - \text{Ci}(2\pi)] \\ &\approx 1.22 \end{aligned} \quad (6.81)$$

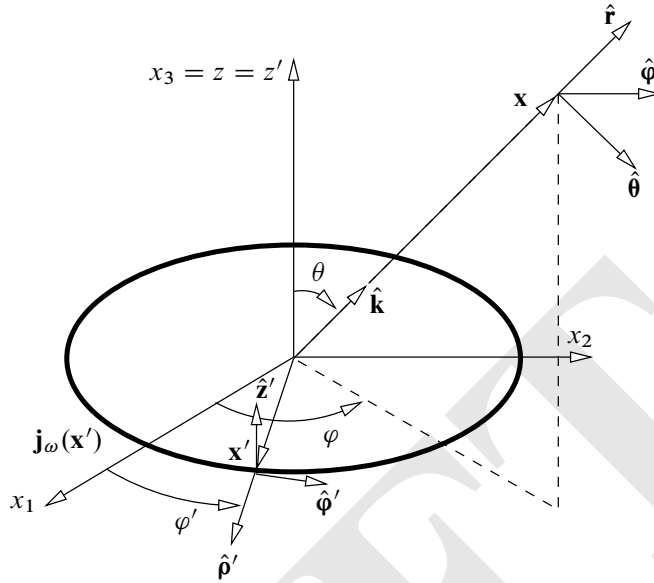
where in the last step the *Euler-Mascheroni constant* $\gamma = 0.5772 \dots$ and the *co-sine integral* $\text{Ci}(x)$ were introduced. Inserting this into the expression equation (6.80) above we obtain the value $R^{\text{rad}}(\lambda/2) \approx 73 \Omega$.

▷ Radiation from a two-dimensional current distribution ————— EXAMPLE 6.2

As an example of a two-dimensional current distribution we consider a circular *loop antenna* of radius a and calculate the far-zone \mathbf{E}^{far} and \mathbf{B}^{far} fields from such an antenna. We choose the Cartesian coordinate system $x_1 x_2 x_3$ with its origin at the centre of the loop as in figure 6.5 on the following page.

According to equation (5.28a) on page 79 the Fourier component of the radiation part of the

Figure 6.5: For the loop antenna the spherical coordinate system (r, θ, φ) describes the field point \mathbf{x} (the radiation field) and the cylindrical coordinate system (ρ', φ', z') describes the source point \mathbf{x}' (the antenna current).



magnetic field generated by an extended, monochromatic current source is

$$\mathbf{B}_{\omega}^{\text{far}} = \frac{-i\mu_0 e^{ik|\mathbf{x}|}}{4\pi |\mathbf{x}|} \int_{V'} d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{j}_{\omega} \times \mathbf{k} \quad (6.82)$$

In our case the generator produces a single frequency ω and we feed the antenna across a small gap where the loop crosses the positive x_1 axis. The circumference of the loop is chosen to be exactly one wavelength $\lambda = 2\pi c/\omega$. This means that the antenna current oscillates in the form of a sinusoidal standing current wave around the circular loop with a Fourier amplitude

$$\mathbf{j}_{\omega} = I_0 \cos \varphi' \delta(\rho' - a) \delta(z') \hat{\varphi}' \quad (6.83)$$

For the spherical coordinate system of the field point, we recall from subsection F.4.1 on page 198 that the following relations between the base vectors hold:

$$\hat{\mathbf{r}} = \sin \theta \cos \varphi \hat{\mathbf{x}}_1 + \sin \theta \sin \varphi \hat{\mathbf{x}}_2 + \cos \theta \hat{\mathbf{x}}_3$$

$$\hat{\boldsymbol{\theta}} = \cos \theta \cos \varphi \hat{\mathbf{x}}_1 + \cos \theta \sin \varphi \hat{\mathbf{x}}_2 - \sin \theta \hat{\mathbf{x}}_3$$

$$\hat{\boldsymbol{\varphi}} = -\sin \varphi \hat{\mathbf{x}}_1 + \cos \varphi \hat{\mathbf{x}}_2$$

and

$$\hat{\mathbf{x}}_1 = \sin \theta \cos \varphi \hat{\mathbf{r}} + \cos \theta \cos \varphi \hat{\boldsymbol{\theta}} - \sin \varphi \hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{x}}_2 = \sin \theta \sin \varphi \hat{\mathbf{r}} + \cos \theta \sin \varphi \hat{\boldsymbol{\theta}} + \cos \varphi \hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{x}}_3 = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}$$

With the use of the above transformations and trigonometric identities, we obtain for the

6.4. Radiation from an extended source volume at rest

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cylindrical coordinate system which describes the source:

$$\hat{\rho}' = \cos \varphi' \hat{\mathbf{x}}_1 + \sin \varphi' \hat{\mathbf{x}}_2 \quad (6.84)$$

$$= \sin \theta \cos(\varphi' - \varphi) \hat{\mathbf{r}} + \cos \theta \cos(\varphi' - \varphi) \hat{\boldsymbol{\theta}} + \sin(\varphi' - \varphi) \hat{\boldsymbol{\phi}}$$

$$\hat{\phi}' = -\sin \varphi' \hat{\mathbf{x}}_1 + \cos \varphi' \hat{\mathbf{x}}_2 \quad (6.85)$$

$$= -\sin \theta \sin(\varphi' - \varphi) \hat{\mathbf{r}} - \cos \theta \sin(\varphi' - \varphi) \hat{\boldsymbol{\theta}} + \cos(\varphi' - \varphi) \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{z}}' = \hat{\mathbf{x}}_3 = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \quad (6.86)$$

This choice of coordinate systems means that $\mathbf{k} = k \hat{\mathbf{r}}$ and $\mathbf{x}' = a \hat{\rho}'$ so that

$$\mathbf{k} \cdot \mathbf{x}' = ka \sin \theta \cos(\varphi' - \varphi) \quad (6.87)$$

and

$$\hat{\rho}' \times \mathbf{k} = k[\cos(\varphi' - \varphi) \hat{\boldsymbol{\theta}} + \sin \theta \sin(\varphi' - \varphi) \hat{\boldsymbol{\phi}}] \quad (6.88)$$

With these expressions inserted, recalling that in cylindrical coordinates $d^3x' = \rho' d\rho' d\varphi' dz'$, the source integral becomes

$$\begin{aligned} \int_{V'} d^3x' e^{-i\mathbf{k} \cdot \mathbf{x}'} \mathbf{j}_\omega \times \mathbf{k} &= a \int_0^{2\pi} d\varphi' e^{-ika \sin \theta \cos(\varphi' - \varphi)} I_0 \cos \varphi' \hat{\rho}' \times \mathbf{k} \\ &= I_0 a k \int_0^{2\pi} e^{-ika \sin \theta \cos(\varphi' - \varphi)} \cos(\varphi' - \varphi) \cos \varphi' d\varphi' \hat{\boldsymbol{\theta}} \\ &\quad + I_0 a k \cos \theta \int_0^{2\pi} e^{-ika \sin \theta \cos(\varphi' - \varphi)} \sin(\varphi' - \varphi) \cos \varphi' d\varphi' \hat{\boldsymbol{\phi}} \end{aligned} \quad (6.89)$$

Utilising the periodicity of the integrands over the integration interval $[0, 2\pi]$, introducing the auxiliary integration variable $\varphi'' = \varphi' - \varphi$, and utilising standard trigonometric identities, the first integral in the RHS of (6.89) can be rewritten

$$\begin{aligned} &\int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \cos \varphi'' \cos(\varphi'' + \varphi) d\varphi'' \\ &= \cos \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \cos^2 \varphi'' d\varphi'' + \text{a vanishing integral} \\ &= \cos \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \left(\frac{1}{2} + \frac{1}{2} \cos 2\varphi'' \right) d\varphi'' \\ &= \frac{1}{2} \cos \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} d\varphi'' \\ &\quad + \frac{1}{2} \cos \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \cos(2\varphi'') d\varphi'' \end{aligned} \quad (6.90)$$

Analogously, the second integral in the RHS of (6.89) can be rewritten

$$\begin{aligned} &\int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \sin \varphi'' \cos(\varphi'' + \varphi) d\varphi'' \\ &= \frac{1}{2} \sin \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} d\varphi'' \\ &\quad - \frac{1}{2} \sin \varphi \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \cos 2\varphi'' d\varphi'' \end{aligned} \quad (6.91)$$

As is well-known from the theory of *Bessel functions*,

$$\begin{aligned} J_n(-\xi) &= (-1)^n J_n(\xi) \\ J_n(-\xi) &= \frac{i^{-n}}{\pi} \int_0^\pi e^{-i\xi \cos \varphi} \cos n\varphi \, d\varphi = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{-i\xi \cos \varphi} \cos n\varphi \, d\varphi \end{aligned} \quad (6.92)$$

which means that

$$\begin{aligned} \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \, d\varphi'' &= 2\pi J_0(ka \sin \theta) \\ \int_0^{2\pi} e^{-ika \sin \theta \cos \varphi''} \cos 2\varphi'' \, d\varphi'' &= -2\pi J_2(ka \sin \theta) \end{aligned} \quad (6.93)$$

Putting everything together, we find that

$$\begin{aligned} \int_{V'} d^3x' e^{-i\mathbf{k} \cdot \mathbf{x}'} \mathbf{j}_\omega \times \mathbf{k} &= \mathcal{I}_\theta \hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi \hat{\boldsymbol{\phi}} \\ &= I_0 a k \pi \cos \varphi [J_0(ka \sin \theta) - J_2(ka \sin \theta)] \hat{\boldsymbol{\theta}} \\ &\quad + I_0 a k \pi \cos \theta \sin \varphi [J_0(ka \sin \theta) + J_2(ka \sin \theta)] \hat{\boldsymbol{\phi}} \end{aligned} \quad (6.94)$$

so that, in spherical coordinates where $|\mathbf{x}| = r$,

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = \frac{-i\mu_0 e^{ikr}}{4\pi r} (\mathcal{I}_\theta \hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi \hat{\boldsymbol{\phi}}) \quad (6.95)$$

To obtain the desired physical magnetic field in the radiation (far) zone we must Fourier transform back to t space and take the real part:

$$\begin{aligned} \mathbf{B}^{\text{far}}(t, \mathbf{x}) &= \text{Re} \left\{ \frac{-i\mu_0 e^{(ikr - \omega t')}}{4\pi r} (\mathcal{I}_\theta \hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi \hat{\boldsymbol{\phi}}) \right\} \\ &= \frac{\mu_0}{4\pi r} \sin(kr - \omega t') (\mathcal{I}_\theta \hat{\boldsymbol{\theta}} + \mathcal{I}_\varphi \hat{\boldsymbol{\phi}}) \\ &= \frac{I_0 a k \mu_0}{4r} \sin(kr - \omega t') \left(\cos \varphi [J_0(ka \sin \theta) - J_2(ka \sin \theta)] \hat{\boldsymbol{\theta}} \right. \\ &\quad \left. + \cos \theta \sin \varphi [J_0(ka \sin \theta) + J_2(ka \sin \theta)] \hat{\boldsymbol{\phi}} \right) \end{aligned} \quad (6.96)$$

From this expression for the radiated \mathbf{B} field, we can obtain the radiated \mathbf{E} field with the help of Maxwell's equations.

—End of example 6.2◁

6.5 Radiation from a localised charge in arbitrary motion

The derivation of the radiation fields for the case of the source moving relative to the observer is considerably more complicated than the stationary cases that we have studied so far. In order to handle this non-stationary situation, we start from the retarded potentials (3.47) on page 40 in chapter 3

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_{V'} d^3x' \frac{\rho(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}}))}{|\mathbf{x}(t) - \mathbf{x}'(t'_{\text{ret}})|} \quad (6.97a)$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int_{V'} d^3x' \frac{\mathbf{j}(t'_{\text{ret}}, \mathbf{x}'(t'_{\text{ret}}))}{|\mathbf{x}(t) - \mathbf{x}'(t'_{\text{ret}})|} \quad (6.97b)$$

and consider a source region with such a limited spatial extent that the charges and currents are well localised. Specifically, we consider a charge q' , for instance an electron, which, classically, can be thought of as a localised, unstructured and rigid ‘charge distribution’ with a small, finite radius. The part of this ‘charge distribution’ dq' which we are considering is located in $dV' = d^3x'$ in the sphere in figure 6.6 on the following page. Since we assume that the electron (or any other similar electric charge) moves with a velocity \mathbf{v}' whose direction is arbitrary and whose magnitude can even be comparable to the speed of light, we cannot say that the charge and current to be used in (6.97) is $\int_{V'} d^3x' \rho(t'_{\text{ret}}, \mathbf{x}')$ and $\int_{V'} d^3x' \mathbf{v} \rho(t'_{\text{ret}}, \mathbf{x}')$, respectively, because in the finite time interval during which the observed signal is generated, part of the charge distribution will ‘leak’ out of the volume element d^3x' .

6.5.1 The Liénard-Wiechert potentials

The charge distribution in figure 6.6 on page 106 which contributes to the field at $\mathbf{x}(t)$ is located at $\mathbf{x}'(t')$ on a sphere with radius $r = |\mathbf{x} - \mathbf{x}'| = c(t - t')$. The radius interval of this sphere from which radiation is received at the field point \mathbf{x} during the time interval $(t', t' + dt')$ is $(r', r' + dr')$ and the net amount of charge in this radial interval is

$$dq' = \rho(t'_{\text{ret}}, \mathbf{x}') dS' dr' - \rho(t'_{\text{ret}}, \mathbf{x}') \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'|} dS' dt' \quad (6.98)$$

where the last term represents the amount of ‘source leakage’ due to the fact that the charge distribution moves with velocity $\mathbf{v}'(t') = d\mathbf{x}'/dt'$. Since $dt' = dr'/c$

$$\begin{aligned} d q' &= \rho(t'_{\text{ret}}, \mathbf{x}') d^3 x' - \rho(t'_{\text{ret}}, \mathbf{x}') \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c |\mathbf{x} - \mathbf{x}'|} d^3 x' \\ &= \rho(t'_{\text{ret}}, \mathbf{x}') \left(1 - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c |\mathbf{x} - \mathbf{x}'|} \right) d^3 x' \end{aligned} \quad (6.99)$$
$$\rho(t'_{\text{ret}}, \mathbf{x}') d^3x' = \frac{dq'}{1 - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c|\mathbf{x} - \mathbf{x}'|}} \quad (6.100)$$
$$\frac{\rho(t'_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \frac{dq'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c}} \quad (6.101)$$
$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c}} \quad (6.102a)$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{v}' dq'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c}} \quad (6.102b)$$

For a sufficiently small and well localised charge distribution we can, assuming that the integrands do not change sign in the integration volume, use the mean value theorem to evaluate these expressions to become

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c}} \int dq' = \frac{q'}{4\pi\epsilon_0} \frac{1}{s} \quad (6.103a)$$

$$\begin{aligned} \mathbf{A}(t, \mathbf{x}) &= \frac{1}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}'}{|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c}} \int dq' = \frac{q'}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}'}{s} \\ &= \frac{\mathbf{v}'}{c^2} \phi \end{aligned} \quad (6.103b)$$

where

$$s = s(t', \mathbf{x}) = |\mathbf{x} - \mathbf{x}'(t')| - \frac{[\mathbf{x} - \mathbf{x}'(t')] \cdot \mathbf{v}'(t')}{c} \quad (6.104a)$$

$$= |\mathbf{x} - \mathbf{x}'(t')| \left(1 - \frac{(\mathbf{x} - \mathbf{x}'(t')) \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'(t')| c} \right) \quad (6.104b)$$

$$= [\mathbf{x} - \mathbf{x}'(t')] \cdot \left(\frac{\mathbf{x} - \mathbf{x}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|} - \frac{\mathbf{v}'(t')}{c} \right) \quad (6.104c)$$

is the *retarded relative distance*. The potentials (6.103) are the *Liénard-Wiechert potentials*.² In section 7.3.2 on page 145 we shall derive them in a more elegant and general way by using a relativistically covariant formalism.

It should be noted that in the complicated derivation presented above, the observer is in a coordinate system which has an ‘absolute’ meaning and the velocity \mathbf{v}' is that of the localised charge q' , whereas, as we shall see later in the covariant derivation, two reference frames of equal standing are moving relative to each other with \mathbf{v}' .

The Liénard-Wiechert potentials are applicable to all problems where a spatially localised charge in arbitrary motion emits electromagnetic radiation, and we shall now study such emission problems. The electric and magnetic fields are calculated from the potentials in the usual way:

$$\mathbf{B}(t, \mathbf{x}) = \nabla \times \mathbf{A}(t, \mathbf{x}) \quad (6.105a)$$

$$\mathbf{E}(t, \mathbf{x}) = -\nabla\phi(t, \mathbf{x}) - \frac{\partial \mathbf{A}(t, \mathbf{x})}{\partial t} \quad (6.105b)$$

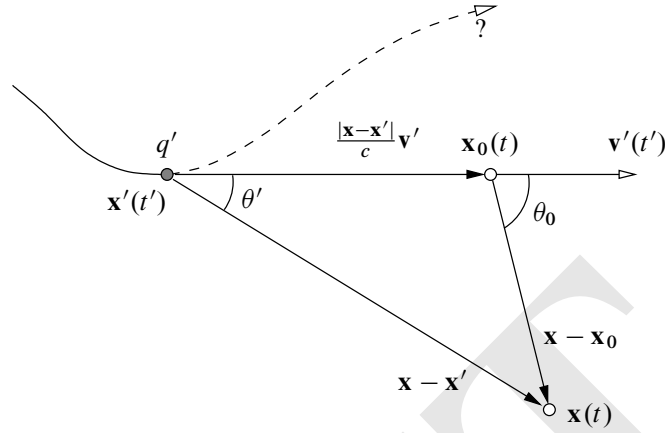
6.5.2 Radiation from an accelerated point charge

Consider a localised charge q' and assume that its trajectory is known experimentally as a function of *retarded time*

$$\mathbf{x}' = \mathbf{x}'(t') \quad (6.106)$$

² These results were derived independently by ALFRED-MARIE LIÉNARD (1869–1958) in 1898 and EMIL JOHANN WIECHERT (1861–1928) in 1900. When $\mathbf{v}' \parallel (\mathbf{x} - \mathbf{x}')$ and $v \rightarrow c$, the potentials become singular. This was first pointed out by ARNOLD JOHANNES WILHELM SOMMERFELD (1868–1951) in 1904.

Figure 6.7: Signals that are observed at time t at the field point \mathbf{x} were generated at time t' at the source point $\mathbf{x}'(t')$. After this time t' the particle, which moves with nonuniform velocity, has followed a so far (*i.e.*, at t) unknown trajectory. Extrapolating tangentially the trajectory from $\mathbf{x}'(t')$, based on the velocity $\mathbf{v}'(t')$, defines the virtual simultaneous coordinate $\mathbf{x}_0(t)$.



(in the interest of simplifying our notation, we drop the subscript 'ret' on t' from now on). This means that we know the trajectory of the charge q' , *i.e.*, \mathbf{x}' , for all times up to the time t' at which a signal was emitted in order to precisely arrive at the field point \mathbf{x} at time t . Because of the finite speed of propagation of the fields, the trajectory at times later than t' cannot be known at time t .

The retarded velocity and acceleration at time t' are given by

$$\mathbf{v}'(t') = \frac{d\mathbf{x}'}{dt'} \quad (6.107a)$$

$$\mathbf{a}'(t') = \dot{\mathbf{v}}'(t') = \frac{d\mathbf{v}'}{dt'} = \frac{d^2\mathbf{x}'}{dt'^2} \quad (6.107b)$$

As for the charge coordinate \mathbf{x}' itself, we have in general no knowledge of the velocity and acceleration at times later than t' , and definitely not at the time of observation t ! If we choose the field point \mathbf{x} as fixed, the application of (6.107) to the relative vector $\mathbf{x} - \mathbf{x}'$ yields

$$\frac{d}{dt'}[\mathbf{x} - \mathbf{x}'(t')] = -\mathbf{v}'(t') \quad (6.108a)$$

$$\frac{d^2}{dt'^2}[\mathbf{x} - \mathbf{x}'(t')] = -\dot{\mathbf{v}}'(t') \quad (6.108b)$$

The retarded time t' can, at least in principle, be calculated from the implicit relation

$$t' = t'(t, \mathbf{x}) = t - \frac{|\mathbf{x} - \mathbf{x}'(t')|}{c} \quad (6.109)$$

and we shall see later how this relation can be taken into account in the calculations.

According to formulæ (6.105) on page 107, the electric and magnetic fields are determined via differentiation of the retarded potentials at the observation time t and at the observation point \mathbf{x} . In these formulæ the unprimed ∇ , *i.e.*, the spatial derivative differentiation operator $\nabla = \hat{\mathbf{x}}_i \partial / \partial x_i$ means that we differentiate with respect to the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ while keeping t fixed, and the unprimed time derivative operator $\partial / \partial t$ means that we differentiate with respect to t while keeping \mathbf{x} fixed. But the Liénard-Wiechert potentials ϕ and \mathbf{A} , equations (6.103) on page 107, are expressed in the charge velocity $\mathbf{v}'(t')$ given by equation (6.107a) on page 108 and the retarded relative distance $s(t', \mathbf{x})$ given by equation (6.104) on page 107. This means that the expressions for the potentials ϕ and \mathbf{A} contain terms that are expressed explicitly in t' , which in turn is expressed implicitly in t via equation (6.109) on the facing page. Despite this complication it is possible, as we shall see below, to determine the electric and magnetic fields and associated quantities at the time of observation t . To this end, we need to investigate carefully the action of differentiation on the potentials.

6.5.2.1 The differential operator method

We introduce the convention that a differential operator embraced by parentheses with an index \mathbf{x} or t means that the operator in question is applied at constant \mathbf{x} and t , respectively. With this convention, we find that

$$\begin{aligned} \left(\frac{\partial}{\partial t'} \right)_{\mathbf{x}} |\mathbf{x} - \mathbf{x}'(t')|^p &= p \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^{2-p}} \cdot \left(\frac{\partial}{\partial t'} \right)_{\mathbf{x}} (\mathbf{x} - \mathbf{x}'(t')) \\ &= -p \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'|^{2-p}} \end{aligned} \quad (6.110)$$

Furthermore, by applying the operator $(\partial / \partial t)_{\mathbf{x}}$ to equation (6.109) on the preceding page we find that

$$\begin{aligned} \left(\frac{\partial t'}{\partial t} \right)_{\mathbf{x}} &= 1 - \left(\frac{\partial}{\partial t} \right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'(t'(t, \mathbf{x}))|}{c} \\ &= 1 - \left[\left(\frac{\partial}{\partial t'} \right)_{\mathbf{x}} \frac{|\mathbf{x} - \mathbf{x}'|}{c} \right] \left(\frac{\partial t'}{\partial t} \right)_{\mathbf{x}} \\ &= 1 + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{c |\mathbf{x} - \mathbf{x}'|} \left(\frac{\partial t'}{\partial t} \right)_{\mathbf{x}} \end{aligned} \quad (6.111)$$

This is an algebraic equation in $(\partial t' / \partial t)_{\mathbf{x}}$ that we can solve to obtain

$$\left(\frac{\partial t'}{\partial t} \right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'| - (\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')/c} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \quad (6.112)$$

where $s = s(t', \mathbf{x})$ is the retarded relative distance given by equation (6.104) on page 107. Making use of equation (6.112) above, we obtain the following useful

operator identity

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} = \left(\frac{\partial t'}{\partial t}\right)_{\mathbf{x}} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} \quad (6.113)$$

Likewise, by applying $(\nabla)_t$ to equation (6.109) on page 108 we obtain

$$\begin{aligned} (\nabla t')_t &= -\left(\nabla \frac{|\mathbf{x} - \mathbf{x}'(t'(t, \mathbf{x}))|}{c}\right)_t = -\frac{\mathbf{x} - \mathbf{x}'}{c |\mathbf{x} - \mathbf{x}'|} \cdot (\nabla(\mathbf{x} - \mathbf{x}'))_t \\ &= -\frac{\mathbf{x} - \mathbf{x}'}{c |\mathbf{x} - \mathbf{x}'|} + \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{c |\mathbf{x} - \mathbf{x}'|} (\nabla t')_t \end{aligned} \quad (6.114)$$

This is an algebraic equation in $(\nabla t')_t$ with the solution

$$(\nabla t')_t = -\frac{\mathbf{x} - \mathbf{x}'}{cs} \quad (6.115)$$

which gives the following operator relation when $(\nabla)_t$ is acting on an arbitrary function of t' and \mathbf{x} :

$$(\nabla)_t = (\nabla t')_t \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\nabla)_{t'} = -\frac{\mathbf{x} - \mathbf{x}'}{cs} \left(\frac{\partial}{\partial t'}\right)_{\mathbf{x}} + (\nabla)_{t'} \quad (6.116)$$

With the help of the rules (6.116) and (6.113) we are now able to replace t by t' in the operations that we need to perform. We find, for instance, that

$$\begin{aligned} \nabla \phi &\equiv (\nabla \phi)_t = \nabla \left(\frac{1}{4\pi\epsilon_0} \frac{q'}{s} \right)_t \\ &= -\frac{q'}{4\pi\epsilon_0 s^2} \left[\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} - \frac{\mathbf{v}'(t')}{c} - \frac{\mathbf{x} - \mathbf{x}'}{cs} \left(\frac{\partial s}{\partial t'} \right)_{\mathbf{x}} \right] \end{aligned} \quad (6.117a)$$

and, from equation (6.103b) on page 107 and equation (6.113),

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial t} &\equiv \left(\frac{\partial \mathbf{A}}{\partial t} \right)_{\mathbf{x}} = \left[\frac{\partial}{\partial t} \left(\frac{\mu_0}{4\pi} \frac{q' \mathbf{v}'(t')}{s} \right) \right]_{\mathbf{x}} \\ &= \frac{q'}{4\pi\epsilon_0 c^2 s^3} \left[|\mathbf{x} - \mathbf{x}'| s \dot{\mathbf{v}}'(t') - |\mathbf{x} - \mathbf{x}'| \mathbf{v}'(t') \left(\frac{\partial s}{\partial t'} \right)_{\mathbf{x}} \right] \end{aligned} \quad (6.117b)$$

Utilising these relations in the calculation of the \mathbf{E} field from the Liénard-Wiechert potentials, equations (6.103) on page 107, we obtain

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= -\nabla \phi(t, \mathbf{x}) - \frac{\partial}{\partial t} \mathbf{A}(t, \mathbf{x}) \\ &= \frac{q'}{4\pi\epsilon_0 s^2(t', \mathbf{x})} \left[\frac{[\mathbf{x} - \mathbf{x}'(t')] - |\mathbf{x} - \mathbf{x}'(t')| \mathbf{v}'(t')/c}{|\mathbf{x} - \mathbf{x}'(t')|} \right. \\ &\quad - \frac{[\mathbf{x} - \mathbf{x}'(t')] - |\mathbf{x} - \mathbf{x}'(t')| \mathbf{v}'(t')/c}{cs(t', \mathbf{x})} \left(\frac{\partial s(t', \mathbf{x})}{\partial t'} \right)_{\mathbf{x}} \\ &\quad \left. - \frac{|\mathbf{x} - \mathbf{x}'(t')| \dot{\mathbf{v}}'(t')}{c^2} \right] \end{aligned} \quad (6.118)$$

Starting from expression (6.104a) on page 107 for the retarded relative distance $s(t', \mathbf{x})$, we see that we can evaluate $(\partial s / \partial t')_{\mathbf{x}}$ in the following way

$$\begin{aligned} \left(\frac{\partial s}{\partial t'} \right)_{\mathbf{x}} &= \left[\frac{\partial}{\partial t'} \left(|\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'(t')}{c} \right) \right]_{\mathbf{x}} \\ &= \left(\frac{\partial}{\partial t'} |\mathbf{x} - \mathbf{x}'(t')| \right)_{\mathbf{x}} \\ &\quad - \frac{1}{c} \left[\left(\frac{\partial [\mathbf{x} - \mathbf{x}'(t')]}{\partial t'} \right)_{\mathbf{x}} \cdot \mathbf{v}'(t') + [\mathbf{x} - \mathbf{x}'(t')] \cdot \left(\frac{\partial \mathbf{v}'(t')}{\partial t'} \right)_{\mathbf{x}} \right] \\ &= - \frac{(\mathbf{x} - \mathbf{x}'(t')) \cdot \mathbf{v}'(t')}{|\mathbf{x} - \mathbf{x}'(t')|} + \frac{v'^2(t')}{c} - \frac{(\mathbf{x} - \mathbf{x}'(t')) \cdot \mathbf{a}'(t')}{c} \end{aligned} \quad (6.119)$$

where equation (6.110) on page 109 and equations (6.107) on page 108, respectively, were used. Hence, the electric field generated by an arbitrarily moving localised charge at $\mathbf{x}'(t')$ is given by the expression

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= \underbrace{\frac{q'}{4\pi\epsilon_0 s^3(t', \mathbf{x})} \left([\mathbf{x} - \mathbf{x}'(t')] - \frac{|\mathbf{x} - \mathbf{x}'(t')| \mathbf{v}'(t')}{c} \right) \left(1 - \frac{v'^2(t')}{c^2} \right)}_{\text{Velocity field (tends to the Coulomb field when } v \rightarrow 0)} \\ &\quad + \underbrace{\frac{q'}{4\pi\epsilon_0 s^3(t', \mathbf{x})} \left\{ \frac{\mathbf{x} - \mathbf{x}'(t')}{c^2} \times \left[\left([\mathbf{x} - \mathbf{x}'(t')] - \frac{|\mathbf{x} - \mathbf{x}'(t')| \mathbf{v}'(t')}{c} \right) \times \mathbf{a}'(t') \right] \right\}}_{\text{Acceleration (radiation) field}} \end{aligned} \quad (6.120)$$

The first part of the field, the *velocity field*, tends to the ordinary Coulomb field when $v' \rightarrow 0$ and does not contribute to the radiation. The second part of the field, the *acceleration field*, is radiated into the far zone and is therefore also called the *radiation field*.

From figure 6.7 on page 108 we see that the position the charged particle would have had if at t' all external forces would have been switched off so that the trajectory from then on would have been a straight line in the direction of the tangent at $\mathbf{x}'(t')$ is $\mathbf{x}_0(t)$, the *virtual simultaneous coordinate*. During the arbitrary motion, we interpret $\mathbf{x} - \mathbf{x}_0(t)$ as the coordinate of the field point \mathbf{x} relative to the virtual simultaneous coordinate $\mathbf{x}_0(t)$. Since the time it takes for a signal to propagate (in the assumed free space) from $\mathbf{x}'(t')$ to \mathbf{x} is $|\mathbf{x} - \mathbf{x}'|/c$, this relative vector is given by

$$\mathbf{x} - \mathbf{x}_0(t) = \mathbf{x} - \mathbf{x}'(t') - \frac{|\mathbf{x} - \mathbf{x}'(t')| \mathbf{v}'(t')}{c} \quad (6.121)$$

This allows us to rewrite equation (6.120) on the preceding page in the following way

$$\mathbf{E}(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0 s^3} \left[(\mathbf{x} - \mathbf{x}_0(t)) \left(1 - \frac{v'^2(t')}{c^2} \right) + (\mathbf{x} - \mathbf{x}'(t')) \times \frac{(\mathbf{x} - \mathbf{x}_0(t)) \times \mathbf{a}'(t')}{c^2} \right] \quad (6.122)$$

The magnetic field can be computed in a similar manner:

$$\begin{aligned} \mathbf{B}(t, \mathbf{x}) &= \nabla \times \mathbf{A}(t, \mathbf{x}) \equiv (\nabla \times \mathbf{A})_t = (\nabla \times \mathbf{A})_{t'} - \frac{\mathbf{x} - \mathbf{x}'}{cs} \times \left(\frac{\partial \mathbf{A}}{\partial t'} \right)_x \\ &= -\frac{q'}{4\pi\epsilon_0 c^2 s^2} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \times \mathbf{v}' - \frac{\mathbf{x} - \mathbf{x}'}{c |\mathbf{x} - \mathbf{x}'|} \times \left(\frac{\partial \mathbf{A}}{\partial t} \right)_x \end{aligned} \quad (6.123)$$

where we made use of equation (6.103) on page 107 and formula (6.113) on page 110. But, according to (6.117a),

$$\frac{\mathbf{x} - \mathbf{x}'}{c |\mathbf{x} - \mathbf{x}'|} \times (\nabla \phi)_t = \frac{q'}{4\pi\epsilon_0 c^2 s^2} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \times \mathbf{v}' \quad (6.124)$$

so that

$$\begin{aligned} \mathbf{B}(t, \mathbf{x}) &= \frac{\mathbf{x} - \mathbf{x}'}{c |\mathbf{x} - \mathbf{x}'|} \times \left[-(\nabla \phi)_t - \left(\frac{\partial \mathbf{A}}{\partial t} \right)_x \right] \\ &= \frac{\mathbf{x} - \mathbf{x}'(t')}{c |\mathbf{x} - \mathbf{x}'(t')|} \times \mathbf{E}(t, \mathbf{x}) \end{aligned} \quad (6.125)$$

The electric far field is obtained from the acceleration field in formula (6.120) on the previous page as

$$\begin{aligned} \mathbf{E}^{\text{far}}(t, \mathbf{x}) &= \frac{q'}{4\pi\epsilon_0 c^2 s^3} (\mathbf{x} - \mathbf{x}') \times \left[\left((\mathbf{x} - \mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'| \mathbf{v}'}{c} \right) \times \mathbf{a}' \right] \\ &= \frac{q'}{4\pi\epsilon_0 c^2 s^3} [\mathbf{x} - \mathbf{x}'(t')] \times \{ [\mathbf{x} - \mathbf{x}_0(t)] \times \mathbf{a}'(t') \} \end{aligned} \quad (6.126)$$

where in the last step we again used formula (6.121) on the preceding page. Combining this formula and formula (6.125) above, the radiation part of the magnetic field can be written

$$\mathbf{B}^{\text{far}}(t, \mathbf{x}) = \frac{\mathbf{x} - \mathbf{x}'(t')}{c |\mathbf{x} - \mathbf{x}'(t')|} \times \mathbf{E}^{\text{far}}(t, \mathbf{x}) \quad (6.127)$$

6.5.2.2 The direct method

An alternative to the differential operator transformation technique just described is to try to express all quantities in the potentials directly in t and \mathbf{x} . An example of such a quantity is the retarded relative distance $s(t', \mathbf{x})$. According to equation (6.104) on page 107, the square of this retarded relative distance can be written

$$s^2(t', \mathbf{x}) = |\mathbf{x} - \mathbf{x}'(t')|^2 - 2 |\mathbf{x} - \mathbf{x}'(t')| \frac{[\mathbf{x} - \mathbf{x}'(t')] \cdot \mathbf{v}'(t')}{c} + \left(\frac{[\mathbf{x} - \mathbf{x}'(t')] \cdot \mathbf{v}'(t')}{c} \right)^2 \quad (6.128)$$

If we use the following handy identity

$$\begin{aligned} & \left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right)^2 + \left(\frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{v}'}{c} \right)^2 \\ &= \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} \cos^2 \theta' + \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} \sin^2 \theta' \\ &= \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} (\cos^2 \theta' + \sin^2 \theta') = \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} \end{aligned} \quad (6.129)$$

we find that

$$\left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right)^2 = \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{v}'}{c} \right)^2 \quad (6.130)$$

Furthermore, from equation (6.121) on page 111, we obtain the identity

$$[\mathbf{x} - \mathbf{x}'(t')] \times \mathbf{v}' = [\mathbf{x} - \mathbf{x}_0(t)] \times \mathbf{v}' \quad (6.131)$$

which, when inserted into equation (6.130), yields the relation

$$\left(\frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right)^2 = \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}'}{c} \right)^2 \quad (6.132)$$

Inserting the above into expression (6.128) above for s^2 , this expression becomes

$$\begin{aligned} s^2 &= |\mathbf{x} - \mathbf{x}'|^2 - 2 |\mathbf{x} - \mathbf{x}'| \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} + \frac{|\mathbf{x} - \mathbf{x}'|^2 v'^2}{c^2} - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}'}{c} \right)^2 \\ &= \left(|\mathbf{x} - \mathbf{x}'| - \frac{|\mathbf{x} - \mathbf{x}'| v'}{c} \right)^2 - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}'}{c} \right)^2 \\ &= (\mathbf{x} - \mathbf{x}_0)^2 - \left(\frac{(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}'}{c} \right)^2 \\ &\equiv |\mathbf{x} - \mathbf{x}_0(t)|^2 - \left(\frac{[\mathbf{x} - \mathbf{x}_0(t)] \times \mathbf{v}'(t')}{c} \right)^2 \end{aligned} \quad (6.133)$$

where in the penultimate step we used equation (6.121) on page 111.

What we have just demonstrated is that if the particle velocity at time t can be calculated or projected from its value at the retarded time t' , the retarded distance s in the Liénard-Wiechert potentials (6.103) can be expressed in terms of the virtual simultaneous coordinate $\mathbf{x}_0(t)$, viz., the point at which the particle will have arrived at time t , i.e., when we obtain the first knowledge of its existence at the source point \mathbf{x}' at the retarded time t' , and in the field coordinate $\mathbf{x} = \mathbf{x}(t)$, where we make our observations. We have, in other words, shown that all quantities in the definition of s , and hence s itself, can, when the motion of the charge is somehow known, be expressed in terms of the time t alone. I.e., in this special case we are able to express the retarded relative distance as $s = s(t, \mathbf{x})$ and we do not have to involve the retarded time t' or any transformed differential operators in our calculations.

Taking the square root of both sides of equation (6.133) on the previous page, we obtain the following alternative final expressions for the retarded relative distance s in terms of the charge's virtual simultaneous coordinate $\mathbf{x}_0(t)$ and velocity $\mathbf{v}'(t')$:

$$s(t', \mathbf{x}) = \sqrt{|\mathbf{x} - \mathbf{x}_0(t)|^2 - \left(\frac{[\mathbf{x} - \mathbf{x}_0(t)] \times \mathbf{v}'(t')}{c} \right)^2} \quad (6.134a)$$

$$= |\mathbf{x} - \mathbf{x}_0(t)| \sqrt{1 - \frac{v'^2(t')}{c^2} \sin^2 \theta_0(t)} \quad (6.134b)$$

$$= \sqrt{|\mathbf{x} - \mathbf{x}_0(t)|^2 \left(1 - \frac{v'^2(t')}{c^2} \right) + \left(\frac{[\mathbf{x} - \mathbf{x}_0(t)] \cdot \mathbf{v}'(t')}{c} \right)^2} \quad (6.134c)$$

If we know what velocity the particle will have at time t , expression (6.134) for s will not be dependent on t' .

Using equation (6.134c) above and standard vector analytic formulae, we obtain

$$\begin{aligned} \nabla s^2 &= \nabla \left[|\mathbf{x} - \mathbf{x}_0|^2 \left(1 - \frac{v'^2}{c^2} \right) + \left(\frac{(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{v}'}{c} \right)^2 \right] \\ &= 2 \left[(\mathbf{x} - \mathbf{x}_0) \left(1 - \frac{v'^2}{c^2} \right) + \frac{\mathbf{v}' \mathbf{v}'}{c^2} \cdot (\mathbf{x} - \mathbf{x}_0) \right] \\ &= 2 \left[(\mathbf{x} - \mathbf{x}_0) + \frac{\mathbf{v}'}{c} \times \left(\frac{\mathbf{v}'}{c} \times (\mathbf{x} - \mathbf{x}_0) \right) \right] \end{aligned} \quad (6.135)$$

We shall use this result in example 6.3 on the facing page for a uniform, unaccelerated motion of the charge.

▷ The fields from a uniformly moving charge

EXAMPLE 6.3

In the special case of uniform motion,³ the localised charge moves in a field-free, isolated space and we know that it will not be affected by any external forces. It will therefore move uniformly in a straight line with the constant velocity \mathbf{v}' . This gives us the possibility to extrapolate its position at the observation time, $\mathbf{x}'(t)$, from its position at the retarded time, $\mathbf{x}'(t')$. Since the particle is not accelerated, $\dot{\mathbf{v}}' \equiv \mathbf{0}$, the virtual simultaneous coordinate \mathbf{x}_0 will be identical to the actual *simultaneous coordinate* of the particle at time t , i.e., $\mathbf{x}_0(t) = \mathbf{x}'(t)$. As depicted in figure 6.7 on page 108, the angle between $\mathbf{x} - \mathbf{x}_0$ and \mathbf{v}' is θ_0 while then angle between $\mathbf{x} - \mathbf{x}'$ and \mathbf{v}' is θ' .

³ This problem was first solved by OLIVER HEAVISIDE in 1888.

In the case of uniform velocity \mathbf{v}' , i.e., a velocity that does not change with time, any physical observable $f(t, \mathbf{x})$ has the same value at time t and position \mathbf{x} as it has at time $t + dt$ and position $\mathbf{x} + \mathbf{v}'dt$. Hence,

$$f(t, \mathbf{x}) = f(t + dt, \mathbf{x} + \mathbf{v}'dt) \quad (6.136)$$

Taylor expanding $f(t + dt, \mathbf{x} + \mathbf{v}'dt)$, keeping only linear terms in the infinitesimally small dt , we obtain

$$f(t + dt, \mathbf{x} + \mathbf{v}'dt) = f(t, \mathbf{x}) + \frac{\partial f}{\partial t} dt + \mathbf{v}' \cdot \nabla f dt + \mathcal{O}((dt)^2) \quad (6.137)$$

From this we conclude that for uniform motion

$$\frac{\partial f}{\partial t} = -\mathbf{v}' \cdot \nabla f \quad (6.138)$$

Since f is an arbitrary physical observable, the time and space derivatives must be related in the following way when they operate on any physical observable dependent on $\mathbf{x}(t)$ [cf. equation (1.36) on page 13]:

$$\frac{\partial}{\partial t} = -\mathbf{v}' \cdot \nabla \quad (6.139)$$

Hence, the \mathbf{E} and \mathbf{B} fields can be obtained from formulæ (6.105) on page 107, with the potentials given by equations (6.103) on page 107 as follows:

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{1}{c^2} \frac{\partial \mathbf{v}'\phi}{\partial t} = -\nabla\phi - \frac{\mathbf{v}'}{c^2} \frac{\partial \phi}{\partial t} \\ &= -\nabla\phi + \frac{\mathbf{v}'}{c} \left(\frac{\mathbf{v}'}{c} \cdot \nabla\phi \right) = -\left(1 - \frac{\mathbf{v}'\mathbf{v}'}{c^2} \right) \nabla\phi \\ &= \left(\frac{\mathbf{v}'\mathbf{v}'}{c^2} - \mathbf{1}_3 \right) \cdot \nabla\phi \end{aligned} \quad (6.140a)$$

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \nabla \times \left(\frac{\mathbf{v}'}{c^2} \phi \right) = \nabla\phi \times \frac{\mathbf{v}'}{c^2} = -\frac{\mathbf{v}'}{c^2} \times \nabla\phi \\ &= \frac{\mathbf{v}'}{c^2} \times \left[\left(\frac{\mathbf{v}'}{c} \cdot \nabla\phi \right) \frac{\mathbf{v}'}{c} - \nabla\phi \right] = \frac{\mathbf{v}'}{c^2} \times \left(\frac{\mathbf{v}'\mathbf{v}'}{c^2} - \mathbf{1}_3 \right) \cdot \nabla\phi \\ &= \frac{\mathbf{v}'}{c^2} \times \mathbf{E} \end{aligned} \quad (6.140b)$$

Here $\mathbf{1}_3 = \hat{\mathbf{x}}_i \hat{\mathbf{x}}_i$ is the unit dyad and we used the fact that $\mathbf{v}' \times \mathbf{v}' \equiv \mathbf{0}$. What remains is just to express $\nabla\phi$ in quantities evaluated at t and \mathbf{x} .

From equation (6.103a) on page 107 and equation (6.135) on the facing page we find

that

$$\begin{aligned}\nabla\phi &= \frac{q'}{4\pi\epsilon_0} \nabla\left(\frac{1}{s}\right) = -\frac{q'}{8\pi\epsilon_0 s^3} \nabla s^2 \\ &= -\frac{q'}{4\pi\epsilon_0 s^3} \left[(\mathbf{x} - \mathbf{x}_0) + \frac{\mathbf{v}'}{c} \times \left(\frac{\mathbf{v}'}{c} \times (\mathbf{x} - \mathbf{x}_0) \right) \right]\end{aligned}\quad (6.141)$$

When this expression for $\nabla\phi$ is inserted into equation (6.140a) on the previous page, the following result

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &= \left(\frac{\mathbf{v}'\mathbf{v}'}{c^2} - \mathbf{1}_3 \right) \cdot \nabla\phi = -\frac{q'}{8\pi\epsilon_0 s^3} \left(\frac{\mathbf{v}'\mathbf{v}'}{c^2} - \mathbf{1}_3 \right) \cdot \nabla s^2 \\ &= \frac{q'}{4\pi\epsilon_0 s^3} \left\{ (\mathbf{x} - \mathbf{x}_0) + \frac{\mathbf{v}'}{c} \times \left(\frac{\mathbf{v}'}{c} \times (\mathbf{x} - \mathbf{x}_0) \right) \right. \\ &\quad \left. - \frac{\mathbf{v}'}{c} \left(\frac{\mathbf{v}'}{c} \cdot (\mathbf{x} - \mathbf{x}_0) \right) - \frac{\mathbf{v}'\mathbf{v}'}{c^2} \cdot \left[\frac{\mathbf{v}'}{c} \times \left(\frac{\mathbf{v}'}{c} \times (\mathbf{x} - \mathbf{x}_0) \right) \right] \right\} \\ &= \frac{q'}{4\pi\epsilon_0 s^3} \left[(\mathbf{x} - \mathbf{x}_0) + \frac{\mathbf{v}'}{c} \left(\frac{\mathbf{v}'}{c} \cdot (\mathbf{x} - \mathbf{x}_0) \right) - (\mathbf{x} - \mathbf{x}_0) \frac{v'^2}{c^2} \right. \\ &\quad \left. - \frac{\mathbf{v}'}{c} \left(\frac{\mathbf{v}'}{c} \cdot (\mathbf{x} - \mathbf{x}_0) \right) \right] \\ &= \frac{q'}{4\pi\epsilon_0 s^3} (\mathbf{x} - \mathbf{x}_0) \left(1 - \frac{v'^2}{c^2} \right)\end{aligned}\quad (6.142)$$

obtains. Of course, the same result also follows from equation (6.122) on page 112 with $\dot{\mathbf{v}}' \equiv \mathbf{0}$ inserted.

From equation (6.142) above we conclude that \mathbf{E} is directed along the vector from the simultaneous coordinate $\mathbf{x}_0(t)$ to the field (observation) coordinate $\mathbf{x}(t)$. In a similar way, the magnetic field can be calculated and one finds that

$$\mathbf{B}(t, \mathbf{x}) = \frac{\mu_0 q'}{4\pi s^3} \left(1 - \frac{v'^2}{c^2} \right) \mathbf{v}' \times (\mathbf{x} - \mathbf{x}_0) = \frac{1}{c^2} \mathbf{v}' \times \mathbf{E} \quad (6.143)$$

From these explicit formulae for the \mathbf{E} and \mathbf{B} fields and formula (6.134b) on page 114 for s , we can discern the following cases:

1. $v' \rightarrow 0 \Rightarrow \mathbf{E}$ goes over into the Coulomb field $\mathbf{E}^{\text{Coulomb}}$
2. $v' \rightarrow 0 \Rightarrow \mathbf{B}$ goes over into the Biot-Savart field
3. $v' \rightarrow c \Rightarrow \mathbf{E}$ becomes dependent on θ_0
4. $v' \rightarrow c, \sin \theta_0 \approx 0 \Rightarrow \mathbf{E} \rightarrow (1 - v'^2/c^2) \mathbf{E}^{\text{Coulomb}}$
5. $v' \rightarrow c, \sin \theta_0 \approx 1 \Rightarrow \mathbf{E} \rightarrow (1 - v'^2/c^2)^{-1/2} \mathbf{E}^{\text{Coulomb}}$

—End of example 6.3<

6.5.2.3 Small velocities

If the charge moves at such low speeds that $v'/c \ll 1$, formula (6.104) on page 107 simplifies to

$$s = |\mathbf{x} - \mathbf{x}'| - \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \approx |\mathbf{x} - \mathbf{x}'|, \quad v' \ll c \quad (6.144)$$

and formula (6.121) on page 111

$$\mathbf{x} - \mathbf{x}_0 = (\mathbf{x} - \mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'| \mathbf{v}'}{c} \approx \mathbf{x} - \mathbf{x}', \quad v' \ll c \quad (6.145)$$

so that the radiation field equation (6.126) on page 112 can be approximated by

$$\mathbf{E}^{\text{far}}(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0 c^2 |\mathbf{x} - \mathbf{x}'|^3} (\mathbf{x} - \mathbf{x}') \times [(\mathbf{x} - \mathbf{x}') \times \dot{\mathbf{v}}'], \quad v' \ll c \quad (6.146)$$

from which we obtain, with the use of formula (6.125) on page 112, the magnetic field

$$\mathbf{B}^{\text{far}}(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0 c^3 |\mathbf{x} - \mathbf{x}'|^2} [\dot{\mathbf{v}}' \times (\mathbf{x} - \mathbf{x}')], \quad v' \ll c \quad (6.147)$$

It is interesting to note the close correspondence that exists between the non-relativistic fields (6.146) and (6.147) and the electric dipole field equations (6.45) on page 93 if we introduce the electric dipole moment for a localised charge [cf. formula (6.41) on page 92]

$$\mathbf{d} = q' \mathbf{x}'(t') \quad (6.148)$$

and at the same time make the transitions

$$q' \dot{\mathbf{v}}' = \ddot{\mathbf{d}} \rightarrow -\omega^2 \mathbf{d}_\omega \quad (6.149a)$$

$$\mathbf{x} - \mathbf{x}' = \mathbf{x} - \mathbf{x}_0 \quad (6.149b)$$

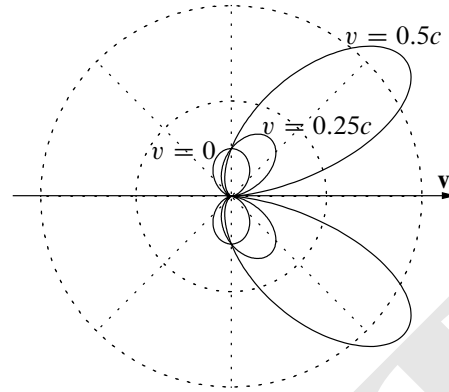
The energy flux in the far zone is described by the Poynting vector as a function of \mathbf{E}^{far} and \mathbf{B}^{far} . We use the close correspondence with the dipole case to find that it becomes

$$\mathbf{S}^{\text{far}} = \frac{\mu_0 q'^2 (\dot{\mathbf{v}}')^2}{16\pi^2 c |\mathbf{x} - \mathbf{x}'|^2} \sin^2 \theta \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \quad (6.150)$$

where θ is the angle between $\dot{\mathbf{v}}'$ and $\mathbf{x} - \mathbf{x}_0$. The total radiated power (integrated over a closed spherical surface) becomes

$$P = \frac{\mu_0 q'^2 (\dot{\mathbf{v}}')^2}{6\pi c} = \frac{q'^2 \dot{v}'^2}{6\pi \epsilon_0 c^3} \quad (6.151)$$

Figure 6.8: Polar diagram of the energy loss angular distribution factor $\sin^2 \theta / (1 - v \cos \theta/c)^5$ during bremsstrahlung for particle speeds $v' = 0$, $v' = 0.25c$, and $v' = 0.5c$.



which is the *Larmor formula for radiated power* from an accelerated charge. Note that here we are treating a charge with $v' \ll c$ but otherwise *totally unspecified motion* while we compare with formulæ derived for a *stationary oscillating dipole*. The electric and magnetic fields, equation (6.146) on the previous page and equation (6.147) on the preceding page, respectively, and the expressions for the Poynting flux and power derived from them, are here *instantaneous* values, dependent on the instantaneous position of the charge at $\mathbf{x}'(t')$. The angular distribution is that which is ‘frozen’ to the point from which the energy is radiated.

6.5.3 Bremsstrahlung

An important special case of radiation is when the velocity \mathbf{v}' and the acceleration $\dot{\mathbf{v}}'$ are collinear (parallel or anti-parallel) so that $\mathbf{v}' \times \dot{\mathbf{v}}' = \mathbf{0}$. This condition (for an arbitrary magnitude of \mathbf{v}') inserted into expression (6.126) on page 112 for the radiation field, yields

$$\mathbf{E}^{\text{far}}(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0 c^2 s^3} (\mathbf{x} - \mathbf{x}') \times [(\mathbf{x} - \mathbf{x}') \times \dot{\mathbf{v}}'], \quad \mathbf{v}' \parallel \dot{\mathbf{v}}' \quad (6.152)$$

from which we obtain, with the use of formula (6.125) on page 112, the magnetic field

$$\mathbf{B}^{\text{far}}(t, \mathbf{x}) = \frac{q' |\mathbf{x} - \mathbf{x}'|}{4\pi\epsilon_0 c^3 s^3} [\dot{\mathbf{v}}' \times (\mathbf{x} - \mathbf{x}')], \quad \mathbf{v}' \parallel \dot{\mathbf{v}}' \quad (6.153)$$

The difference between this case and the previous case of $v' \ll c$ is that the approximate expression (6.144) on the previous page for s is no longer valid; instead we must use the correct expression (6.104) on page 107. The angular

distribution of the energy flux (Poynting vector) far away from the source therefore becomes

$$\mathbf{S}^{\text{far}} = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c |\mathbf{x} - \mathbf{x}'|^2} \frac{\sin^2 \theta}{\left(1 - \frac{v'}{c} \cos \theta\right)^6} \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \quad (6.154)$$

It is interesting to note that the magnitudes of the electric and magnetic fields are the same whether \mathbf{v}' and $\dot{\mathbf{v}}'$ are parallel or anti-parallel.

We must be careful when we compute the energy (\mathbf{S} integrated over time). The Poynting vector is related to the time t when it is measured and to a *fixed* surface in space. The radiated power into a solid angle element $d\Omega$, measured relative to the particle's retarded position, is given by the formula

$$\frac{dU^{\text{rad}}(\theta)}{dt} d\Omega = \mathbf{S} \cdot (\mathbf{x} - \mathbf{x}') |\mathbf{x} - \mathbf{x}'| d\Omega = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{\sin^2 \theta}{\left(1 - \frac{v'}{c} \cos \theta\right)^6} d\Omega \quad (6.155)$$

On the other hand, the radiation loss due to radiation from the charge at retarded time t' :

$$\frac{dU^{\text{rad}}}{dt'} d\Omega = \frac{dU^{\text{rad}}}{dt} \left(\frac{\partial t}{\partial t'} \right) d\Omega \quad (6.156)$$

Using formula (6.112) on page 109, we obtain

$$\frac{dU^{\text{rad}}}{dt'} d\Omega = \frac{dU^{\text{rad}}}{dt} \frac{s}{|\mathbf{x} - \mathbf{x}'|} d\Omega = \mathbf{S} \cdot (\mathbf{x} - \mathbf{x}') s d\Omega \quad (6.157)$$

Inserting equation (6.154) for \mathbf{S} into (6.157), we obtain the explicit expression for the energy loss due to radiation evaluated at the retarded time

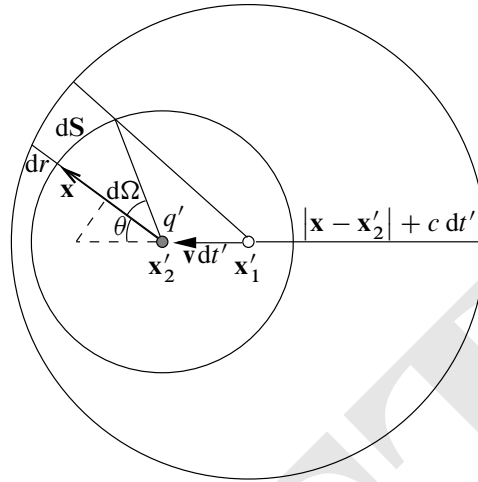
$$\frac{dU^{\text{rad}}(\theta)}{dt'} d\Omega = \frac{\mu_0 q'^2 \dot{v}^2}{16\pi^2 c} \frac{\sin^2 \theta}{\left(1 - \frac{v'}{c} \cos \theta\right)^5} d\Omega \quad (6.158)$$

The angular factors of this expression, for three different particle speeds, are plotted in figure 6.8 on the preceding page.

Comparing expression (6.155) with expression (6.158) above, we see that they differ by a factor $1 - v' \cos \theta / c$ that comes from the extra factor $s / |\mathbf{x} - \mathbf{x}'|$ introduced in (6.157). Let us explain this in geometrical terms.

During the interval $(t', t' + dt')$ and within the solid angle element $d\Omega$ the particle radiates an energy $[dU^{\text{rad}}(\theta)/dt'] dt' d\Omega$. As shown in figure 6.9 on the following page this energy is at time t located between two spheres, one outer with its origin at $\mathbf{x}'_1(t')$ and radius $c(t - t')$, and one inner with its origin at $\mathbf{x}'_2(t' + dt') = \mathbf{x}'_1(t') + \mathbf{v}' dt'$ and radius $c[t - (t' + dt')] = c(t - t' - dt')$.

Figure 6.9: Location of radiation between two spheres as the charge moves with velocity \mathbf{v}' from \mathbf{x}'_1 to \mathbf{x}'_2 during the time interval $(t', t' + dt')$. The observation point (field point) is at the fixed location \mathbf{x} .



From Figure 6.9 we see that the volume element subtending the solid angle element

$$d\Omega = \frac{d^2x}{|\mathbf{x} - \mathbf{x}'_2|^2} \quad (6.159)$$

is

$$d^3x = d^2x dr = |\mathbf{x} - \mathbf{x}'_2|^2 d\Omega dr \quad (6.160)$$

Here, dr denotes the differential distance between the two spheres and can be evaluated in the following way

$$\begin{aligned} dr &= |\mathbf{x} - \mathbf{x}'_2| + c dt' - |\mathbf{x} - \mathbf{x}'_2| - \underbrace{\frac{\mathbf{x} - \mathbf{x}'_2}{|\mathbf{x} - \mathbf{x}'_2|} \cdot \mathbf{v}' dt'}_{v' \cos \theta} \\ &= \left(c - \frac{\mathbf{x} - \mathbf{x}'_2}{|\mathbf{x} - \mathbf{x}'_2|} \cdot \mathbf{v}' \right) dt' = \frac{cs}{|\mathbf{x} - \mathbf{x}'_2|} dt' \end{aligned} \quad (6.161)$$

where formula (6.104) on page 107 was used in the last step. Hence, the volume element under consideration is

$$d^3x = d^2x dr = \frac{s}{|\mathbf{x} - \mathbf{x}'_2|} d^2x c dt' \quad (6.162)$$

We see that the energy that is radiated per unit solid angle during the time interval $(t', t' + dt')$ is located in a volume element whose size is θ dependent. This

explains the difference between expression (6.155) on page 119 and expression (6.158) on page 119.

Let the radiated energy, integrated over Ω , be denoted \tilde{U}_{rad} . After tedious, but relatively straightforward integration of formula (6.158) on page 119, one obtains

$$\frac{d\tilde{U}_{\text{rad}}}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{6\pi c} \frac{1}{\left(1 - \frac{v'^2}{c^2}\right)^3} = \frac{2}{3} \frac{q'^2 \dot{v}'^2}{4\pi \epsilon_0 c^3} \left(1 - \frac{v'^2}{c^2}\right)^{-3} \quad (6.163)$$

If we know $\mathbf{v}'(t')$, we can integrate this expression over t' and obtain the total energy radiated during the acceleration or deceleration of the particle. This way we obtain a classical picture of *bremssstrahlung* (*braking radiation, free-free radiation*). Often, an atomistic treatment is required for obtaining an acceptable result.

►Bremssstrahlung for low speeds and short acceleration times

EXAMPLE 6.4

Calculate the bremsstrahlung when a charged particle, moving at a non-relativistic speed, is accelerated or decelerated during an infinitely short time interval.

We approximate the velocity change at time $t' = t_0$ by a delta function:

$$\dot{\mathbf{v}}'(t') = \Delta \mathbf{v}' \delta(t' - t_0) \quad (6.164)$$

which means that

$$\Delta \mathbf{v}'(t_0) = \int_{-\infty}^{\infty} dt' \dot{\mathbf{v}}' \quad (6.165)$$

Also, we assume $v/c \ll 1$ so that, according to formula (6.104) on page 107,

$$s \approx |\mathbf{x} - \mathbf{x}'| \quad (6.166)$$

and, according to formula (6.121) on page 111,

$$\mathbf{x} - \mathbf{x}_0 \approx \mathbf{x} - \mathbf{x}' \quad (6.167)$$

From the general expression (6.125) on page 112 we conclude that $\mathbf{E} \perp \mathbf{B}$ and that it suffices to consider $E \equiv |\mathbf{E}^{\text{far}}|$. According to the ‘bremssstrahlung expression’ for \mathbf{E}^{far} , equation (6.152) on page 118,

$$E = \frac{q' \sin \theta'}{4\pi \epsilon_0 c^2 |\mathbf{x} - \mathbf{x}'|} \Delta v' \delta(t' - t_0) \quad (6.168)$$

In this simple case $B \equiv |\mathbf{B}^{\text{far}}|$ is given by

$$B = \frac{E}{c} \quad (6.169)$$

Beacuse of the Dirac δ behaviour in time, Fourier transforming expression (6.168) above

for E is trivial, yielding

$$\begin{aligned} E_\omega &= \frac{q' \sin \theta'}{4\pi\epsilon_0 c^2} \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \frac{\Delta \mathbf{v}' \delta(t' - t_0)}{|\mathbf{x} - \mathbf{x}'(t')|} e^{i\omega t'} \\ &= \frac{q' \sin \theta'}{8\pi^2 \epsilon_0 c^2 |\mathbf{x} - \mathbf{x}'(t_0)|} \Delta v'(t_0) e^{i\omega t_0} \end{aligned} \quad (6.170)$$

We note that the magnitude of this Fourier component is independent of ω . This is a consequence of the infinitely short ‘impulsive step’ $\delta(t' - t_0)$ in the time domain which produces an infinite spectrum in the frequency domain.

The total radiation energy is given by the expression

$$\begin{aligned} \tilde{U}^{\text{rad}} &= \int_{-\infty}^{\infty} dt' \frac{d\tilde{U}^{\text{rad}}}{dt'} = \int_{-\infty}^{\infty} dt' \oint_{S'} d^2x' \hat{\mathbf{n}}' \cdot \left(\mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) \\ &= \frac{1}{\mu_0} \oint_{S'} d^2x' \int_{-\infty}^{\infty} dt' EB = \frac{1}{\mu_0 c} \oint_{S'} d^2x' \int_{-\infty}^{\infty} dt' E^2 \\ &= \epsilon_0 c \oint_{S'} d^2x' \int_{-\infty}^{\infty} dt' E^2 \end{aligned} \quad (6.171)$$

According to Parseval’s identity [cf. equation (6.8) on page 84] the following equality holds:

$$\int_{-\infty}^{\infty} dt' E^2 = 4\pi \int_0^{\infty} d\omega |E_\omega|^2 \quad (6.172)$$

which means that the radiated energy in the frequency interval $(\omega, \omega + d\omega)$ is

$$\tilde{U}_\omega^{\text{rad}} d\omega = 4\pi \epsilon_0 c \left(\oint_{S'} d^2x' |E_\omega|^2 \right) d\omega \quad (6.173)$$

For our infinite spectrum, equation (6.170) above, we obtain

$$\begin{aligned} \tilde{U}_\omega^{\text{rad}} d\omega &= \frac{q'^2 (\Delta v')^2}{16\pi^3 \epsilon_0 c^3} \oint_{S'} d^2x' \frac{\sin^2 \theta'}{|\mathbf{x} - \mathbf{x}'|^2} d\omega \\ &= \frac{q'^2 (\Delta v')^2}{16\pi^3 \epsilon_0 c^3} \int_0^{2\pi} d\varphi' \int_0^\pi d\theta' \sin \theta' \sin^2 \theta' d\omega \\ &= \frac{q'^2}{3\pi \epsilon_0 c} \left(\frac{\Delta v'}{c} \right)^2 \frac{d\omega}{2\pi} \end{aligned} \quad (6.174)$$

We see that the energy spectrum $\tilde{U}_\omega^{\text{rad}}$ is independent of frequency ω . This means that if we would integrate it over all frequencies $\omega \in [0, \infty)$, a divergent integral would result.

In reality, all spectra have finite widths, with an upper *cutoff* limit set by the quantum condition

$$\hbar\omega_{\text{max}} = \frac{1}{2}m(v' + \Delta v')^2 - \frac{1}{2}mv'^2 \quad (6.175)$$

which expresses that the highest possible frequency ω_{max} in the spectrum is that for which all kinetic energy difference has gone into one single *field quantum (photon)* with energy $\hbar\omega_{\text{max}}$. If we adopt the picture that the total energy is quantised in terms of N_ω photons radiated during the process, we find that

$$\frac{\tilde{U}_\omega^{\text{rad}} d\omega}{\hbar\omega} = dN_\omega \quad (6.176)$$

or, for an electron where $q' = -|e|$, where e is the elementary charge,

$$dN_\omega = \frac{e^2}{4\pi\epsilon_0\hbar c} \frac{2}{3\pi} \left(\frac{\Delta v'}{c}\right)^2 \frac{d\omega}{\omega} \approx \frac{1}{137} \frac{2}{3\pi} \left(\frac{\Delta v'}{c}\right)^2 \frac{d\omega}{\omega} \quad (6.177)$$

where we used the value of the *fine structure constant* $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$.

Even if the number of photons becomes infinite when $\omega \rightarrow 0$, these photons have negligible energies so that the total radiated energy is still finite.

—End of example 6.4◁

6.5.4 Cyclotron and synchrotron radiation (magnetic bremsstrahlung)

Formula (6.125) and formula (6.126) on page 112 for the magnetic field and the radiation part of the electric field are general, valid for any kind of motion of the localised charge. A very important special case is circular motion, *i.e.*, the case $\mathbf{v}' \perp \dot{\mathbf{v}}'$.

With the charged particle orbiting in the x_1x_2 plane as in figure 6.10 on the next page, an orbit radius a , and an angular frequency ω_0 , we obtain

$$\varphi(t') = \omega_0 t' \quad (6.178a)$$

$$\mathbf{x}'(t') = a[\hat{\mathbf{x}}_1 \cos \varphi(t') + \hat{\mathbf{x}}_2 \sin \varphi(t')] \quad (6.178b)$$

$$\mathbf{v}'(t') = \dot{\mathbf{x}}'(t') = a\omega_0[-\hat{\mathbf{x}}_1 \sin \varphi(t') + \hat{\mathbf{x}}_2 \cos \varphi(t')] \quad (6.178c)$$

$$v' = |\mathbf{v}'| = a\omega_0 \quad (6.178d)$$

$$\dot{\mathbf{v}}'(t') = \ddot{\mathbf{x}}'(t') = -a\omega_0^2[\hat{\mathbf{x}}_1 \cos \varphi(t') + \hat{\mathbf{x}}_2 \sin \varphi(t')] \quad (6.178e)$$

$$\dot{v}' = |\dot{\mathbf{v}}'| = a\omega_0^2 \quad (6.178f)$$

Because of the rotational symmetry we can, without loss of generality, rotate our coordinate system around the x_3 axis so the relative vector $\mathbf{x} - \mathbf{x}'$ from the source point to an arbitrary field point always lies in the x_2x_3 plane, *i.e.*,

$$\mathbf{x} - \mathbf{x}' = |\mathbf{x} - \mathbf{x}'| (\hat{\mathbf{x}}_2 \sin \alpha + \hat{\mathbf{x}}_3 \cos \alpha) \quad (6.179)$$

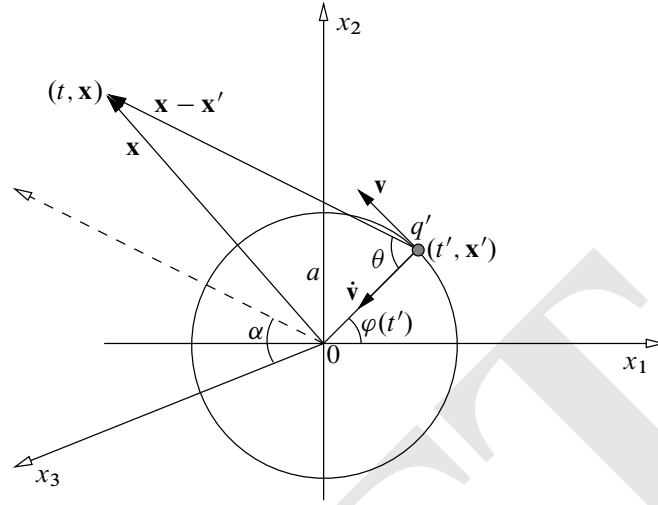
where α is the angle between $\mathbf{x} - \mathbf{x}'$ and the normal to the plane of the particle orbit (see Figure 6.10). From the above expressions we obtain

$$(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}' = |\mathbf{x} - \mathbf{x}'| v' \sin \alpha \cos \varphi \quad (6.180a)$$

$$(\mathbf{x} - \mathbf{x}') \cdot \dot{\mathbf{v}}' = -|\mathbf{x} - \mathbf{x}'| \dot{v}' \sin \alpha \sin \varphi = |\mathbf{x} - \mathbf{x}'| \dot{v}' \cos \theta \quad (6.180b)$$

where in the last step we simply used the definition of a scalar product and the fact that the angle between $\dot{\mathbf{v}}'$ and $\mathbf{x} - \mathbf{x}'$ is θ .

Figure 6.10: Coordinate system for the radiation from a charged particle at $\mathbf{x}'(t')$ in circular motion with velocity $\mathbf{v}'(t')$ along the tangent and constant acceleration $\dot{\mathbf{v}}'(t')$ toward the origin. The x_1x_2 axes are chosen so that the relative field point vector $\mathbf{x} - \mathbf{x}'$ makes an angle α with the x_3 axis, which is normal to the plane of the orbital motion. The radius of the orbit is a .



The energy flux is given by the Poynting vector, which, with the help of formula (6.125) on page 112, can be written

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) = \frac{1}{c\mu_0} |\mathbf{E}|^2 \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \quad (6.181)$$

Inserting this into equation (6.157) on page 119, we obtain

$$\frac{dU^{\text{rad}}(\alpha, \varphi)}{dt'} = \frac{|\mathbf{x} - \mathbf{x}'| s}{c\mu_0} |\mathbf{E}|^2 \quad (6.182)$$

where the retarded distance s is given by expression (6.104) on page 107. With the radiation part of the electric field, expression (6.126) on page 112, inserted, and using (6.180a) and (6.180b) on the preceding page, one finds, after some algebra, that

$$\frac{dU^{\text{rad}}(\alpha, \varphi)}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} \frac{\left(1 - \frac{v'}{c} \sin \alpha \cos \varphi\right)^2 - \left(1 - \frac{v'^2}{c^2}\right) \sin^2 \alpha \sin^2 \varphi}{\left(1 - \frac{v'}{c} \sin \alpha \cos \varphi\right)^5} \quad (6.183)$$

The angles θ and φ vary in time during the rotation, so that θ refers to a *moving* coordinate system. But we can parametrise the solid angle $d\Omega$ in the angle φ and the angle α so that $d\Omega = \sin \alpha d\alpha d\varphi$. Integration of equation (6.183) above over this $d\Omega$ gives, after some cumbersome algebra, the angular integrated expression

$$\frac{d\tilde{U}^{\text{rad}}}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{6\pi c} \frac{1}{\left(1 - \frac{v'^2}{c^2}\right)^2} \quad (6.184)$$

In equation (6.183), two limits are particularly interesting:

1. $v'/c \ll 1$ which corresponds to *cyclotron radiation*.
2. $v'/c \lesssim 1$ which corresponds to *synchrotron radiation*.

6.5.4.1 Cyclotron radiation

For a non-relativistic speed $v' \ll c$, equation (6.183) on the preceding page reduces to

$$\frac{dU^{\text{rad}}(\alpha, \varphi)}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} (1 - \sin^2 \alpha \sin^2 \varphi) \quad (6.185)$$

But, according to equation (6.180b) on page 123

$$\sin^2 \alpha \sin^2 \varphi = \cos^2 \theta \quad (6.186)$$

where θ is defined in figure 6.10 on the preceding page. This means that we can write

$$\frac{dU^{\text{rad}}(\theta)}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} (1 - \cos^2 \theta) = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} \sin^2 \theta \quad (6.187)$$

Consequently, a fixed observer near the orbit plane ($\alpha \approx \pi/2$) will observe cyclotron radiation twice per revolution in the form of two equally broad pulses of radiation with alternating polarisation.

6.5.4.2 Synchrotron radiation

When the particle is relativistic, $v' \lesssim c$, the denominator in equation (6.183) on the facing page becomes very small if $\sin \alpha \cos \varphi \approx 1$, which defines the forward direction of the particle motion ($\alpha \approx \pi/2$, $\varphi \approx 0$). The equation (6.183) on the preceding page becomes

$$\frac{dU^{\text{rad}}(\pi/2, 0)}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} \frac{1}{\left(1 - \frac{v'}{c}\right)^3} \quad (6.188)$$

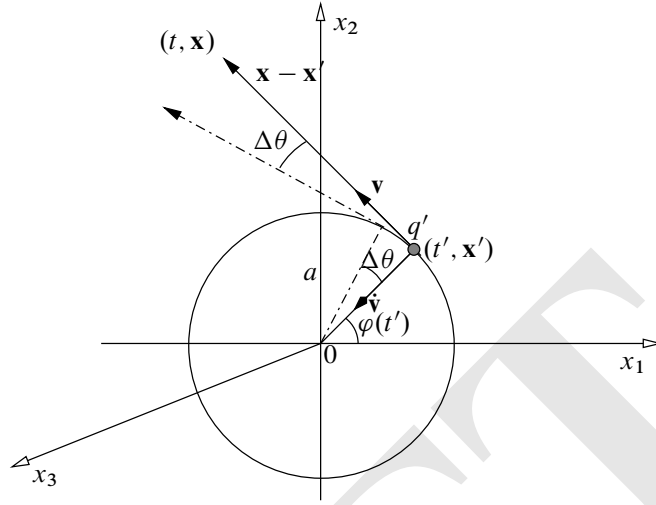
which means that an observer near the orbit plane sees a very strong pulse followed, half an orbit period later, by a much weaker pulse.

The two cases represented by equation (6.187) and equation (6.188) above are very important results since they can be used to determine the characteristics of the particle motion both in particle accelerators and in astrophysical objects where a direct measurement of particle velocities are impossible.

In the orbit plane ($\alpha = \pi/2$), equation (6.183) on the facing page gives

$$\frac{dU^{\text{rad}}(\pi/2, \varphi)}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{16\pi^2 c} \frac{\left(1 - \frac{v'}{c} \cos \varphi\right)^2 - \left(1 - \frac{v'^2}{c^2}\right) \sin^2 \varphi}{\left(1 - \frac{v'}{c} \cos \varphi\right)^5} \quad (6.189)$$

Figure 6.11: If the observation point \mathbf{x} is in the plane of the particle orbit, i.e., if $\alpha = \pi/2$, the lobe width is given by $\Delta\theta$.



This vanishes for angles φ_0 such that

$$\cos \varphi_0 = \frac{v'}{c} \quad (6.190a)$$

$$\sin \varphi_0 = \sqrt{1 - \frac{v'^2}{c^2}} \quad (6.190b)$$

Hence, the angle φ_0 is a measure of the *synchrotron radiation lobe width* $\Delta\theta$; see figure 6.11. For ultra-relativistic particles, defined by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}} \gg 1, \quad \sqrt{1 - \frac{v'^2}{c^2}} \ll 1, \quad (6.191)$$

one can approximate

$$\varphi_0 \approx \sin \varphi_0 = \sqrt{1 - \frac{v'^2}{c^2}} = \frac{1}{\gamma} \quad (6.192)$$

Hence, synchrotron radiation from ultra-relativistic charges is characterized by a radiation lobe width which is approximately

$$\Delta\theta \approx \frac{1}{\gamma} \quad (6.193)$$

This angular interval is swept by the charge during the time interval

$$\Delta t' = \frac{\Delta\theta}{\omega_0} \quad (6.194)$$

during which the particle moves a length interval

$$\Delta l' = v' \Delta t' = v' \frac{\Delta \theta}{\omega_0} \quad (6.195)$$

in the direction toward the observer who therefore measures a compressed pulse width of length

$$\begin{aligned} \Delta t &= \Delta t' - \frac{\Delta l'}{c} = \Delta t' - \frac{v' \Delta t'}{c} = \left(1 - \frac{v'}{c}\right) \Delta t' \\ &= \left(1 - \frac{v'}{c}\right) \frac{\Delta \theta}{\omega_0} \approx \left(1 - \frac{v'}{c}\right) \frac{1}{\gamma \omega_0} = \frac{\left(1 - \frac{v'}{c}\right) \left(1 + \frac{v'}{c}\right)}{\underbrace{1 + \frac{v'}{c}}_{\approx 2}} \frac{1}{\gamma \omega_0} \\ &\approx \underbrace{\left(1 - \frac{v'^2}{c^2}\right)}_{1/\gamma^2} \frac{1}{2\gamma \omega_0} = \frac{1}{2\gamma^3} \frac{1}{\omega_0} \end{aligned} \quad (6.196)$$

Typically, the spectral width of a pulse of length Δt is $\Delta \omega \lesssim 1/\Delta t$. In the ultra-relativistic synchrotron case one can therefore expect frequency components up to

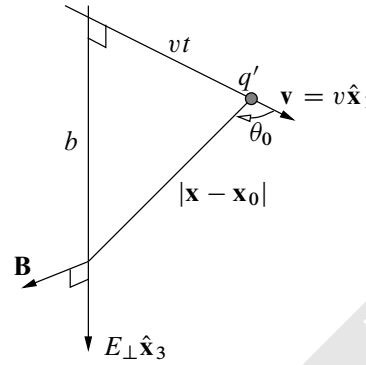
$$\omega_{\max} \approx \frac{1}{\Delta t} = 2\gamma^3 \omega_0 \quad (6.197)$$

A spectral analysis of the radiation pulse will therefore exhibit a (broadened) line spectrum of Fourier components $n\omega_0$ from $n = 1$ up to $n \approx 2\gamma^3$.

When many charged particles, N say, contribute to the radiation, we can have three different situations depending on the relative phases of the radiation fields from the individual particles:

1. All N radiating particles are spatially much closer to each other than a typical wavelength. Then the relative phase differences of the individual electric and magnetic fields radiated are negligible and the total radiated fields from all individual particles will add up to become N times that from one particle. This means that the power radiated from the N particles will be N^2 higher than for a single charged particle. This is called *coherent radiation*.
2. The charged particles are perfectly evenly distributed in the orbit. In this case the phases of the radiation fields cause a complete cancellation of the fields themselves. No radiation escapes.
3. The charged particles are somewhat unevenly distributed in the orbit. This happens for an open ring current, carried initially by evenly distributed charged particles, which is subject to thermal fluctuations. From statistical mechanics

Figure 6.12: The perpendicular electric field of a charge q' moving with velocity $\mathbf{v}' = v'\hat{\mathbf{x}}$ is $E_{\perp}\hat{\mathbf{z}}$.



we know that this happens for all open systems and that the particle densities exhibit fluctuations of order \sqrt{N} . This means that out of the N particles, \sqrt{N} will exhibit deviation from perfect randomness—and thereby perfect radiation field cancellation—and give rise to net radiation fields which are proportional to \sqrt{N} . As a result, the radiated power will be proportional to N , and we speak about *incoherent radiation*. Examples of this can be found both in earthly laboratories and under cosmic conditions.

6.5.4.3 Radiation in the general case

We recall that the general expression for the radiation \mathbf{E} field from a moving charge concentration is given by expression (6.126) on page 112. This expression in equation (6.182) on page 124 yields the general formula

$$\frac{dU^{\text{rad}}(\theta, \varphi)}{dt'} = \frac{\mu_0 q'^2 |\mathbf{x} - \mathbf{x}'|}{16\pi^2 c s^5} (\mathbf{x} - \mathbf{x}') \times \left[\left((\mathbf{x} - \mathbf{x}') - \frac{|\mathbf{x} - \mathbf{x}'| \mathbf{v}'}{c} \right) \times \dot{\mathbf{v}}' \right]^2 \quad (6.198)$$

Integration over the solid angle Ω gives the totally radiated power as

$$\frac{d\tilde{U}^{\text{rad}}}{dt'} = \frac{\mu_0 q'^2 \dot{v}'^2}{6\pi c} \frac{1 - \frac{v'^2}{c^2} \sin^2 \psi}{\left(1 - \frac{v'^2}{c^2}\right)^3} \quad (6.199)$$

where ψ is the angle between \mathbf{v}' and $\dot{\mathbf{v}}'$.

If \mathbf{v}' is collinear with $\dot{\mathbf{v}}'$, so that $\sin \psi = 0$, we get *bremsstrahlung*. For $\mathbf{v}' \perp \dot{\mathbf{v}}'$, $\sin \psi = 1$, which corresponds to *cyclotron radiation* or *synchrotron radiation*.

6.5.4.4 Virtual photons

Let us consider a charge q' moving with constant, high velocity $\mathbf{v}'(t')$ along the x_1 axis. According to formula (6.142) on page 116 and figure 6.12 on the preceding page, the perpendicular component along the \mathbf{x}_3 axis of the electric field from this moving charge is

$$E_{\perp} = E_3 = \frac{q'}{4\pi\epsilon_0 s^3} \left(1 - \frac{v'^2}{c^2}\right) (\mathbf{x} - \mathbf{x}_0) \cdot \hat{\mathbf{x}}_3 \quad (6.200)$$

Utilising expression (6.134) on page 114 and simple geometrical relations, we can rewrite this as

$$E_{\perp} = \frac{q'}{4\pi\epsilon_0} \frac{b}{\gamma^2 [(v't')^2 + b^2/\gamma^2]^{3/2}} \quad (6.201)$$

This represents a contracted Coulomb field, approaching the field of a plane wave. The passage of this field ‘pulse’ corresponds to a frequency distribution of the field energy. Fourier transforming, we obtain

$$E_{\omega,\perp} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt E_{\perp}(t) e^{i\omega t} = \frac{q'}{4\pi^2\epsilon_0 b v'} \left[\left(\frac{b\omega}{v'\gamma}\right) K_1\left(\frac{b\omega}{v'\gamma}\right) \right] \quad (6.202)$$

Here, K_1 is the *Kelvin function* (Bessel function of the second kind with imaginary argument) which behaves in such a way for small and large arguments that

$$E_{\omega,\perp} \sim \frac{q'}{4\pi^2\epsilon_0 b v'}, \quad b\omega \ll v'\gamma \Leftrightarrow \frac{b}{v'\gamma}\omega \ll 1 \quad (6.203a)$$

$$E_{\omega,\perp} \sim 0, \quad b\omega \gg v'\gamma \Leftrightarrow \frac{b}{v'\gamma}\omega \gg 1 \quad (6.203b)$$

showing that the ‘pulse’ length is of the order $b/(v'\gamma)$.

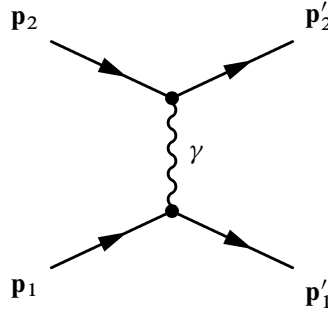
Due to the equipartitioning of the field energy into the electric and magnetic fields, the total field energy can be written

$$\tilde{U} = \epsilon_0 \int_V d^3x E_{\perp}^2 = \epsilon_0 \int_{b_{\min}}^{b_{\max}} db 2\pi b \int_{-\infty}^{\infty} dt v' E_{\perp}^2 \quad (6.204)$$

where the volume integration is over the plane perpendicular to \mathbf{v}' . With the use of *Parseval's identity* for Fourier transforms, formula (6.8) on page 84, we can rewrite this as

$$\begin{aligned} \tilde{U} &= \int_0^{\infty} d\omega \tilde{U}_{\omega} = 4\pi\epsilon_0 v' \int_{b_{\min}}^{b_{\max}} db 2\pi b \int_0^{\infty} d\omega E_{\omega,\perp}^2 \\ &\approx \frac{q'^2}{2\pi^2\epsilon_0 v'} \int_{-\infty}^{\infty} d\omega \int_{b_{\min}}^{v'\gamma/\omega} \frac{db}{b} \end{aligned} \quad (6.205)$$

Figure 6.13: Diagrammatic representation of the semi-classical electron-electron interaction (Møller scattering).



from which we conclude that

$$\tilde{U}_\omega \approx \frac{q'^2}{2\pi^2 \epsilon_0 v'} \ln \left(\frac{v' \gamma}{b_{\min} \omega} \right) \quad (6.206)$$

where an explicit value of b_{\min} can be calculated in quantum theory only.

As in the case of bremsstrahlung, it is intriguing to quantise the energy into photons [cf. equation (6.176) on page 122]. Then we find that

$$N_\omega d\omega \approx \frac{2\alpha}{\pi} \ln \left(\frac{c\gamma}{b_{\min} \omega} \right) \frac{d\omega}{\omega} \quad (6.207)$$

where $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$ is the *fine structure constant*.

Let us consider the interaction of two (classical) electrons, 1 and 2. The result of this interaction is that they change their linear momenta from \mathbf{p}_1 to \mathbf{p}'_1 and \mathbf{p}_2 to \mathbf{p}'_2 , respectively. Heisenberg's uncertainty principle gives $b_{\min} \sim \hbar/|\mathbf{p}_1 - \mathbf{p}'_1|$ so that the number of photons exchanged in the process is of the order

$$N_\omega d\omega \approx \frac{2\alpha}{\pi} \ln \left(\frac{c\gamma}{\hbar\omega} |\mathbf{p}_1 - \mathbf{p}'_1| \right) \frac{d\omega}{\omega} \quad (6.208)$$

Since this change in momentum corresponds to a change in energy $\hbar\omega = E_1 - E'_1$ and $E_1 = m_0\gamma c^2$, we see that

$$N_\omega d\omega \approx \frac{2\alpha}{\pi} \ln \left(\frac{E_1}{m_0 c^2} \frac{|c\mathbf{p}_1 - c\mathbf{p}'_1|}{E_1 - E'_1} \right) \frac{d\omega}{\omega} \quad (6.209)$$

a formula which gives a reasonable semi-classical account of a photon-induced electron-electron interaction process. In quantum theory, including only the lowest order contributions, this process is known as *Møller scattering*. A diagrammatic representation of (a semi-classical approximation of) this process is given in figure 6.13.

6.6 Bibliography

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DRAFT

7

RELATIVISTIC ELECTRODYNAMICS

We saw in chapter 3 how the introduction of electrodynamic potentials led, in a most natural way, to the existence of a characteristic, finite speed of propagation of electromagnetic fields (and related quantities) in free space (vacuum) that equals the speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$ and which can be considered a constant of Nature. To take this finite speed of propagation of information into account, and to ensure that our laws of physics be independent of any specific coordinate frame, requires a treatment of electrodynamics in a relativistically covariant (coordinate independent) form. This is the objective of this chapter.¹

The technique we shall use to study relativity is the mathematical apparatus developed for non-Euclidean spaces of arbitrary dimensions, here specialised to four dimensions. It turns out that this theory of Riemannian spaces, derived for more or less purely mathematical reasons only, is ideal for a formal description of relativistic physics. For the simple case of the *special theory of relativity*, the mathematics is quite simple, whereas for the *general theory of relativity* it becomes more complicated.

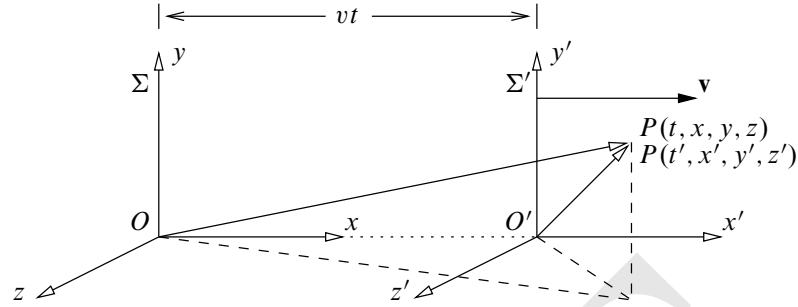
7.1 The special theory of relativity

An *inertial system*, or *inertial reference frame*, is a system of reference, or rigid coordinate system, in which the *law of inertia* (*Galileo's law*, *Newton's first law*) holds. In other words, an inertial system is a system in which free bodies

¹ *The Special Theory of Relativity*, by the physicist and philosopher DAVID JOSEPH BOHM (1917–1992), opens with the following paragraph:

‘The theory of relativity is not merely a scientific development of great importance in its own right. It is even more significant as the first stage of a radical change in our basic concepts, which began in physics, and which is spreading into other fields of science, and indeed, even into a great deal of thinking outside of science. For as is well known, the modern trend is away from the notion of sure ‘absolute’ truth, (*i.e.*, one which holds independently of all conditions, contexts, degrees, and types of approximation *etc.*) and toward the idea that a given concept has significance only in relation to suitable broader forms of reference, within which that concept can be given its full meaning.’

Figure 7.1: Two inertial systems Σ and Σ' in relative motion with velocity \mathbf{v} along the $x = x'$ axis. At time $t = t' = 0$ the origin O' of Σ' coincided with the origin O of Σ . At time t , the inertial system Σ' has been translated a distance vt along the x axis in Σ . An event represented by $P(t, x, y, z)$ in Σ is represented by $P(t', x', y', z')$ in Σ' .



move uniformly and do not experience any acceleration. The *special theory of relativity* describes how physical processes are interrelated when observed in different inertial systems in uniform, rectilinear motion relative to each other and is based on two postulates:

POSTULATE 7.1 (Relativity principle; POINCARÉ, 1905) *All laws of physics (except the laws of gravitation) are independent of the uniform translational motion of the system on which they operate.*

POSTULATE 7.2 (EINSTEIN, 1905) *The velocity of light in empty space is independent of the motion of the source that emits the light.*

A consequence of the first postulate is that all geometrical objects (vectors, tensors) in an equation describing a physical process must transform in a *covariant* manner, *i.e.*, in the same way.

7.1.1 The Lorentz transformation

Let us consider two three-dimensional inertial systems Σ and Σ' in free space. They are in rectilinear motion relative to each other in such a way that Σ' moves with constant velocity \mathbf{v} along the x axis of the Σ system. The times and the spatial coordinates as measured in the two systems are t and (x, y, z) , and t' and (x', y', z') , respectively. At time $t = t' = 0$ the origins O and O' and the x and x' axes of the two inertial systems coincide and at a later time t they have the relative location as depicted in figure 7.1, referred to as the *standard configuration*.

For convenience, let us introduce the two quantities

$$\beta = \frac{v}{c} \quad (7.1)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (7.2)$$

where $v = |\mathbf{v}|$. In the following, we shall make frequent use of these shorthand notations.

As shown by Einstein, the two postulates of special relativity require that the spatial coordinates and times as measured by an observer in Σ and Σ' , respectively, are connected by the following transformation:

$$ct' = \gamma(ct - x\beta) \quad (7.3a)$$

$$x' = \gamma(x - vt) \quad (7.3b)$$

$$y' = y \quad (7.3c)$$

$$z' = z \quad (7.3d)$$

Taking the difference between the square of (7.3a) and the square of (7.3b) we find that

$$\begin{aligned} c^2t'^2 - x'^2 &= \gamma^2 (c^2t^2 - 2xc\beta t + x^2\beta^2 - x^2 + 2xvt - v^2t^2) \\ &= \frac{1}{1 - \frac{v^2}{c^2}} \left[c^2t^2 \left(1 - \frac{v^2}{c^2} \right) - x^2 \left(1 - \frac{v^2}{c^2} \right) \right] \\ &= c^2t^2 - x^2 \end{aligned} \quad (7.4)$$

From equations (7.3) above we see that the y and z coordinates are unaffected by the translational motion of the inertial system Σ' along the x axis of system Σ . Using this fact, we find that we can generalise the result in equation (7.4) to

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 \quad (7.5)$$

which means that if a light wave is transmitted from the coinciding origins O and O' at time $t = t' = 0$ it will arrive at an observer at (x, y, z) at time t in Σ and an observer at (x', y', z') at time t' in Σ' in such a way that both observers conclude that the speed (spatial distance divided by time) of light in vacuum is c . Hence, the speed of light in Σ and Σ' is the same. A linear coordinate transformation which has this property is called a (homogeneous) *Lorentz transformation*.

7.1.2 Lorentz space

Let us introduce an ordered quadruple of real numbers, enumerated with the help of upper indices $\mu = 0, 1, 2, 3$, where the zeroth component is ct (c is the speed

of light and t is time), and the remaining components are the components of the ordinary \mathbb{R}^3 position vector \mathbf{x} defined in equation (M.1) on page 205:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \equiv (ct, \mathbf{x}) \quad (7.6)$$

In order that this quadruple x^μ represent a *physical observable*, it must transform as (the component form of) a *position four-vector* (*radius four-vector*) in a real, linear, *four-dimensional vector space*.² We require that this four-dimensional space be a *Riemannian space*, *i.e.*, a metric space where a ‘distance’ and a scalar product are defined. In this space we therefore define a *metric tensor*, also known as the *fundamental tensor*, which we denote by $g_{\mu\nu}$.

² The British mathematician and philosopher ALFRED NORTH WHITEHEAD (1861–1947) writes in his book *The Concept of Nature*:

‘I regret that it has been necessary for me in this lecture to administer a large dose of four-dimensional geometry. I do not apologise, because I am really not responsible for the fact that Nature in its most fundamental aspect is four-dimensional. Things are what they are....’

7.1.2.1 Radius four-vector in contravariant and covariant form

The position four-vector $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x})$, as defined in equation (7.6), is, by definition, the prototype of a *contravariant vector* (or, more accurately, a vector in *contravariant component form*). To every such vector there exists a *dual vector*. The vector dual to x^μ is the *covariant vector* x_μ , obtained as

$$x_\mu = g_{\mu\nu} x^\nu \quad (7.7)$$

where the upper index μ in x^μ is summed over and is therefore a *dummy index* and may be replaced by another dummy index ν , say. This summation process is an example of *index contraction* and is often referred to as *index lowering*.

7.1.2.2 Scalar product and norm

The scalar product of x^μ with itself in a Riemannian space is defined as

$$g_{\mu\nu} x^\nu x^\mu = x_\mu x^\mu \quad (7.8)$$

This scalar product acts as an invariant ‘distance’, or *norm*, in this space.

In order to put the physical property of Lorentz transformation invariance, described by equation (7.5) on the previous page, into a convenient mathematical framework, we perceive this invariance as the manifestation of the conservation of the norm in a 4D Riemannian space.

7.1.2.3 Metric tensor

In \mathbb{L}^4 one can choose the metric tensor $g_{\mu\nu}$ to take the simple form

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (7.9)$$

or, in matrix representation,

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (7.10)$$

i.e., a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+, -, -, -\}$, the index lowering operation in our flat 4D space \mathbb{L}^4 becomes nearly trivial:

In matrix representation the lowering of the indices of x^μ becomes

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix} \quad (7.11)$$

In four-tensor notation, this can be written

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -\mathbf{x}) \quad (7.12)$$

Hence, if the metric tensor is defined according to expression (7.9) on the facing page, the covariant position four-vector x_μ is obtained from the contravariant position four-vector x^μ simply by changing the sign of the last three components. These components are referred to as the *space components*; the zeroth component is referred to as the *time component*.

As we see, for this particular choice of metric, the scalar product of x^μ with itself becomes

$$x_\mu x^\mu = (ct, \mathbf{x}) \cdot (ct, -\mathbf{x}) = c^2 t^2 - x^2 - y^2 - z^2 \quad (7.13)$$

which indeed is the desired Lorentz transformation invariance as required by equation (7.13) above. Without changing the physics, one can alternatively choose a signature $\{-, +, +, +\}$. The latter has the advantage that the transition from 3D to 4D becomes smooth, while it will introduce some annoying minus signs in the theory. In current physics literature, the signature $\{+, -, -, -\}$ seems to be the most commonly used one. Note that our space, regardless of signature chosen, will have an *indefinite norm*, *i.e.*, a norm which can be positive definite, negative definite or even zero. This means that we deal with a *non-Euclidean space* and we call our four-dimensional space (or *space-time*) with this property *Lorentz space* and denote it \mathbb{L}^4 . A corresponding real, linear 4D space with a *positive definite norm* which is conserved during ordinary rotations is a *Euclidean vector space*. We denote such a space \mathbb{R}^4 .

The \mathbb{L}^4 metric tensor equation (7.9) on the facing page has a number of interesting properties: firstly, we see that this tensor has a trace $\text{Tr}(\eta_{\mu\nu}) = -2$

whereas in \mathbb{R}^4 , as in any vector space with definite norm, the trace equals the space dimensionality. Secondly, we find, after trivial algebra, that the following relations between the contravariant, covariant and mixed forms of the metric tensor hold:

$$\eta_{\mu\nu} = \eta_{\nu\mu} \quad (7.14a)$$

$$\eta^{\mu\nu} = \eta_{\mu\nu} \quad (7.14b)$$

$$\eta_{\nu\kappa} \eta^{\kappa\mu} = \eta_{\nu}^{\mu} = \delta_{\nu}^{\mu} \quad (7.14c)$$

$$\eta^{\nu\kappa} \eta_{\kappa\mu} = \eta_{\mu}^{\nu} = \delta_{\mu}^{\nu} \quad (7.14d)$$

Here we have introduced the 4D version of the Kronecker delta δ_{ν}^{μ} , a mixed four-tensor of rank 2 that fulfils

$$\delta_{\nu}^{\mu} = \delta_{\mu}^{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (7.15)$$

Clearly, the matrix representation of this tensor is

$$(\delta_{\nu}^{\mu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.16)$$

i.e., the 4×4 unit matrix.

7.1.2.4 Invariant line element and proper time

The *differential distance* ds between the two points x^{μ} and $x^{\mu} + dx^{\mu}$ in \mathbb{L}^4 can be calculated from the *Riemannian metric*, given by the *quadratic differential form*

$$ds^2 = \eta_{\mu\nu} dx^{\nu} dx^{\mu} = dx_{\mu} dx^{\mu} = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (7.17)$$

where the metric tensor is as in equation (7.9) on page 136. As we see, this form is *indefinite* as expected for a non-Euclidean space. The square root of this expression is the *invariant line element*

$$\begin{aligned} ds &= c \, dt \sqrt{1 - \frac{1}{c^2} \left[\left(\frac{dx^1}{dt} \right)^2 + \left(\frac{dx^2}{dt} \right)^2 + \left(\frac{dx^3}{dt} \right)^2 \right]} \\ &= c \, dt \sqrt{1 - \frac{1}{c^2} [(v_x)^2 + (v_y)^2 + (v_z)^2]} = c \, dt \sqrt{1 - \frac{v^2}{c^2}} \\ &= c \, dt \sqrt{1 - \beta^2} = c \frac{dt}{\gamma} = c \, d\tau \end{aligned} \quad (7.18)$$

where we introduced

$$d\tau = dt/\gamma \quad (7.19)$$

Since $d\tau$ measures the time when no spatial changes are present, *i.e.*, by a clock that is fixed relative the given frame of reference, it is called the *proper time*. As equation (7.19) shows, the proper time of a moving object is always less than the corresponding interval in the rest system. One may say that moving clocks go slower than those at rest.

Expressing the property of the Lorentz transformation described by equations (7.5) on page 135 in terms of the differential interval ds and comparing with equation (7.17) on the facing page, we find that

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad (7.20)$$

is invariant, *i.e.*, remains unchanged, during a Lorentz transformation. Conversely, we may say that every coordinate transformation which preserves this differential interval is a Lorentz transformation.

If in some inertial system

$$dx^2 + dy^2 + dz^2 < c^2 dt^2 \quad (7.21)$$

ds is a *time-like interval*, but if

$$dx^2 + dy^2 + dz^2 > c^2 dt^2 \quad (7.22)$$

ds is a *space-like interval*, whereas

$$dx^2 + dy^2 + dz^2 = c^2 dt^2 \quad (7.23)$$

is a *light-like interval*; we may also say that in this case we are on the *light cone*. A vector which has a light-like interval is called a *null vector*. The time-like, space-like or light-like aspects of an interval ds are invariant under a Lorentz transformation. *I.e.*, it is not possible to change a time-like interval into a space-like one or *vice versa* via a Lorentz transformation.

7.1.2.5 Four-vector fields

Any quantity which relative to any coordinate system has a quadruple of real numbers and transforms in the same way as the position four-vector x^μ does, is called a *four-vector*. In analogy with the notation for the position four-vector we introduce the notation $a^\mu = (a^0, \mathbf{a})$ for a general *contravariant four-vector field* in \mathbb{L}^4 and find that the ‘lowering of index’ rule, formula (7.7) on page 136, for such an arbitrary four-vector yields the dual *covariant four-vector field*

$$a_\mu(x^\kappa) = \eta_{\mu\nu} a^\nu(x^\kappa) = (a^0(x^\kappa), -\mathbf{a}(x^\kappa)) \quad (7.24)$$

The scalar product between this four-vector field and another one $b^\mu(x^\kappa)$ is

$$\eta_{\mu\nu}a^\nu(x^\kappa)b^\mu(x^\kappa) = (a^0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a^0b^0 - \mathbf{a} \cdot \mathbf{b} \quad (7.25)$$

which is a *scalar field*, *i.e.*, an invariant scalar quantity $\alpha(x^\kappa)$ which depends on time and space, as described by $x^\kappa = (ct, x, y, z)$.

7.1.2.6 The Lorentz transformation matrix

Introducing the transformation matrix

$$(\Lambda^\mu_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7.26)$$

the linear Lorentz transformation (7.3) on page 135, *i.e.*, the coordinate transformation $x^\mu \rightarrow x'^\mu = x'^\mu(x^0, x^1, x^2, x^3)$, from one inertial system Σ to another inertial system Σ' in the standard configuration, can be written

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad (7.27)$$

7.1.2.7 The Lorentz group

It is easy to show, by means of direct algebra, that two successive Lorentz transformations of the type in equation (7.27) above, and defined by the speed parameters β_1 and β_2 , respectively, correspond to a single transformation with speed parameter

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \quad (7.28)$$

This means that the nonempty set of Lorentz transformations constitutes a *closed algebraic structure* with a binary operation (multiplication) that is *associative*. Furthermore, one can show that this set possesses at least one *identity element* and at least one *inverse element*. In other words, this set of Lorentz transformations constitutes a *mathematical group*. However tempting, we shall not make any further use of *group theory*.

7.1.3 Minkowski space

Specifying a point $x^\mu = (x^0, x^1, x^2, x^3)$ in 4D space-time is a way of saying that ‘something takes place at a certain time $t = x^0/c$ and at a certain place $(x, y, z) = (x^1, x^2, x^3)$ ’. Such a point is therefore called an *event*. The trajectory for an event as a function of time and space is called a *world line*. For

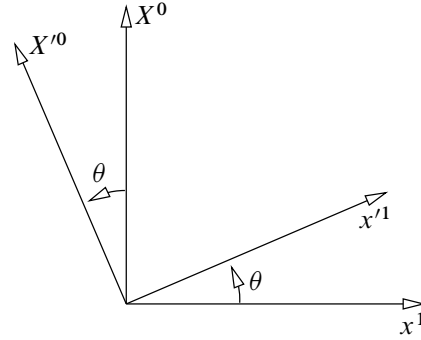


Figure 7.2: Minkowski space can be considered an ordinary Euclidean space where a Lorentz transformation from $(x^1, X^0 = ict)$ to $(x'^1, X'^0 = ict')$ corresponds to an ordinary rotation through an angle θ . This rotation leaves the Euclidean distance $(x^1)^2 + (X^0)^2 = x^2 - c^2 t^2$ invariant.

instance, the world line for a light ray that propagates in vacuum (free space) is the trajectory $x^0 = x^1$.

Introducing

$$X^0 = ix^0 = ict \quad (7.29a)$$

$$X^1 = x^1 \quad (7.29b)$$

$$X^2 = x^2 \quad (7.29c)$$

$$X^3 = x^3 \quad (7.29d)$$

$$dS = ids \quad (7.29e)$$

where $i = \sqrt{-1}$, we see that equation (7.17) on page 138 transforms into

$$dS^2 = (dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \quad (7.30)$$

i.e., into a 4D differential form that is *positive definite* just as is ordinary 3D Euclidean space \mathbb{R}^3 . We shall call the 4D Euclidean space constructed in this way the *Minkowski space* \mathbb{M}^4 .³

As before, it suffices to consider the simplified case where the relative motion between Σ and Σ' is along the x axes. Then

$$dS^2 = (dX^0)^2 + (dX^1)^2 = (dX^0)^2 + (dx^1)^2 \quad (7.31)$$

and we consider the X^0 and $X^1 = x^1$ axes as orthogonal axes in a Euclidean space. As in all Euclidean spaces, every interval is invariant under a rotation of the $X^0 x^1$ plane through an angle θ into $X'^0 x'^1$:

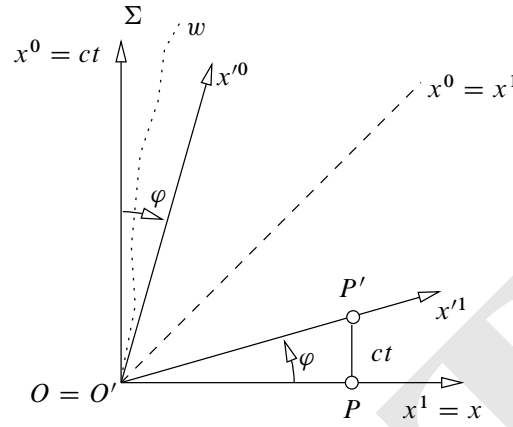
$$X'^0 = -x^1 \sin \theta + X^0 \cos \theta \quad (7.32a)$$

$$x'^1 = x^1 \cos \theta + X^0 \sin \theta \quad (7.32b)$$

See figure 7.2.

³ The fact that our Riemannian space can be transformed in this way into a Euclidean one means that it is, strictly speaking, a *pseudo-Riemannian space*.

Figure 7.3: Minkowski diagram depicting geometrically the transformation (7.33) from the unprimed system to the primed system. Here w denotes the world line for an event and the line $x^0 = x^1 \Leftrightarrow x = ct$ the world line for a light ray in vacuum. Note that the event P is simultaneous with all points on the x^1 axis ($t = 0$), including the origin O . The event P' , which is simultaneous with all points on the x' axis, including $O' = O$, to an observer at rest in the primed system, is not simultaneous with O in the unprimed system but occurs there at time $|P - P'|/c$.



If we introduce the angle $\varphi = -i\theta$, often called the *rapidity* or the *Lorentz boost parameter*, and transform back to the original space and time variables by using equation (7.29) on the preceding page backwards, we obtain

$$ct' = -x \sinh \varphi + ct \cosh \varphi \quad (7.33a)$$

$$x' = x \cosh \varphi - ct \sinh \varphi \quad (7.33b)$$

which are identical to the original transformation equations (7.3) on page 135 if we let

$$\sinh \varphi = \gamma \beta \quad (7.34a)$$

$$\cosh \varphi = \gamma \quad (7.34b)$$

$$\tanh \varphi = \beta \quad (7.34c)$$

It is therefore possible to envisage the Lorentz transformation as an ‘ordinary’ rotation in the 4D Euclidean space \mathbb{M}^4 . Such a rotation in \mathbb{M}^4 corresponds to a coordinate change in \mathbb{L}^4 as depicted in figure 7.3. equation (7.28) on page 140 for successive Lorentz transformation then corresponds to the tanh addition formula

$$\tanh(\varphi_1 + \varphi_2) = \frac{\tanh \varphi_1 + \tanh \varphi_2}{1 + \tanh \varphi_1 \tanh \varphi_2} \quad (7.35)$$

The use of ict and \mathbb{M}^4 , which leads to the interpretation of the Lorentz transformation as an ‘ordinary’ rotation, may, at best, be illustrative, but is not very physical. Besides, if we leave the flat \mathbb{L}^4 space and enter the curved space of general relativity, the ‘ ict ’ trick will turn out to be an impasse. Let us therefore immediately return to \mathbb{L}^4 where all components are real valued.

7.2 Covariant classical mechanics

The invariance of the differential ‘distance’ ds in \mathbb{L}^4 , and the associated differential proper time $d\tau$ [see equation (7.18) on page 138] allows us to define the *four-velocity*

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \mathbf{v}) = \left(\frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = (u^0, \mathbf{u}) \quad (7.36)$$

which, when multiplied with the scalar invariant m_0 yields the *four-momentum*

$$p^\mu = m_0 \frac{dx^\mu}{d\tau} = m_0 \gamma(c, \mathbf{v}) = \left(\frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = (p^0, \mathbf{p}) \quad (7.37)$$

From this we see that we can write

$$\mathbf{p} = m\mathbf{v} \quad (7.38)$$

where

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (7.39)$$

We can interpret this such that the Lorentz covariance implies that the mass-like term in the ordinary 3D linear momentum is not invariant. A better way to look at this is that $\mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$ is the covariantly correct expression for the kinetic three-momentum.

Multiplying the zeroth (time) component of the four-momentum p^μ by the scalar invariant c , we obtain

$$cp^0 = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 \quad (7.40)$$

Since this component has the dimension of energy and is the result of a Lorentz-covariant description of the motion of a particle with its kinetic momentum described by the spatial components of the four-momentum, equation (7.37) above, we interpret cp^0 as the total energy E . Hence,

$$cp^\mu = (cp^0, c\mathbf{p}) = (E, c\mathbf{p}) \quad (7.41)$$

Scalar multiplying this four-vector with itself, we obtain

$$\begin{aligned} cp_\mu cp^\mu &= c^2 \eta_{\mu\nu} p^\nu p^\mu = c^2 [(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2] \\ &= (E, -c\mathbf{p}) \cdot (E, c\mathbf{p}) = E^2 - c^2 \mathbf{p}^2 \\ &= \frac{(m_0 c^2)^2}{1 - \frac{v^2}{c^2}} \left(1 - \frac{v^2}{c^2} \right) = (m_0 c^2)^2 \end{aligned} \quad (7.42)$$

Since this is an invariant, this equation holds in any inertial frame, particularly in the frame where $\mathbf{p} = \mathbf{0}$ and there we have

$$E = m_0 c^2 \quad (7.43)$$

This is probably the most famous formula in physics history.

7.3 Covariant classical electrodynamics

Let us consider a charge density which in its rest inertial system is denoted by ρ_0 . The four-vector (in contravariant component form)

$$j^\mu = \rho_0 \frac{dx^\mu}{d\tau} = \rho_0 u^\mu = \rho_0 \gamma(c, \mathbf{v}) = (\rho c, \rho \mathbf{v}) \quad (7.44)$$

with

$$\rho = \gamma \rho_0 \quad (7.45)$$

is the *four-current*.

The contravariant form of the four-del operator $\partial^\mu = \partial/\partial x_\mu$ is defined in equation (M.93) on page 221 and its covariant counterpart $\partial_\mu = \partial/\partial x^\mu$ in equation (M.94) on page 221, respectively. As is shown in example M.8 on page 223, the *d'Alembert operator* is the scalar product of the four-del with itself:

$$\square^2 = \partial^\mu \partial_\mu = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (7.46)$$

Since it has the characteristics of a four-scalar, the d'Alembert operator is invariant and, hence, the homogeneous wave equation $\square^2 f(t, \mathbf{x}) = 0$ is Lorentz covariant.

7.3.1 The four-potential

If we introduce the *four-potential*

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A} \right) \quad (7.47)$$

where ϕ is the scalar potential and \mathbf{A} the vector potential, defined in section 3.3 on page 35, we can write the uncoupled inhomogeneous wave equations, equations (3.29) on page 37, in the following compact (and covariant) way:

$$\square^2 A^\mu = \mu_0 j^\mu \quad (7.48)$$

With the help of the above, we can formulate our electrodynamic equations covariantly. For instance, the covariant form of the *equation of continuity*, equation (1.25) on page 10 is

$$\partial_\mu j^\mu = 0 \quad (7.49)$$

and the *Lorenz-Lorentz gauge condition*, equation (3.28) on page 37, can be written

$$\partial_\mu A^\mu = 0 \quad (7.50)$$

The Lorenz-Lorentz gauge is sometimes called the *covariant gauge*. The gauge transformations (3.58) on page 43 in covariant form are

$$A^\mu \mapsto A'^\mu = A^\mu + \partial^\mu \Gamma(x^\nu) \quad (7.51)$$

If only one dimension Lorentz contracts (for instance, due to relative motion along the x direction), a 3D spatial volume element transforms according to

$$dV = d^3x = \frac{1}{\gamma} dV_0 = dV_0 \sqrt{1 - \beta^2} = dV_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (7.52)$$

where dV_0 denotes the volume element as measured in the rest system, then from equation (7.45) on the facing page we see that

$$\rho dV = \rho_0 dV_0 \quad (7.53)$$

i.e., the charge in a given volume is conserved. We can therefore conclude that the elementary electric charge is a *universal constant*.

7.3.2 The Liénard-Wiechert potentials

Let us now solve the the inhomogeneous wave equations (3.29) on page 37 in vacuum for the case of a well-localised charge q' at a source point defined by the position four-vector $x'^\mu \equiv (x'^0 = ct', x'^1, x'^2, x'^3)$. The field point (observation point) is denoted by the position four-vector $x^\mu = (x^0 = ct, x^1, x^2, x^3)$.

In the rest system we know that the solution is simply

$$(A^\mu)_0 = \left(\frac{\phi}{c}, \mathbf{A} \right)_{\mathbf{v}=\mathbf{0}} = \left(\frac{q'}{4\pi\epsilon_0 c} \frac{1}{|\mathbf{x} - \mathbf{x}'|_0}, \mathbf{0} \right) \quad (7.54)$$

where $|\mathbf{x} - \mathbf{x}'|_0$ is the usual distance from the source point to the field point, evaluated in the rest system (signified by the index '0').

Let us introduce the relative position four-vector between the source point and the field point:

$$R^\mu = x^\mu - x'^\mu = (c(t - t'), \mathbf{x} - \mathbf{x}') \quad (7.55)$$

Scalar multiplying this relative four-vector with itself, we obtain

$$R^\mu R_\mu = (c(t - t'), \mathbf{x} - \mathbf{x}') \cdot (c(t - t'), -(\mathbf{x} - \mathbf{x}')) = c^2(t - t')^2 - |\mathbf{x} - \mathbf{x}'|^2 \quad (7.56)$$

We know that in vacuum the signal (field) from the charge q' at x'^μ propagates to x^μ with the speed of light c so that

$$|\mathbf{x} - \mathbf{x}'| = c(t - t') \quad (7.57)$$

Inserting this into equation (7.56), we see that

$$R^\mu R_\mu = 0 \quad (7.58)$$

or that equation (7.55) above can be written

$$R^\mu = (|\mathbf{x} - \mathbf{x}'|, \mathbf{x} - \mathbf{x}') \quad (7.59)$$

Now we want to find the correspondence to the rest system solution, equation (7.54) on the preceding page, in an arbitrary inertial system. We note from equation (7.36) on page 143 that in the rest system

$$(u^\mu)_0 = \left(\frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)_{\mathbf{v}=\mathbf{0}} = (c, \mathbf{0}) \quad (7.60)$$

and

$$(R^\mu)_0 = (|\mathbf{x} - \mathbf{x}'|, \mathbf{x} - \mathbf{x}')_0 = (|\mathbf{x} - \mathbf{x}'|_0, (\mathbf{x} - \mathbf{x}')_0) \quad (7.61)$$

As all scalar products, $u^\mu R_\mu$ is invariant, which means that we can evaluate it in any inertial system and it will have the same value in all other inertial systems. If we evaluate it in the rest system the result is:

$$\begin{aligned} u^\mu R_\mu &= (u^\mu R_\mu)_0 = (u^\mu)_0 (R_\mu)_0 \\ &= (c, \mathbf{0}) \cdot (|\mathbf{x} - \mathbf{x}'|_0, -(\mathbf{x} - \mathbf{x}')_0) = c |\mathbf{x} - \mathbf{x}'|_0 \end{aligned} \quad (7.62)$$

We therefore see that the expression

$$A^\mu = \frac{q'}{4\pi\epsilon_0} \frac{u^\mu}{cu^\nu R_\nu} \quad (7.63)$$

subject to the condition $R^\mu R_\mu = 0$ has the proper transformation properties (proper tensor form) and reduces, in the rest system, to the solution equation (7.54) on page 145. It is therefore the correct solution, valid in any inertial system.

According to equation (7.36) on page 143 and equation (7.59) on the facing page

$$u^\nu R_\nu = \gamma(c, \mathbf{v}) \cdot (|\mathbf{x} - \mathbf{x}'|, -(\mathbf{x} - \mathbf{x}')) = \gamma(c|\mathbf{x} - \mathbf{x}'| - \mathbf{v} \cdot (\mathbf{x} - \mathbf{x}')) \quad (7.64)$$

Generalising expression (7.1) on page 135 to vector form:

$$\boldsymbol{\beta} = \beta \hat{\mathbf{v}} \stackrel{\text{def}}{=} \frac{\mathbf{v}}{c} \quad (7.65)$$

and introducing

$$s \stackrel{\text{def}}{=} |\mathbf{x} - \mathbf{x}'| - \frac{\mathbf{v} \cdot (\mathbf{x} - \mathbf{x}')}{c} \equiv |\mathbf{x} - \mathbf{x}'| - \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{x}') \quad (7.66)$$

we can write

$$u^\nu R_\nu = \gamma cs \quad (7.67)$$

and

$$\frac{u^\mu}{cu^\nu R_\nu} = \left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s} \right) \quad (7.68)$$

from which we see that the solution (7.63) can be written

$$A^\mu(x^\kappa) = \frac{q'}{4\pi\epsilon_0} \left(\frac{1}{cs}, \frac{\mathbf{v}}{c^2 s} \right) = \left(\frac{\phi}{c}, \mathbf{A} \right) \quad (7.69)$$

where in the last step the definition of the four-potential, equation (7.47) on page 144, was used. Writing the solution in the ordinary 3D way, we conclude that for a very localised charge volume, moving relative an observer with a velocity \mathbf{v} , the scalar and vector potentials are given by the expressions

$$\phi(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0} \frac{1}{s} = \frac{q'}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'| - \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{x}')} \quad (7.70a)$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{q'}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{s} = \frac{q'}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{|\mathbf{x} - \mathbf{x}'| - \boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{x}')} \quad (7.70b)$$

These potentials are the *Liénard-Wiechert potentials* that we derived in a more complicated and restricted way in subsection 6.5.1 on page 105.

7.3.3 The electromagnetic field tensor

Consider a vectorial (cross) product \mathbf{c} between two ordinary vectors \mathbf{a} and \mathbf{b} :

$$\begin{aligned}\mathbf{c} &= \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \hat{\mathbf{x}}_k \\ &= (a_2 b_3 - a_3 b_2) \hat{\mathbf{x}}_1 + (a_3 b_1 - a_1 b_3) \hat{\mathbf{x}}_2 + (a_1 b_2 - a_2 b_1) \hat{\mathbf{x}}_3\end{aligned}\quad (7.71)$$

We notice that the k th component of the vector \mathbf{c} can be represented as

$$c_k = a_i b_j - a_j b_i = c_{ij} = -c_{ji}, \quad i, j \neq k \quad (7.72)$$

In other words, the *pseudovector* $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ can be considered as an *antisymmetric tensor* of rank two. The same is true for the curl operator $\nabla \times$ operating on a polar vector. For instance, the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.73)$$

can in this tensor notation be written

$$\frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j} = -\frac{\partial B_{ij}}{\partial t} \quad (7.74)$$

We know from chapter 3 that the fields can be derived from the electromagnetic potentials in the following way:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (7.75a)$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \quad (7.75b)$$

In component form, this can be written

$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = \partial_i A_j - \partial_j A_i \quad (7.76a)$$

$$E_i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} = -\partial_i \phi - \partial_t A_i \quad (7.76b)$$

From this, we notice the clear difference between the *axial vector* (pseudovector) \mathbf{B} and the *polar vector* ('ordinary vector') \mathbf{E} .

Our goal is to express the electric and magnetic fields in a tensor form where the components are functions of the covariant form of the four-potential, equation (7.47) on page 144:

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A} \right) \quad (7.77)$$

Inspection of (7.77) and equation (7.76) above makes it natural to define the four-tensor

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (7.78)$$

This anti-symmetric (*skew-symmetric*), four-tensor of rank 2 is called the *electromagnetic field tensor* or the *Faraday tensor*. In matrix representation, the *contravariant field tensor* can be written

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \quad (7.79)$$

We note that the field tensor is a sort of four-dimensional curl of the four-potential vector A^μ .

The *covariant field tensor* is obtained from the contravariant field tensor in the usual manner by index lowering

$$F_{\mu\nu} = \eta_{\mu\kappa}\eta_{\nu\lambda}F^{\kappa\lambda} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (7.80)$$

which in matrix representation becomes

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \quad (7.81)$$

Comparing formula (7.81) with formula (7.79) above we see that the covariant field tensor is obtained from the contravariant one by a transformation $\mathbf{E} \rightarrow -\mathbf{E}$.

That the two Maxwell source equations can be written

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad (7.82)$$

is immediately observed by explicitly solving this covariant equation. Setting $\nu = 0$, corresponding to the first/leftmost column in the matrix representation of the covariant component form of the electromagnetic field tensor, $F^{\mu\nu}$, *i.e.*, equation (7.79), we see that

$$\begin{aligned} \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} &= 0 + \frac{1}{c} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \\ &= \frac{1}{c} \nabla \cdot \mathbf{E} = \mu_0 j^0 = \mu_0 c \rho \end{aligned} \quad (7.83)$$

or, equivalently (recalling that $\epsilon_0 \mu_0 = 1/c^2$),

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (7.84)$$

which we recognise as the Maxwell source equation for the electric field, equation (1.49a) on page 15.

For $\nu = 1$ [the second column in equation (7.79) on the preceding page], equation (7.82) on the previous page yields

$$\begin{aligned} \frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} &= -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + 0 + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ &= \mu_0 j^1 = \mu_0 \rho v_x \end{aligned} \quad (7.85)$$

This result can be rewritten as

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} = \mu_0 j_x \quad (7.86)$$

or, equivalently, as

$$(\nabla \times \mathbf{B})_x = \mu_0 j_x + \varepsilon_0 \mu_0 \frac{\partial E_x}{\partial t} \quad (7.87)$$

and similarly for $\nu = 2, 3$. In summary, we can write the result in three-vector form as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}(t, \mathbf{x}) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (7.88)$$

which we recognise as the Maxwell source equation for the magnetic field, equation (1.49d) on page 15.

With the help of the fully antisymmetric pseudotensor of rank 4

$$\epsilon^{\mu\nu\kappa\lambda} = \begin{cases} 1 & \text{if } \mu, \nu, \kappa, \lambda \text{ is an even permutation of } 0,1,2,3 \\ 0 & \text{if at least two of } \mu, \nu, \kappa, \lambda \text{ are equal} \\ -1 & \text{if } \mu, \nu, \kappa, \lambda \text{ is an odd permutation of } 0,1,2,3 \end{cases} \quad (7.89)$$

which can be viewed as a generalisation of the Levi-Civita tensor, formula (M.22) on page 209, we can introduce the *dual electromagnetic tensor*

$${}^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = -{}^*F^{\nu\mu} \quad (7.90)$$

with the further property

$${}^*({}^*F^{\mu\nu}) = -F^{\mu\nu} \quad (7.91)$$

In matrix representation the dual field tensor is

$$({}^*F^{\mu\nu}) = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \quad (7.92)$$

i.e., the dual field tensor is obtained from the ordinary field tensor by the *duality transformation* $\mathbf{E} \rightarrow c\mathbf{B}$ and $\mathbf{B} \rightarrow -\mathbf{E}/c$.

The covariant form of the two Maxwell field equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.93)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (7.94)$$

can then be written

$$\partial_\mu {}^\star F^{\mu\nu} = 0 \quad (7.95)$$

Explicit evaluation shows that this corresponds to (no summation!)

$$\partial_\kappa F_{\mu\nu} + \partial_\mu F_{\nu\kappa} + \partial_\nu F_{\kappa\mu} = 0 \quad (7.96)$$

sometimes referred to as the *Jacobi identity*. Hence, equation (7.82) on page 149 and equation (7.96) above constitute Maxwell's equations in four-dimensional formalism.

It is interesting to note that equation (7.82) on page 149 and

$$\partial_\mu {}^\star F^{\mu\nu} = \mu_0 j_m^\nu \quad (7.97)$$

where j_m is the *magnetic four-current*, represent the covariant form of Dirac's symmetrised Maxwell equations (1.50) on page 16.

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8

ELECTROMAGNETIC FIELDS AND PARTICLES

In previous chapters, we calculated the electromagnetic fields and potentials from arbitrary, but prescribed distributions of charges and currents. In this chapter we first study the opposite situation, *viz.*, the dynamics of charged particles in arbitrary, but prescribed electromagnetic fields. Then we go on to consider the general problem of interaction between electric and magnetic fields and electrically charged particles. The analysis is based on Lagrangian and Hamiltonian methods, is fully covariant, and yields results which are relativistically correct.

8.1 Charged particles in an electromagnetic field

We first establish a relativistically correct theory describing the motion of charged particles in prescribed electric and magnetic fields. From these equations we may then calculate the charged particle dynamics in the most general case.

8.1.1 Covariant equations of motion

We will show that for our problem we can derive the correct equations of motion by using in four-dimensional \mathbb{L}^4 a function with similar properties as a Lagrange function in 3D and then apply a variational principle. We will also show that we can find a Hamiltonian-type function in 4D and solve the corresponding

Hamilton-type equations to obtain the correct covariant formulation of classical electrodynamics.

8.1.1.1 Lagrangian formalism

In analogy with particle dynamics in 3D Euclidean space, we introduce a generalised 4D action

$$S_4 = \int L_4(x^\mu, u^\mu) d\tau \quad (8.1)$$

where $d\tau$ is the proper time defined via equation (7.18) on page 138, and L_4 acts as a kind of generalisation to the common 3D Lagrangian. As a result, the variational principle

$$\delta S_4 = \delta \int_{\tau_0}^{\tau_1} L_4(x^\mu, u^\mu) d\tau = 0 \quad (8.2)$$

with fixed endpoints τ_0, τ_1 must be fulfilled. We require that L_4 is a scalar invariant which does not contain higher than the second power of the four-velocity u^μ in order that the resulting equations of motion be linear.

According to formula (M.125) on page 227 the ordinary 3D Lagrangian is the difference between the kinetic and potential energies. A free particle has only kinetic energy. If the particle mass is m_0 then in 3D the kinetic energy is $m_0 v^2/2$. This suggests that in 4D the Lagrangian for a free particle should be

$$L_4^{\text{free}} = \frac{1}{2} m_0 u^\mu u_\mu \quad (8.3)$$

Again drawing inferences from analytical mechanics in 3D, we introduce a generalised interaction between the particles and the electromagnetic field with the help of the four-potential given by equation (7.77) on page 148 in the following way

$$L_4 = \frac{1}{2} m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu) \quad (8.4)$$

We call this the *four-Lagrangian* and shall now show how this function, together with the variation principle, formula (8.2) above, yields Lorentz covariant results which are physically correct.

The variation principle (8.2) with the 4D Lagrangian (8.4) inserted, leads to

$$\begin{aligned} \delta S_4 &= \delta \int_{\tau_0}^{\tau_1} \left(\frac{m_0}{2} u^\mu u_\mu + q u_\mu A^\mu(x^\nu) \right) d\tau \\ &= \int_{\tau_0}^{\tau_1} \left[\frac{m_0}{2} \frac{\partial(u^\mu u_\mu)}{\partial u^\mu} \delta u^\mu + q \left(A_\mu \delta u^\mu + u^\mu \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu \right) \right] d\tau \\ &= \int_{\tau_0}^{\tau_1} [m_0 u_\mu \delta u^\mu + q (A_\mu \delta u^\mu + u^\mu \partial_\nu A_\mu \delta x^\nu)] d\tau = 0 \end{aligned} \quad (8.5)$$

According to equation (7.36) on page 143, the four-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} \quad (8.6)$$

which means that we can write the variation of u^μ as a total derivative with respect to τ :

$$\delta u^\mu = \delta \left(\frac{dx^\mu}{d\tau} \right) = \frac{d}{d\tau} (\delta x^\mu) \quad (8.7)$$

Inserting this into the first two terms in the last integral in equation (8.5) on the preceding page, we obtain

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left(m_0 u_\mu \frac{d}{d\tau} (\delta x^\mu) + q A_\mu \frac{d}{d\tau} (\delta x^\mu) + q u^\mu \partial_\nu A_\mu \delta x^\nu \right) d\tau \quad (8.8)$$

Partial integration in the two first terms in the right hand member of (8.8) gives

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left(-m_0 \frac{du_\mu}{d\tau} \delta x^\mu - q \frac{dA_\mu}{d\tau} \delta x^\mu + q u^\mu \partial_\nu A_\mu \delta x^\nu \right) d\tau \quad (8.9)$$

where the integrated parts do not contribute since the variations at the endpoints vanish. A change of irrelevant summation index from μ to ν in the first two terms of the right hand member of (8.9) yields, after moving the ensuing common factor δx^ν outside the parenthesis, the following expression:

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left(-m_0 \frac{du_\nu}{d\tau} - q \frac{dA_\nu}{d\tau} + q u^\mu \partial_\nu A_\mu \right) \delta x^\nu d\tau \quad (8.10)$$

Applying well-known rules of differentiation and the expression (7.36) for the four-velocity, we can express $dA_\nu/d\tau$ as follows:

$$\frac{dA_\nu}{d\tau} = \frac{\partial A_\nu}{\partial x^\mu} \frac{dx^\mu}{d\tau} = \partial_\mu A_\nu u^\mu \quad (8.11)$$

By inserting this expression (8.11) into the second term in right-hand member of equation (8.10), and noting the common factor $q u^\mu$ of the resulting term and the last term, we obtain the final variational principle expression

$$\delta S_4 = \int_{\tau_0}^{\tau_1} \left[-m_0 \frac{du_\nu}{d\tau} + q u^\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) \right] \delta x^\nu d\tau \quad (8.12)$$

Since, according to the variational principle, this expression shall vanish and δx^ν is arbitrary between the fixed end points τ_0 and τ_1 , the expression inside $[]$ in the integrand in the right hand member of equation (8.12) above must vanish. In other words, we have found an equation of motion for a charged particle in a prescribed electromagnetic field:

$$m_0 \frac{du_\nu}{d\tau} = q u^\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) \quad (8.13)$$

With the help of formula (7.80) on page 149 for the covariant component form of the field tensor, we can express this equation in terms of the electromagnetic field tensor in the following way:

$$m_0 \frac{du_\nu}{d\tau} = qu^\mu F_{\nu\mu} \quad (8.14)$$

This is the sought-for covariant equation of motion for a particle in an electromagnetic field. It is often referred to as the *Minkowski equation*. As the reader may easily verify, the spatial part of this 4-vector equation is the covariant (relativistically correct) expression for the *Newton-Lorentz force equation*.

8.1.1.2 Hamiltonian formalism

The usual *Hamilton equations* for a 3D space are given by equation (M.136) on page 228 in appendix M. These six first-order partial differential equations are

$$\frac{\partial H}{\partial p_i} = \frac{dq_i}{dt} \quad (8.15a)$$

$$\frac{\partial H}{\partial q_i} = -\frac{dp_i}{dt} \quad (8.15b)$$

where $H(p_i, q_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$ is the ordinary 3D Hamiltonian, q_i is a *generalised coordinate* and p_i is its *canonically conjugate momentum*.

We seek a similar set of equations in 4D space. To this end we introduce a *canonically conjugate four-momentum* p^μ in an analogous way as the ordinary 3D conjugate momentum

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} \quad (8.16)$$

and utilise the four-velocity u^μ , as given by equation (7.36) on page 143, to define the *four-Hamiltonian*

$$H_4 = p^\mu u_\mu - L_4 \quad (8.17)$$

With the help of these, the position four-vector x^μ , considered as the *generalised four-coordinate*, and the invariant line element ds , defined in equation (7.18) on page 138, we introduce the following eight partial differential equations:

$$\frac{\partial H_4}{\partial p^\mu} = \frac{dx_\mu}{d\tau} \quad (8.18a)$$

$$\frac{\partial H_4}{\partial x^\mu} = -\frac{dp_\mu}{d\tau} \quad (8.18b)$$

which form the *four-dimensional Hamilton equations*.

Our strategy now is to use equation (8.16) on the facing page and equations (8.18) on the preceding page to derive an explicit algebraic expression for the canonically conjugate momentum four-vector. According to equation (7.41) on page 143, c times a four-momentum has a zeroth (time) component which we can identify with the total energy. Hence we require that the component p^0 of the conjugate four-momentum vector defined according to equation (8.16) on the preceding page be identical to the ordinary 3D Hamiltonian H divided by c and hence that this cp^0 solves the Hamilton equations, equations (8.15) on the facing page.¹ This latter consistency check is left as an exercise to the reader.

¹ Recall that in 3D, the Hamiltonian equals the total energy.

Using the definition of H_4 , equation (8.17) on the preceding page, and the expression for L_4 , equation (8.4) on page 154, we obtain

$$H_4 = p^\mu u_\mu - L_4 = p^\mu u_\mu - \frac{1}{2} m_0 u^\mu u_\mu - q u_\mu A^\mu(x^\nu) \quad (8.19)$$

Furthermore, from the definition (8.16) of the canonically conjugate four-momentum p^μ , we see that

$$p^\mu = \frac{\partial L_4}{\partial u_\mu} = \frac{\partial}{\partial u_\mu} \left(\frac{1}{2} m_0 u^\mu u_\mu + q u_\mu A^\mu(x^\nu) \right) = m_0 u^\mu + q A^\mu \quad (8.20)$$

Inserting this into (8.19), we obtain

$$H_4 = m_0 u^\mu u_\mu + q A^\mu u_\mu - \frac{1}{2} m_0 u^\mu u_\mu - q u^\mu A_\mu(x^\nu) = \frac{1}{2} m_0 u^\mu u_\mu \quad (8.21)$$

Since the four-velocity scalar-multiplied by itself is $u^\mu u_\mu = c^2$, we clearly see from equation (8.21) above that H_4 is indeed a scalar invariant, whose value is simply

$$H_4 = \frac{m_0 c^2}{2} \quad (8.22)$$

However, at the same time (8.20) provides the algebraic relationship

$$u^\mu = \frac{1}{m_0} (p^\mu - q A^\mu) \quad (8.23)$$

and if this is used in (8.21) to eliminate u^μ , one gets

$$\begin{aligned} H_4 &= \frac{m_0}{2} \left(\frac{1}{m_0} (p^\mu - q A^\mu) \frac{1}{m_0} (p_\mu - q A_\mu) \right) \\ &= \frac{1}{2m_0} (p^\mu - q A^\mu) (p_\mu - q A_\mu) \\ &= \frac{1}{2m_0} (p^\mu p_\mu - 2q A^\mu p_\mu + q^2 A^\mu A_\mu) \end{aligned} \quad (8.24)$$

That this four-Hamiltonian yields the correct covariant equation of motion can be seen by inserting it into the four-dimensional Hamilton equations (8.18) and using the relation (8.23):

$$\begin{aligned}
 \frac{\partial H_4}{\partial x^\mu} &= -\frac{q}{m_0}(p^\nu - qA^\nu)\frac{\partial A_\nu}{\partial x^\mu} \\
 &= -\frac{q}{m_0}m_0u^\nu\frac{\partial A_\nu}{\partial x^\mu} \\
 &= -qu^\nu\frac{\partial A_\nu}{\partial x^\mu} \\
 &= -\frac{dp_\mu}{d\tau} = -m_0\frac{du_\mu}{d\tau} - q\frac{\partial A_\mu}{\partial x^\nu}u^\nu
 \end{aligned} \tag{8.25}$$

where in the last step equation (8.20) on the preceding page was used. Rearranging terms, and using equation (7.80) on page 149, we obtain

$$m_0\frac{du_\mu}{d\tau} = qu^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = qu^\nu F_{\mu\nu} \tag{8.26}$$

which is identical to the covariant equation of motion equation (8.14) on page 156. We can therefore safely conclude that the Hamiltonian in question yields correct results.

Recalling expression (7.47) on page 144 for the four-potential, and representing the canonically conjugate four-momentum as $p^\mu = (p^0, \mathbf{p})$, we obtain the following scalar products:

$$p^\mu p_\mu = (p^0)^2 - (\mathbf{p})^2 \tag{8.27a}$$

$$A^\mu p_\mu = \frac{1}{c}\phi p^0 - (\mathbf{p} \cdot \mathbf{A}) \tag{8.27b}$$

$$A^\mu A_\mu = \frac{1}{c^2}\phi^2 - (\mathbf{A})^2 \tag{8.27c}$$

Inserting these explicit expressions into equation (8.24) on the preceding page, and using the fact that H_4 is equal to the scalar value $m_0c^2/2$, as derived in equation (8.22) on the previous page, we obtain the equation

$$\frac{m_0c^2}{2} = \frac{1}{2m_0} \left[(p^0)^2 - (\mathbf{p})^2 - \frac{2}{c}q\phi p^0 + 2q(\mathbf{p} \cdot \mathbf{A}) + \frac{q^2}{c^2}\phi^2 - q^2(\mathbf{A})^2 \right] \tag{8.28}$$

which is a second-order algebraic equation in p^0 :

$$(p^0)^2 - \frac{2q}{c}\phi p^0 - \underbrace{[(\mathbf{p})^2 - 2q\mathbf{p} \cdot \mathbf{A} + q^2(\mathbf{A})^2]}_{(\mathbf{p}-q\mathbf{A})^2} + \frac{q^2}{c^2}\phi^2 - m_0^2c^2 = 0 \tag{8.29}$$

with two possible solutions

$$p^0 = \frac{q}{c}\phi \pm \sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2 c^2} \quad (8.30)$$

Since the zeroth component (time component) p^0 of a four-momentum vector p^μ multiplied by c represents the energy [cf. equation (7.41) on page 143], the positive solution in equation (8.30) above must be identified with the ordinary Hamilton function H divided by c . Consequently,

$$H \equiv cp^0 = q\phi + c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2 c^2} \quad (8.31)$$

is the ordinary 3D Hamilton function for a charged particle moving in scalar and vector potentials associated with prescribed electric and magnetic fields.

The ordinary Lagrange and Hamilton functions L and H are related to each other by the 3D transformation [cf. the 4D transformation (8.17) between L_4 and H_4]

$$L = \mathbf{p} \cdot \mathbf{v} - H \quad (8.32)$$

Using the the explicit expressions given by equation (8.31) and equation (8.32), we obtain the explicit expression for the ordinary 3D Lagrange function

$$L = \mathbf{p} \cdot \mathbf{v} - q\phi - c\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m_0^2 c^2} \quad (8.33)$$

and if we make the identification

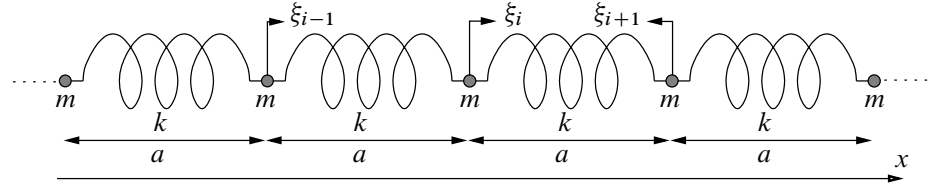
$$\mathbf{p} - q\mathbf{A} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\mathbf{v} \quad (8.34)$$

where the quantity $m\mathbf{v}$ is the usual *kinetic momentum*, we can rewrite this expression for the ordinary Lagrangian as follows:

$$\begin{aligned} L &= q\mathbf{A} \cdot \mathbf{v} + mv^2 - q\phi - c\sqrt{m^2 v^2 + m_0^2 c^2} \\ &= mv^2 - q(\phi - \mathbf{A} \cdot \mathbf{v}) - mc^2 = -q\phi + q\mathbf{A} \cdot \mathbf{v} - m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \quad (8.35)$$

What we have obtained is the relativistically correct (covariant) expression for the Lagrangian describing the mechanical motion of a charged particle in scalar and vector potentials associated with prescribed electric and magnetic fields.

Figure 8.1: A one-dimensional chain consisting of N discrete, identical mass points m , connected to their neighbours with identical, ideal springs with spring constants k . The equilibrium distance between the neighbouring mass points is a and $\xi_{i-1}(t)$, $\xi_i(t)$, $\xi_{i+1}(t)$ are the instantaneous deviations, along the x axis, of positions of the $(i-1)$ th, i th, and $(i+1)$ th mass point, respectively.



8.2 Covariant field theory

So far, we have considered two classes of problems. Either we have calculated the fields from given, prescribed distributions of charges and currents, or we have derived the equations of motion for charged particles in given, prescribed fields. Let us now put the fields and the particles on an equal footing and present a theoretical description which treats the fields, the particles, and their interactions in a unified way. This involves transition to a field picture with an infinite number of degrees of freedom. We shall first consider a simple mechanical problem whose solution is well known. Then, drawing inferences from this model problem, we apply a similar view on the electromagnetic problem.

8.2.1 Lagrange-Hamilton formalism for fields and interactions

Consider the situation, illustrated in figure 8.1, with N identical mass points, each with mass m and connected to its neighbour along a one-dimensional straight line, which we choose to be the x axis, by identical ideal springs with spring constants k (*Hooke's law*). At equilibrium the mass points are at rest, distributed evenly with a distance a to their two nearest neighbours so that the equilibrium coordinate for the i th particle is $\mathbf{x}_i = ia\hat{\mathbf{x}}$. After perturbation, the motion of mass point i will be a one-dimensional oscillatory motion along $\hat{\mathbf{x}}$. Let us denote the deviation for mass point i from its equilibrium position by $\xi_i(t)\hat{\mathbf{x}}$.

As is well known, the solution to this mechanical problem can be obtained if we can find a *Lagrangian* (*Lagrange function*) L which satisfies the variational equation

$$\delta \int L(\xi_i, \dot{\xi}_i, t) dt = 0 \quad (8.36)$$

According to equation (M.125) on page 227, the Lagrangian is $L = T - V$ where T denotes the *kinetic energy* and V the *potential energy* of a classical

mechanical system with *conservative forces*. In our case the Lagrangian is

$$L = \frac{1}{2} \sum_{i=1}^N \left[m \dot{\xi}_i^2 - k(\xi_{i+1} - \xi_i)^2 \right] \quad (8.37)$$

Let us write the Lagrangian, as given by equation (8.37) above, in the following way:

$$L = \sum_{i=1}^N a \mathcal{L}_i \quad (8.38)$$

where

$$\mathcal{L}_i = \frac{1}{2} \left[\frac{m}{a} \dot{\xi}_i^2 - ka \left(\frac{\xi_{i+1} - \xi_i}{a} \right)^2 \right] \quad (8.39)$$

is the so called linear *Lagrange density*, measured in J m^{-1} . If we now let $N \rightarrow \infty$ and, at the same time, let the springs become infinitesimally short according to the following scheme:

$$a \rightarrow dx \quad (8.40a)$$

$$\frac{m}{a} \rightarrow \frac{dm}{dx} = \mu \quad \text{linear mass density} \quad (8.40b)$$

$$ka \rightarrow Y \quad \text{Young's modulus} \quad (8.40c)$$

$$\frac{\xi_{i+1} - \xi_i}{a} \rightarrow \frac{\partial \xi}{\partial x} \quad (8.40d)$$

we obtain

$$L = \int \mathcal{L} dx \quad (8.41)$$

where

$$\mathcal{L} \left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x}, t \right) = \frac{1}{2} \left[\mu \left(\frac{\partial \xi}{\partial t} \right)^2 - Y \left(\frac{\partial \xi}{\partial x} \right)^2 \right] \quad (8.42)$$

Notice how we made a transition from a discrete description, in which the mass points were identified by a discrete integer variable $i = 1, 2, \dots, N$, to a continuous description, where the infinitesimal mass points were instead identified by a continuous real parameter x , namely their position along $\hat{\mathbf{x}}$.

A consequence of this transition is that the number of degrees of freedom for the system went from the finite number N to infinity! Another consequence is that \mathcal{L} has now become dependent also on the partial derivative with respect to x of the ‘field coordinate’ ξ . But, as we shall see, the transition is well worth the cost because it allows us to treat all fields, be it classical scalar or vectorial fields,

or wave functions, spinors and other fields that appear in quantum physics, on an equal footing.

Under the assumption of time independence and fixed endpoints, the variation principle (8.36) on page 160 yields:

$$\begin{aligned}
 & \delta \int L dt \\
 &= \delta \iint \mathcal{L} \left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x} \right) dx dt \\
 &= \iint \left[\frac{\partial \mathcal{L}}{\partial \xi} \delta \xi + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial t} \right)} \delta \left(\frac{\partial \xi}{\partial t} \right) + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x} \right)} \delta \left(\frac{\partial \xi}{\partial x} \right) \right] dx dt \\
 &= 0
 \end{aligned} \tag{8.43}$$

As before, the last integral can be integrated by parts. This results in the expression

$$\iint \left[\frac{\partial \mathcal{L}}{\partial \xi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial t} \right)} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x} \right)} \right) \right] \delta \xi dx dt = 0 \tag{8.44}$$

where the variation is arbitrary (and the endpoints fixed). This means that the integrand itself must vanish. If we introduce the *functional derivative*

$$\frac{\delta \mathcal{L}}{\delta \xi} = \frac{\partial \mathcal{L}}{\partial \xi} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x} \right)} \right) \tag{8.45}$$

we can express this as

$$\frac{\delta \mathcal{L}}{\delta \xi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial t} \right)} \right) = 0 \tag{8.46}$$

which is the one-dimensional *Euler-Lagrange equation*.

Inserting the linear mass point chain Lagrangian density, equation (8.42) on the previous page, into equation (8.46), we obtain the equation of motion for our one-dimensional linear mechanical structure. It is:

$$\mu \frac{\partial^2 \xi}{\partial t^2} - Y \frac{\partial^2 \xi}{\partial x^2} = 0 \tag{8.47a}$$

or

$$\left(\frac{1}{v_\phi^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \xi = 0 \tag{8.47b}$$

i.e., the one-dimensional wave equation for compression waves which propagate with phase speed $v_\varphi = \sqrt{Y/\mu}$ along the linear structure.

A generalisation of the above 1D results to a three-dimensional continuum is straightforward. For this 3D case we get the variational principle

$$\begin{aligned} \delta \int L \, dt &= \delta \iiint \mathcal{L} \, d^3x \, dt = \delta \int \mathcal{L} \left(\xi, \frac{\partial \xi}{\partial x^\mu} \right) d^4x \\ &= \iiint \left[\frac{\partial \mathcal{L}}{\partial \xi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x^\mu} \right)} \right) \right] \delta \xi \, d^4x = 0 \end{aligned} \quad (8.48)$$

where the variation $\delta \xi$ is arbitrary and the endpoints are fixed. This means that the integrand itself must vanish:

$$\frac{\partial \mathcal{L}}{\partial \xi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x^\mu} \right)} \right) = \frac{\partial \mathcal{L}}{\partial \xi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \xi)} \right) = 0 \quad (8.49)$$

This constitutes the four-dimensional *Euler-Lagrange equations*.

Introducing the *three-dimensional functional derivative*

$$\frac{\delta \mathcal{L}}{\delta \xi} = \frac{\partial \mathcal{L}}{\partial \xi} - \frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial x^i} \right)} \right) \quad (8.50)$$

we can express this as

$$\frac{\delta \mathcal{L}}{\delta \xi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial t} \right)} \right) = 0 \quad (8.51)$$

In analogy with particle mechanics (finite number of degrees of freedom), we may introduce the *canonically conjugate momentum density*

$$\pi(x^\mu) = \pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \xi}{\partial t} \right)} \quad (8.52)$$

and define the *Hamilton density*

$$\mathcal{H} \left(\pi, \xi, \frac{\partial \xi}{\partial x^i}; t \right) = \pi \frac{\partial \xi}{\partial t} - \mathcal{L} \left(\xi, \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x^i} \right) \quad (8.53)$$

If, as usual, we differentiate this expression and identify terms, we obtain the following *Hamilton density equations*

$$\frac{\partial \mathcal{H}}{\partial \pi} = \frac{\partial \xi}{\partial t} \quad (8.54a)$$

$$\frac{\delta \mathcal{H}}{\delta \xi} = -\frac{\partial \pi}{\partial t} \quad (8.54b)$$

The Hamilton density functions are in many ways similar to the ordinary Hamilton functions for a system of a finite number of particles and lead to similar results. However, they describe the dynamics of a continuous system of infinitely many degrees of freedom.

8.2.1.1 The electromagnetic field

Above, when we described the mechanical field, we used a scalar field $\xi(t, \mathbf{x})$. If we want to describe the electromagnetic field in terms of a Lagrange density \mathcal{L} and Euler-Lagrange equations, it comes natural to express \mathcal{L} in terms of the four-potential $A^\mu(x^\kappa)$.

The entire system of particles and fields consists of a mechanical part, a field part and an interaction part. We therefore assume that the total Lagrange density \mathcal{L}^{tot} for this system can be expressed as

$$\mathcal{L}^{\text{tot}} = \mathcal{L}^{\text{mech}} + \mathcal{L}^{\text{inter}} + \mathcal{L}^{\text{em}} \quad (8.55)$$

where the mechanical part has to do with the particle motion (kinetic energy). It is given by L_4/V where L_4 is given by equation (8.3) on page 154 and V is the volume. Expressed in the *rest mass density* ϱ_0 , the *mechanical Lagrange density* can be written

$$\mathcal{L}^{\text{mech}} = \frac{1}{2} \varrho_0 u^\mu u_\mu \quad (8.56)$$

The $\mathcal{L}^{\text{inter}}$ part describes the interaction between the charged particles and the external electromagnetic field. A convenient expression for this *interaction Lagrange density* is

$$\mathcal{L}^{\text{inter}} = j^\mu A_\mu \quad (8.57)$$

For the field part \mathcal{L}^{em} we choose the difference between magnetic and electric energy density (in analogy with the difference between kinetic and potential energy in a mechanical field). With the help of the field tensor, we express this *field Lagrange density* as

$$\mathcal{L}^{\text{em}} = \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad (8.58)$$

so that the total Lagrangian density can be written

$$\mathcal{L}^{\text{tot}} = \frac{1}{2} \varrho_0 u^\mu u_\mu + j^\mu A_\mu + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad (8.59)$$

From this we can calculate all physical quantities.

Using \mathcal{L}^{tot} in the 3D Euler-Lagrange equations, equation (8.49) on page 163 (with ξ replaced by A_ν), we can derive the dynamics for the whole system. For instance, the electromagnetic part of the Lagrangian density

$$\mathcal{L}^{\text{EM}} = \mathcal{L}^{\text{inter}} + \mathcal{L}^{\text{em}} = j^\nu A_\nu + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad (8.60)$$

inserted into the Euler-Lagrange equations, expression (8.49) on page 163, yields two of Maxwell's equations. To see this, we note from equation (8.60) and the results in Example 8.1 that

$$\frac{\partial \mathcal{L}^{\text{EM}}}{\partial A_\nu} = j^\nu \quad (8.61)$$

Furthermore,

$$\begin{aligned} \partial_\mu \left[\frac{\partial \mathcal{L}^{\text{EM}}}{\partial (\partial_\mu A_\nu)} \right] &= \frac{1}{4\mu_0} \partial_\mu \left[\frac{\partial}{\partial (\partial_\mu A_\nu)} (F^{\kappa\lambda} F_{\kappa\lambda}) \right] \\ &= \frac{1}{4\mu_0} \partial_\mu \left\{ \frac{\partial}{\partial (\partial_\mu A_\nu)} [(\partial^\kappa A^\lambda - \partial^\lambda A^\kappa)(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa)] \right\} \\ &= \frac{1}{4\mu_0} \partial_\mu \left\{ \frac{\partial}{\partial (\partial_\mu A_\nu)} \left[\partial^\kappa A^\lambda \partial_\kappa A_\lambda - \partial^\kappa A^\lambda \partial_\lambda A_\kappa \right. \right. \\ &\quad \left. \left. - \partial^\lambda A^\kappa \partial_\kappa A_\lambda + \partial^\lambda A^\kappa \partial_\lambda A_\kappa \right] \right\} \\ &= \frac{1}{2\mu_0} \partial_\mu \left[\frac{\partial}{\partial (\partial_\mu A_\nu)} (\partial^\kappa A^\lambda \partial_\kappa A_\lambda - \partial^\kappa A^\lambda \partial_\lambda A_\kappa) \right] \end{aligned} \quad (8.62)$$

But

$$\begin{aligned} \frac{\partial}{\partial (\partial_\mu A_\nu)} (\partial^\kappa A^\lambda \partial_\kappa A_\lambda) &= \partial^\kappa A^\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial^\kappa A^\lambda \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\kappa A_\lambda + \partial_\kappa A_\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \eta^{\kappa\alpha} \partial_\alpha \eta^{\lambda\beta} A_\beta \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\kappa A_\lambda + \eta^{\kappa\alpha} \eta^{\lambda\beta} \partial_\kappa A_\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\alpha A_\beta \\ &= \partial^\kappa A^\lambda \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\kappa A_\lambda + \partial^\alpha A^\beta \frac{\partial}{\partial (\partial_\mu A_\nu)} \partial_\alpha A_\beta \\ &= 2\partial^\mu A^\nu \end{aligned} \quad (8.63)$$

Similarly,

$$\frac{\partial}{\partial (\partial_\mu A_\nu)} (\partial^\kappa A^\lambda \partial_\lambda A_\kappa) = 2\partial^\nu A^\mu \quad (8.64)$$

so that

$$\partial_\mu \left[\frac{\partial \mathcal{L}^{\text{EM}}}{\partial (\partial_\mu A_\nu)} \right] = \frac{1}{\mu_0} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{\mu_0} \partial_\mu F^{\mu\nu} \quad (8.65)$$

This means that the Euler-Lagrange equations, expression (8.49) on page 163, for the Lagrangian density \mathcal{L}^{EM} and with A_ν as the field quantity become

$$\frac{\partial \mathcal{L}^{\text{EM}}}{\partial A_\nu} - \partial_\mu \left[\frac{\partial \mathcal{L}^{\text{EM}}}{\partial (\partial_\mu A_\nu)} \right] = j^\nu - \frac{1}{\mu_0} \partial_\mu F^{\mu\nu} = 0 \quad (8.66)$$

or

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad (8.67)$$

which, according to equation (7.82) on page 149, is a Lorentz covariant formulation of Maxwell's source equations.

EXAMPLE 8.1 ▷ Field energy difference expressed in the field tensor

Show, by explicit calculation, that

$$\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} \left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2 \right) \quad (8.68)$$

i.e., the difference between the magnetic and electric field energy densities.

From formula (7.79) on page 149 we recall that

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \quad (8.69)$$

and from formula (7.81) on page 149 that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \quad (8.70)$$

where μ denotes the row number and ν the column number. Then, Einstein summation and direct substitution yields

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} &= F^{00} F_{00} + F^{01} F_{01} + F^{02} F_{02} + F^{03} F_{03} \\ &\quad + F^{10} F_{10} + F^{11} F_{11} + F^{12} F_{12} + F^{13} F_{13} \\ &\quad + F^{20} F_{20} + F^{21} F_{21} + F^{22} F_{22} + F^{23} F_{23} \\ &\quad + F^{30} F_{30} + F^{31} F_{31} + F^{32} F_{32} + F^{33} F_{33} \\ &= 0 - E_x^2/c^2 - E_y^2/c^2 - E_z^2/c^2 \\ &\quad - E_x^2/c^2 + 0 + B_z^2 + B_y^2 \\ &\quad - E_y^2/c^2 + B_z^2 + 0 + B_x^2 \\ &\quad - E_z^2/c^2 + B_y^2 + B_x^2 + 0 \\ &= -2E_x^2/c^2 - 2E_y^2/c^2 - 2E_z^2/c^2 + 2B_x^2 + 2B_y^2 + 2B_z^2 \\ &= -2E^2/c^2 + 2B^2 = 2(B^2 - E^2/c^2) \end{aligned} \quad (8.71)$$

or [cf. equation (2.16a) on page 23]

$$\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} \left(\frac{B^2}{\mu_0} - \frac{1}{c^2 \mu_0} E^2 \right) = \frac{1}{2} \left(\frac{B^2}{\mu_0} - \varepsilon_0 E^2 \right) = -\frac{\varepsilon_0}{2} (E^2 - c^2 B^2) \quad (8.72)$$

where, in the last step, the identity $\varepsilon_0 \mu_0 = 1/c^2$ was used.

QED ■

—End of example 8.1◁

8.2.1.2 Other fields

In general, the dynamic equations for most any fields, and not only electromagnetic ones, can be derived from a Lagrangian density together with a variational principle (the Euler-Lagrange equations). Both linear and non-linear fields are studied with this technique. As a simple example, consider a real, scalar field ξ which has the following Lagrange density:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \xi \partial^\mu \xi - m^2 \xi^2) \quad (8.73)$$

Insertion into the 1D Euler-Lagrange equation, equation (8.46) on page 162, yields the dynamic equation

$$(\Box^2 - m^2)\xi = 0 \quad (8.74)$$

with the solution

$$\xi = e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \quad (8.75)$$

which describes the *Yukawa meson field* for a scalar meson with mass m . With

$$\pi = \frac{1}{c^2} \frac{\partial \xi}{\partial t} \quad (8.76)$$

we obtain the Hamilton density

$$\mathcal{H} = \frac{1}{2} [c^2 \pi^2 + (\nabla \xi)^2 + m^2 \xi^2] \quad (8.77)$$

which is positive definite.

Another Lagrangian density which has attracted quite some interest is the *Proca Lagrangian*

$$\mathcal{L}^{\text{EM}} = \mathcal{L}^{\text{inter}} + \mathcal{L}^{\text{em}} = j^\nu A_\nu + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + m^2 A^\mu A_\mu \quad (8.78)$$

which leads to the dynamic equation

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = \mu_0 j^\nu \quad (8.79)$$

This equation describes an electromagnetic field with a mass, or, in other words, *massive photons*. If massive photons do exist, large-scale magnetic fields, including those of the earth and galactic spiral arms, should be significantly modified from what they are to yield measurable discrepancies from their usual form. Space experiments of this kind on board satellites have led to stringent upper bounds on the photon mass. If the photon really has a mass, it will have an impact on electrodynamics as well as on cosmology and astrophysics.

8.3 Bibliography

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9

ELECTROMAGNETIC FIELDS AND MATTER

The microscopic Maxwell equations derived in chapter 1, which in chapter 2 were chosen as the axiomatic basis for the treatment in the remainder of the book, are valid on all scales where a classical description is good. They provide a correct physical picture for arbitrary field and source distributions, on macroscopic and, under certain assumptions, microscopic scales. A more complete and accurate theory, valid also when quantum effects are significant, is provided by *quantum electrodynamics*. *QED* gives a consistent description of how electromagnetic fields are quantised into *photons* and describes their intrinsic and extrinsic properties. However, this theory is beyond the scope of the current book.

In a material medium, be it in a solid, fluid or gaseous state or a combination thereof, it is sometimes convenient to replace the Maxwell-Lorentz equations (2.1) on page 20 by the corresponding macroscopic Maxwell equations in which auxiliary, derived fields are introduced. These auxiliary fields, *viz.*, the *electric displacement vector* \mathbf{D} (measured in C m^{-2}) and the *magnetising field* \mathbf{H} (measured in A m^{-1}), incorporate intrinsic electromagnetic properties of macroscopic matter, or properties that appear when the medium is immersed fully or partially in an electromagnetic field. Consequently, they represent, respectively, electric and magnetic field quantities in which, in an average sense, the material properties of the substances are already included. In the most general case, these derived fields are complicated, possibly non-local and nonlinear, functions of

the primary fields \mathbf{E} and \mathbf{B} :

$$\mathbf{D} = \mathbf{D}(t, \mathbf{x}; \mathbf{E}, \mathbf{B}) \quad (9.1a)$$

$$\mathbf{H} = \mathbf{H}(t, \mathbf{x}; \mathbf{E}, \mathbf{B}) \quad (9.1b)$$

An example of this are *chiral media*.

A general treatment of these fields will not be included here. Only simplified, but important and illuminating examples will be given.

9.1 Maxwell's macroscopic theory

Under certain conditions, for instance for small magnitudes of the primary field strengths \mathbf{E} and \mathbf{B} , we may assume that the response of a substance to the fields can be approximated by a linear one so that

$$\mathbf{D} \approx \varepsilon \mathbf{E} \quad (9.2)$$

$$\mathbf{H} \approx \mu^{-1} \mathbf{B} \quad (9.3)$$

i.e., that the *electric displacement vector* $\mathbf{D}(t, \mathbf{x})$ is only linearly dependent on the *electric field* $\mathbf{E}(t, \mathbf{x})$, and the *magnetising field* $\mathbf{H}(t, \mathbf{x})$ is only linearly dependent on the *magnetic field* $\mathbf{B}(t, \mathbf{x})$. In this chapter we derive these linearised forms, and then consider a simple, explicit linear model for a medium from which we derive the expression for the *dielectric permittivity* $\varepsilon(t, \mathbf{x})$, the *magnetic susceptibility* $\mu(t, \mathbf{x})$, and the *refractive index* or *index of refraction* $n(t, \mathbf{x})$ of this medium. Using this simple model, we study certain interesting aspects of the propagation of electromagnetic particles and waves in the medium.

9.1.1 Polarisation and electric displacement

By writing the first microscopic Maxwell-Lorentz equation (2.1a) on page 20 as in equation (6.25) on page 88, *i.e.*, in a form where the total charge density $\rho(t, \mathbf{x})$ is split into the charge density for free, ‘true’ charges, ρ^{true} , and the charge density, ρ^{pol} , for bound *polarisation charges* induced by the applied field \mathbf{E} , as

$$\nabla \cdot \mathbf{E} = \frac{\rho^{\text{total}}(t, \mathbf{x})}{\varepsilon_0} = \frac{\rho^{\text{true}}(t, \mathbf{x}) + \rho^{\text{pol}}(t, \mathbf{x})}{\varepsilon_0} = \frac{\rho^{\text{true}}(t, \mathbf{x}) - \nabla \cdot \mathbf{P}(t, \mathbf{x})}{\varepsilon_0} \quad (9.4)$$

and at the same time introducing the *electric displacement vector* (C m^{-2})

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon_0 \mathbf{E}(t, \mathbf{x}) + \mathbf{P}(t, \mathbf{x}) \quad (9.5)$$

one can reshuffle expression (9.4) on the preceding page to obtain

$$\nabla \cdot [\varepsilon_0 \mathbf{E}(t, \mathbf{x}) + \mathbf{P}(t, \mathbf{x})] = \nabla \cdot \mathbf{D}(t, \mathbf{x}) = \rho^{\text{true}}(t, \mathbf{x}) \quad (9.6)$$

This is one of the original macroscopic Maxwell equations. It is important to remember that only the induced electric dipole moment of matter, subject to the field \mathbf{E} , was included in the above separation into true and induced charge densities. Contributions to \mathbf{D} from higher-order electric moments were neglected. This is one of the approximations assumed.

Another approximation is the assumption that there exists a simple linear relationship between \mathbf{P} and \mathbf{E} in the material medium under consideration

$$\mathbf{P}(t, \mathbf{x}) = \varepsilon_0 \chi_e(t, \mathbf{x}) \mathbf{E}(t, \mathbf{x}) \quad (9.7)$$

This approximation is often valid for regular media if the field strength $|\mathbf{E}|$ is low enough. Here the variations in time and space of the the material dependent *electric susceptibility*, χ_e , are usually on much slower and longer scales than for \mathbf{E} itself.¹ Inserting the approximation (9.7) into equation (9.5) on the facing page, we can write the latter

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon(t, \mathbf{x}) \mathbf{E}(t, \mathbf{x}) \quad (9.8)$$

where, approximately,

$$\varepsilon(t, \mathbf{x}) = \varepsilon_0 [1 + \chi_e(t, \mathbf{x})] = \varepsilon_0 \kappa_e(t, \mathbf{x}) \quad (9.9)$$

For an electromagnetically *anisotropic medium* such as a *magnetised plasma* or a *birefringent crystal*, the susceptibility χ_e or, equivalently the *relative dielectric permittivity*

$$\kappa_e(t, \mathbf{x}) = \frac{\varepsilon(t, \mathbf{x})}{\varepsilon_0} = 1 + \chi_e(t, \mathbf{x}) \quad (9.10)$$

will have to be replaced by a tensor. This would still describe a linear relationship between \mathbf{E} and \mathbf{P} but one where the linear proportionality factor, or, as we shall call it, the *dispersive property* of the medium, is dependent on the direction in space.

In general, however, the relationship is not of a simple linear form as in equation (9.7) above but non-linear terms are important. In such a situation the principle of superposition is no longer valid and non-linear effects such as frequency conversion and mixing can be expected.²

9.1.2 Magnetisation and the magnetising field

An analysis of the properties of magnetic media and the associated currents shows that three such types of currents exist:

¹ The fact that the relation between the dipole moment per unit volume \mathbf{P} and the applied electric field \mathbf{E} is local in time and space is yet another approximation assumed in macroscopic Maxwell theory.

² The nonlinearity of semiconductor diodes is used, *e.g.*, in radio receivers to convert high radio frequencies into lower ones, or into the audible spectrum. These techniques are called *heterodyning* and *demodulation*, respectively. Another example of the nonlinear response of a medium is the *Kerr effect*.

1. In analogy with true charges for the electric case, we may have true currents \mathbf{j}^{true} , *i.e.*, a physical transport of true (free) charges.
2. In analogy with the electric polarisation \mathbf{P} there may be a form of charge transport associated with the changes of the polarisation with time. Such currents, induced by an external field, are called *polarisation currents* and are identified with $\partial\mathbf{P}/\partial t$.
3. There may also be intrinsic currents of a microscopic, often atomistic, nature that are inaccessible to direct observation, but which may produce net effects at discontinuities and boundaries. These *magnetisation currents* are denoted \mathbf{j}^{M} .

Magnetic monopoles have not yet been unambiguously identified in experiments. So there is no correspondence in the magnetic case to the electric monopole moment, formula (6.16a) on page 86. The lowest order magnetic moment, corresponding to the electric dipole moment, formula (6.16b) on page 86, is the *magnetic dipole moment* [*cf.* the Fourier component expression (6.64) on page 96]

$$\mathbf{m}(t) = \frac{1}{2} \int_{V'} d^3x' (\mathbf{x}' - \mathbf{x}_0) \times \mathbf{j}(t', \mathbf{x}') \quad (9.11)$$

Analogously to the electric case, one may, for a distribution of magnetic dipole moments in a volume, describe this volume in terms of its *magnetisation*, or *magnetic dipole moment per unit volume*, \mathbf{M} . Via the definition of the vector potential \mathbf{A} one can show that the magnetisation current and the magnetisation is simply related:

$$\mathbf{j}^{\text{M}} = \nabla \times \mathbf{M} \quad (9.12)$$

In a stationary medium we therefore have a total current which is (approximately) the sum of the three currents enumerated above:

$$\mathbf{j}^{\text{total}} = \mathbf{j}^{\text{true}} + \frac{\partial\mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \quad (9.13)$$

One might then be led to think that the right-hand side (RHS) of the $\nabla \times \mathbf{B}$ Maxwell equation (2.1d) on page 20 should be

$$\text{RHS} = \mu_0 \left(\mathbf{j}^{\text{true}} + \frac{\partial\mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right)$$

However, moving the term $\nabla \times \mathbf{M}$ from the right hand side (RHS) to the left hand side (LHS) and introducing the *magnetising field* (*magnetic field intensity*, *Ampère-turn density*) as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (9.14)$$

and using the definition for \mathbf{D} , equation (9.5) on page 170, we find that

$$\begin{aligned}\text{LHS} &= \nabla \times \mathbf{H} \\ \text{RHS} &= \mathbf{j}^{\text{true}} + \frac{\partial \mathbf{P}}{\partial t} = \mathbf{j}^{\text{true}} + \frac{\partial \mathbf{D}}{\partial t} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

Hence, in this simplistic view, we would pick up a term $-\epsilon_0 \partial \mathbf{E} / \partial t$ which makes the equation inconsistent: the divergence of the left hand side vanishes while the divergence of the right hand side does not! Maxwell realised this and to overcome this inconsistency he was forced to add his famous displacement current term which precisely compensates for the last term the RHS expression.³ In chapter 1, we discussed an alternative way, based on the postulate of conservation of electric charge, to introduce the displacement current.

We may, in analogy with the electric case, introduce a *magnetic susceptibility* for the medium. Denoting it χ_m , we can write

$$\mathbf{H}(t, \mathbf{x}) = \mu^{-1}(t, \mathbf{x}) \mathbf{B}(t, \mathbf{x}) \quad (9.15)$$

where, approximately,

$$\mu(t, \mathbf{x}) = \mu_0 [1 + \chi_m(t, \mathbf{x})] = \mu_0 \kappa_m(t, \mathbf{x}) \quad (9.16)$$

and

$$\kappa_m(t, \mathbf{x}) = \frac{\mu(t, \mathbf{x})}{\mu_0} = 1 + \chi_m(t, \mathbf{x}) \quad (9.17)$$

is the *relative permeability*. In the case of anisotropy, κ_m will be a tensor, but it is still only a linear approximation.⁴

9.1.3 Macroscopic Maxwell equations

Field equations, expressed in terms of the derived, and therefore in principle superfluous, field quantities \mathbf{D} and \mathbf{H} are obtained from the Maxwell-Lorentz microscopic equations (2.1) on page 20, by replacing the \mathbf{E} and \mathbf{B} in the two *source equations* by using the approximate relations formula (9.8) on page 171 and formula (9.15) above, respectively:

$$\nabla \cdot \mathbf{D} = \rho^{\text{true}} \quad (9.18a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (9.18b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (9.18c)$$

$$\nabla \times \mathbf{H} = \mathbf{j}^{\text{true}} + \frac{\partial \mathbf{D}}{\partial t} \quad (9.18d)$$

³ This term, which ensures that electric charge is conserved also in non-stationary problems, is the one that makes it possible to turn the Maxwell equations into wave equations (see chapter 2) and, hence, the term that, in a way, is the basis for radio communications and other engineering applications of the theory.

⁴ This is the case for the *Hall effect* which produces a potential difference across an electric conduction current channel, orthogonal to this current, in the presence of an external magnetic field which is likewise perpendicular to the current. This effect was discovered 1879 by the US physicist EDWIN HERBERT HALL (1855–1938).

This set of differential equations, originally derived by Maxwell himself, are called *Maxwell's macroscopic equations*. Together with the boundary conditions and the constitutive relations, they describe uniquely (but only approximately) the properties of the electric and magnetic fields in matter and are convenient to use in certain simple cases, particularly in engineering applications. However, the structure of these equations rely on certain linear approximations and there are many situations where they are not useful or even applicable. Therefore, these equations, which are the original Maxwell equations (albeit expressed in their modern vector form as introduced by OLIVER HEAVISIDE), should be used with some care.⁵

⁵ It should be recalled that Maxwell formulated these macroscopic equations before it was known that matter has an atomistic structure and that there exist electrically charged particles such as electrons and protons, which possess a quantum mechanical property called spin that gives rise to magnetism!

9.2 Phase velocity, group velocity and dispersion

If we introduce the *phase velocity* in the medium as

$$v_\varphi = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\kappa_e\epsilon_0\kappa_m\mu_0}} = \frac{c}{\sqrt{\kappa_e\kappa_m}} \quad (9.19)$$

where, according to equation (1.13) on page 6, $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light, *i.e.*, the phase speed of electromagnetic waves, in vacuum. Associated with the phase speed of a medium for a wave of a given frequency ω we have a *wave vector*, defined as

$$\mathbf{k} \stackrel{\text{def}}{=} k\hat{\mathbf{k}} = k\hat{\mathbf{v}}_\varphi = \frac{\omega}{v_\varphi} \frac{\mathbf{v}_\varphi}{v_\varphi} \quad (9.20)$$

The ratio of the phase speed in vacuum and in the medium

$$\frac{c}{v_\varphi} = \sqrt{\kappa_e\kappa_m} = c\sqrt{\epsilon\mu} \stackrel{\text{def}}{=} n \quad (9.21)$$

where the material dependent quantity

$$n(t, \mathbf{x}) \stackrel{\text{def}}{=} \frac{c}{v_\varphi} = \sqrt{\kappa_e(t, \mathbf{x})\kappa_m(t, \mathbf{x})} \quad (9.22)$$

⁶ In fact, there exist *metamaterials* where κ_e and κ_m are negative. For such materials, the refractive index becomes negative:

$$\begin{aligned} n &= i\sqrt{|\kappa_e|} i\sqrt{|\kappa_m|} \\ &= -|\kappa_e\kappa_m|^{1/2} \end{aligned}$$

Such *negative refractive index* materials, have quite remarkable electromagnetic properties.

is called the *refractive index* of the medium and describes its refractive and reflective properties.⁶ In general n is a function of frequency. If the medium is *anisotropic* or *birefringent*, the refractive index is a rank-two tensor field. Under our simplifying assumptions, in the material medium that we consider $n = \text{Const}$ for each frequency component of the fields. In certain materials, the refractive index is larger than unity (*e.g.*, glass and water at optical frequencies), in others, it can be smaller than unity (*e.g.*, plasma and metals at radio and optical frequencies).

It is important to notice that depending on the electric and magnetic properties of a medium, and, hence, on the value of the refractive index n , the phase speed in the medium can be smaller or larger than the speed of light:

$$v_\varphi = \frac{c}{n} = \frac{\omega}{k} \quad (9.23)$$

where, in the last step, we used equation (9.20) on the preceding page.

If the medium has a refractive index which, as is usually the case, dependent on frequency ω , we say that the medium is *dispersive*. Because in this case also $\mathbf{k}(\omega)$ and $\omega(\mathbf{k})$, so that the *group velocity*

$$v_g = \frac{\partial \omega}{\partial k} \quad (9.24)$$

has a unique value for each frequency component, and is different from v_φ . Except in regions of *anomalous dispersion*, v_g is always smaller than c . In a gas of free charges, such as a *plasma*, the refractive index is given by the expression

$$n^2(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad (9.25)$$

where

$$\omega_p^2 = \sum_{\sigma} \frac{N_{\sigma} q_{\sigma}^2}{\epsilon_0 m_{\sigma}} \quad (9.26)$$

is the square of the *plasma frequency* ω_p . Here m_{σ} and N_{σ} denote the mass and number density, respectively, of charged particle species σ . In an inhomogeneous plasma, $N_{\sigma} = N_{\sigma}(\mathbf{x})$ so that the refractive index and also the phase and group velocities are space dependent. As can be easily seen, for each given frequency, the phase and group velocities in a plasma are different from each other. If the frequency ω is such that it coincides with ω_p at some point in the medium, then at that point $v_\varphi \rightarrow \infty$ while $v_g \rightarrow 0$ and the wave Fourier component at ω is reflected there.

9.3 Radiation from charges in a material medium

When electromagnetic radiation is propagating through matter, new phenomena may appear which are (at least classically) not present in vacuum. As mentioned earlier, one can under certain simplifying assumptions include, to some extent, the influence from matter on the electromagnetic fields by introducing new, derived field quantities \mathbf{D} and \mathbf{H} according to

$$\mathbf{D} = \epsilon(t, \mathbf{x})\mathbf{E} = \kappa_e(t, \mathbf{x})\epsilon_0\mathbf{E} \quad (9.27)$$

$$\mathbf{B} = \mu(t, \mathbf{x})\mathbf{H} = \kappa_m(t, \mathbf{x})\mu_0\mathbf{H} \quad (9.28)$$

9.3.1 Vavilov-Čerenkov radiation

As we saw in subsection 6.3, a charge in uniform, rectilinear motion *in vacuum* does not give rise to any radiation; see in particular equation (6.140a) on page 115. Let us now consider a charge in uniform, rectilinear motion *in a medium* with electric properties which are different from those of a (classical) vacuum. Specifically, consider a medium where

$$\varepsilon = \text{Const} > \varepsilon_0 \quad (9.29a)$$

$$\mu = \mu_0 \quad (9.29b)$$

This implies that in this medium the phase speed is

$$v_\phi = \frac{c}{n} = \frac{1}{\sqrt{\varepsilon\mu_0}} < c \quad (9.30)$$

Hence, in this particular medium, the speed of propagation of (the phase planes of) electromagnetic waves is less than the speed of light in vacuum, which we know is an absolute limit for the motion of anything, including particles. A medium of this kind has the interesting property that particles, entering into the medium at high speeds $|\mathbf{v}'|$, which, of course, are below the phase speed *in vacuum*, can experience that the particle speeds are *higher* than the phase speed *in the medium*. This is the basis for the *Vavilov-Čerenkov radiation*, more commonly known in the western literature as *Cherenkov radiation*, that we shall now study.

If we recall the general derivation, in the vacuum case, of the retarded (and advanced) potentials in chapter 3 and the Liénard-Wiechert potentials, equations (6.103) on page 107, we realise that we obtain the latter in the medium by a simple formal replacement $c \rightarrow c/n$ in the expression (6.104) on page 107 for s . Hence, the Liénard-Wiechert potentials in a medium characterized by a refractive index n , are

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q'}{\left| |\mathbf{x} - \mathbf{x}'| - n \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right|} = \frac{1}{4\pi\varepsilon_0} \frac{q'}{s} \quad (9.31a)$$

$$\mathbf{A}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \frac{q' \mathbf{v}'}{\left| |\mathbf{x} - \mathbf{x}'| - n \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right|} = \frac{1}{4\pi\varepsilon_0 c^2} \frac{q' \mathbf{v}'}{s} \quad (9.31b)$$

where now

$$s = \left| |\mathbf{x} - \mathbf{x}'| - n \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{v}'}{c} \right| \quad (9.32)$$

The need for the absolute value of the expression for s is obvious in the case when $v'/c \geq 1/n$ because then the second term can be larger than the first term;

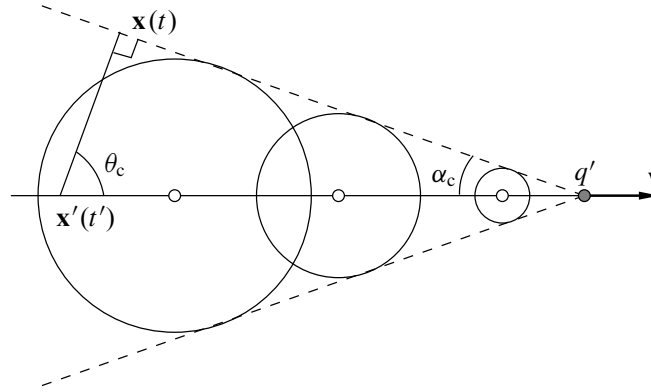


Figure 9.1: Instantaneous picture of the expanding field spheres from a point charge moving with constant speed $v'/c > 1/n$ in a medium where $n > 1$. This generates a Vavilov-Čerenkov shock wave in the form of a cone.

if $v'/c \ll 1/n$ we recover the well-known vacuum case but with modified phase speed. We also note that the retarded and advanced times in the medium are [cf. equation (3.45) on page 40]

$$t'_{\text{ret}} = t'_{\text{ret}}(t, |\mathbf{x} - \mathbf{x}'|) = t - \frac{k |\mathbf{x} - \mathbf{x}'|}{\omega} = t - \frac{|\mathbf{x} - \mathbf{x}'| n}{c} \quad (9.33a)$$

$$t'_{\text{adv}} = t'_{\text{adv}}(t, |\mathbf{x} - \mathbf{x}'|) = t + \frac{k |\mathbf{x} - \mathbf{x}'|}{\omega} = t + \frac{|\mathbf{x} - \mathbf{x}'| n}{c} \quad (9.33b)$$

so that the usual time interval $t - t'$ between the time measured at the point of observation and the retarded time *in a medium* becomes

$$t - t' = \frac{|\mathbf{x} - \mathbf{x}'| n}{c} \quad (9.34)$$

For $v'/c \geq 1/n$, the retarded distance s , and therefore the denominators in equations (9.31) on the facing page, vanish when

$$n(\mathbf{x} - \mathbf{x}') \cdot \frac{\mathbf{v}'}{c} = |\mathbf{x} - \mathbf{x}'| \frac{nv'}{c} \cos \theta_c = |\mathbf{x} - \mathbf{x}'| \quad (9.35)$$

or, equivalently, when

$$\cos \theta_c = \frac{c}{nv'} \quad (9.36)$$

In the direction defined by this angle θ_c , the potentials become singular. During the time interval $t - t'$ given by expression (9.34) above, the field exists within a sphere of radius $|\mathbf{x} - \mathbf{x}'|$ around the particle while the particle moves a distance

$$l' = (t - t')v' \quad (9.37)$$

along the direction of \mathbf{v}' .

In the direction θ_c where the potentials are singular, all field spheres are tangent to a straight cone with its apex at the instantaneous position of the particle and with the apex half angle α_c defined according to

$$\sin \alpha_c = \cos \theta_c = \frac{c}{nv'} \quad (9.38)$$

This is illustrated in figure 9.1 on the preceding page.

The cone of potential singularities and field sphere circumferences propagates with speed c/n in the form of a *shock front*. The first observation of this type of radiation was reported by MARIE SKŁODOWSKA CURIE in 1910, but she never pursued the exploration of it. This radiation in question is therefore called *Vavilov-Čerenkov radiation*.⁷

In order to make some quantitative estimates of this radiation, we note that we can describe the motion of each charged particle q' as a current density:

$$\mathbf{j} = q' \mathbf{v}' \delta(\mathbf{x}' - \mathbf{v}' t') = q' v' \delta(x' - v' t') \delta(y') \delta(z') \hat{\mathbf{x}}_1 \quad (9.39)$$

which has the trivial Fourier transform

$$\mathbf{j}_\omega = \frac{q'}{2\pi} e^{i\omega x'/v'} \delta(y') \delta(z') \hat{\mathbf{x}}_1 \quad (9.40)$$

This Fourier component can be used in the formulæ derived for a linear current in subsection 6.4.1 if only we make the replacements

$$\varepsilon_0 \rightarrow \varepsilon = n^2 \varepsilon_0 \quad (9.41a)$$

$$k \rightarrow \frac{n\omega}{c} \quad (9.41b)$$

In this manner, using \mathbf{j}_ω from equation (9.40) above, the resulting Fourier transforms of the Vavilov-Čerenkov magnetic and electric radiation fields can be calculated from the expressions (5.10) on page 74) and (5.21) on page 76, respectively.

The total energy content is then obtained from equation (6.8) on page 84 (integrated over a closed sphere at large distances). For a Fourier component one obtains [cf. equation (6.11) on page 85]

$$\begin{aligned} U_\omega^{\text{rad}} d\Omega &\approx \frac{1}{4\pi \varepsilon_0 n c} \left| \int_{V'} d^3x' (\mathbf{j}_\omega \times \mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}'} \right|^2 d\Omega \\ &= \frac{q'^2 n \omega^2}{16\pi^3 \varepsilon_0 c^3} \left| \int_{-\infty}^{\infty} \exp \left[i x' \left(\frac{\omega}{v'} - k \cos \theta \right) \right] dx' \right|^2 \sin^2 \theta d\Omega \end{aligned} \quad (9.42)$$

where θ is the angle between the direction of motion, $\hat{\mathbf{x}}'_1$, and the direction to the observer, $\hat{\mathbf{k}}$. The integral in (9.42) is singular of a ‘Dirac delta type’. If we limit

⁷ The first systematic exploration of this radiation was made in 1934 by PAVEL ALEKSEEVICH ČERENKOV (1904–1990), who was then a doctoral student in SERGEY IVANOVICH VAVILOV’s (1891–1951) research group at the Lebedev Physical Institute in Moscow. Vavilov wrote a manuscript with the experimental findings, put Čerenkov as the author, and submitted it to *Nature*. In the manuscript, Vavilov explained the results in terms of radioactive particles creating Compton electrons which gave rise to the radiation. This was indeed the correct interpretation, but the paper was rejected. The paper was then sent to *Physical Review* and was, after some controversy with the American editors, who claimed the results to be wrong, eventually published in 1937. In the same year, IGOR’ EVGEN’EVICH TAMM (1895–1975) and ILYA MIKHAILOVICH FRANK (1908–1990) published the theory for the effect (‘the singing electron’).

In fact, predictions of a similar effect had been made as early as 1888 by OLIVER HEAVISIDE (1850–1925), and by ARNOLD JOHANNES WILHELM SOMMERFELD (1868–1951) in his 1904 paper ‘Radiating body moving with velocity of light’. On 8 May, 1937, Sommerfeld sent a letter to Tamm via Austria, saying that he was surprised that his old 1904 ideas were now becoming interesting. Tamm, Frank and Čerenkov received the Nobel Prize in 1958 ‘for the discovery and the interpretation of the Čerenkov effect’ [VITALIY LAZAREVICH GINZBURG (1916–2009), *private communication*]. The Vavilov-Čerenkov cone is similar in nature to the Mach cone in acoustics.

the spatial extent of the motion of the particle to the closed interval $[-X, X]$ on the x' axis we can evaluate the integral to obtain

$$U_{\omega}^{\text{rad}} d\Omega = \frac{q'^2 n \omega^2 \sin^2 \theta}{4\pi^3 \epsilon_0 c^3} \frac{\sin^2 \left[\left(1 - \frac{nv'}{c} \cos \theta\right) \frac{X\omega}{v'} \right]}{\left[\left(1 - \frac{nv'}{c} \cos \theta\right) \frac{\omega}{v'} \right]^2} d\Omega \quad (9.43)$$

which has a maximum in the direction θ_c as expected. The magnitude of this maximum grows and its width narrows as $X \rightarrow \infty$. The integration of (9.43) over Ω therefore picks up the main contributions from $\theta \approx \theta_c$. Consequently, we can set $\sin^2 \theta \approx \sin^2 \theta_c$ and the result of the integration is

$$\begin{aligned} \tilde{U}_{\omega}^{\text{rad}} &= 2\pi \int_0^{\pi} U_{\omega}^{\text{rad}}(\theta) \sin \theta d\theta = [\cos \theta = -\xi] = 2\pi \int_{-1}^1 U_{\omega}^{\text{rad}}(\xi) d\xi \\ &\approx \frac{q'^2 n \omega^2 \sin^2 \theta_c}{2\pi^2 \epsilon_0 c^3} \int_{-1}^1 \frac{\sin^2 \left[\left(1 + \frac{nv'\xi}{c}\right) \frac{X\omega}{v'} \right]}{\left[\left(1 + \frac{nv'\xi}{c}\right) \frac{\omega}{v'} \right]^2} d\xi \end{aligned} \quad (9.44)$$

The integrand in (9.44) is strongly peaked near $\xi = -c/(nv')$, or, equivalently, near $\cos \theta_c = c/(nv')$. This means that the integrand function is practically zero outside the integration interval $\xi \in [-1, 1]$. Consequently, one may extend the ξ integration interval to $(-\infty, \infty)$ without introducing too much an error. Via yet another variable substitution we can therefore approximate

$$\begin{aligned} \sin^2 \theta_c \int_{-1}^1 \frac{\sin^2 \left[\left(1 + \frac{nv'\xi}{c}\right) \frac{X\omega}{v'} \right]}{\left[\left(1 + \frac{nv'\xi}{c}\right) \frac{\omega}{v'} \right]^2} d\xi &\approx \left(1 - \frac{c^2}{n^2 v'^2}\right) \frac{cX}{\omega n} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx \\ &= \frac{cX\pi}{\omega n} \left(1 - \frac{c^2}{n^2 v'^2}\right) \end{aligned} \quad (9.45)$$

leading to the final approximate result for the total energy loss in the frequency interval $(\omega, \omega + d\omega)$

$$\tilde{U}_{\omega}^{\text{rad}} d\omega = \frac{q'^2 X}{2\pi \epsilon_0 c^2} \left(1 - \frac{c^2}{n^2 v'^2}\right) \omega d\omega \quad (9.46)$$

As mentioned earlier, the refractive index is usually frequency dependent. Realising this, we find that the radiation energy per frequency unit and *per unit length* is

$$\frac{\tilde{U}_{\omega}^{\text{rad}} d\omega}{2X} = \frac{q'^2 \omega}{4\pi \epsilon_0 c^2} \left(1 - \frac{c^2}{n^2(\omega) v'^2}\right) d\omega \quad (9.47)$$

This result was derived under the assumption that $v'/c > 1/n(\omega)$, *i.e.*, under the condition that the expression inside the parentheses in the right hand side is positive. For all media it is true that $n(\omega) \rightarrow 1$ when $\omega \rightarrow \infty$, so there exist always a highest frequency for which we can obtain Vavilov-Čerenkov radiation from a fast charge in a medium. Our derivation above for a fixed value of n is valid for each individual Fourier component.

9.4 Electromagnetic waves in a medium

In section 2.3 in chapter 2 we derived the wave equations for the electric and magnetic fields, \mathbf{E} and \mathbf{B} , respectively,

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = -\frac{\nabla \rho}{\epsilon_0} - \mu_0 \frac{\partial \mathbf{j}}{\partial t} \quad (9.48a)$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \mu_0 \nabla \times \mathbf{j} \quad (9.48b)$$

where the charge density ρ and the current density \mathbf{j} were viewed as the sources of the wave fields. As we recall, these wave equations were derived from the Maxwell-Lorentz equations (2.1) on page 20, taken as an axiomatic foundation, or postulates, of electromagnetic theory. As such, these equations just state what relations exist between (the second order derivatives of) the fields, *i.e.*, essentially the Coulomb and Ampère forces, and the dynamics of the charges (charge and current densities) in the region under study.

Even if the ρ and \mathbf{j} terms in the Maxwell-Lorentz equations are often referred to as the source terms, they can equally well be viewed as terms that describe the impact on matter in a particular region upon which an electromagnetic wave, produced in another region with its own charges and currents, impinges. In order to do so, one needs to find the *constitutive relations* that describe how charge and current densities are induced by the impinging fields. Then one can solve the wave equations (9.48) above. In general, this is a formidable task, and one must often resort to numerical methods.

Let us, for simplicity, assume that the linear relations, as given by formula (9.8) on page 171 and formula (9.15) on page 173, hold, and that there is also a linear relation between the electric field \mathbf{E} and the current density, known as *Ohm's law*:

$$\mathbf{j}(t, \mathbf{x}) = \sigma(t, \mathbf{x}) \mathbf{E}(t, \mathbf{x}) \quad (9.49)$$

where σ is the *conductivity* of the medium. Let us make the further assumption that $\epsilon = \epsilon(\mathbf{x})$, $\mu = \mu(\mathbf{x})$, and $\sigma(\mathbf{x})$ are not explicitly dependent on time and are

local in space. Then we obtain the coupled wave equations

$$\nabla^2 \mathbf{E} - \mu(\mathbf{x})\sigma(\mathbf{x})\frac{\partial \mathbf{E}}{\partial t} - \varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2 \mathbf{E}}{\partial t^2} = (\nabla \times \mathbf{E}) \times \nabla \ln \mu(\mathbf{x}) - \nabla[\nabla \ln \varepsilon(\mathbf{x}) \cdot \mathbf{E}] + \nabla \frac{\rho}{\varepsilon} \quad (9.50a)$$

$$\nabla^2 \mathbf{H} - \mu(\mathbf{x})\sigma(\mathbf{x})\frac{\partial \mathbf{H}}{\partial t} - \varepsilon(\mathbf{x})\mu(\mathbf{x})\frac{\partial^2 \mathbf{H}}{\partial t^2} = (\nabla \times \mathbf{H}) \times \nabla \ln \varepsilon(\mathbf{x}) - \nabla[\nabla \ln \mu(\mathbf{x}) \cdot \mathbf{H}] + \nabla \ln \varepsilon(\mathbf{x}) \times (\sigma(\mathbf{x})\mathbf{E}) - [\nabla \sigma(\mathbf{x})] \times \mathbf{E} \quad (9.50b)$$

For the case $\mu = \mu_0$ (no magnetisation) and $\rho = \text{Const}$ in the medium, equations (9.50) simplify to

$$\nabla^2 \mathbf{E} - \mu_0\sigma(\mathbf{x})\frac{\partial \mathbf{E}}{\partial t} - \varepsilon(\mathbf{x})\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla[\nabla \ln \varepsilon(\mathbf{x}) \cdot \mathbf{E}] \quad (9.51a)$$

$$\nabla^2 \mathbf{B} - \mu_0\sigma(\mathbf{x})\frac{\partial \mathbf{B}}{\partial t} - \varepsilon(\mathbf{x})\mu_0\frac{\partial^2 \mathbf{B}}{\partial t^2} = (\nabla \times \mathbf{B}) \times \nabla \ln \varepsilon(\mathbf{x}) + \mu_0 \nabla \ln \varepsilon(\mathbf{x}) \times (\sigma \mathbf{E}) - \mu_0 [\nabla \sigma(\mathbf{x})] \times \mathbf{E} \quad (9.51b)$$

Making the further assumption that the medium is not conductive, *i.e.*, that $\sigma = 0$, the uncoupled wave equations

$$\nabla^2 \mathbf{E} - \frac{\varepsilon(\mathbf{x})}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla\{[\nabla \ln \varepsilon(\mathbf{x})] \cdot \mathbf{E}\} \quad (9.52a)$$

$$\nabla^2 \mathbf{B} - \frac{\varepsilon(\mathbf{x})}{\varepsilon_0 c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = (\nabla \times \mathbf{B}) \times [\nabla \ln \varepsilon(\mathbf{x})] \quad (9.52b)$$

are obtained.

9.4.1 Constitutive relations

In a solid, fluid or gaseous medium the source terms in the microscopic Maxwell equations (2.1) on page 20 must include all charges and currents in the medium, *i.e.*, also the intrinsic ones (*e.g.*, the polarisation charges in electrets, and atomistic magnetisation currents in magnets) and the self-consistently imposed ones (*e.g.*, polarisation currents). This is of course also true for the inhomogeneous wave equations derived from Maxwell's equations.

From now on we assume that ρ and \mathbf{j} represent only the charge and current densities (*i.e.*, polarisation and conduction charges and currents, respectively) that are induced by the \mathbf{E} and \mathbf{B} fields of the waves impinging upon the medium of interest.⁸

Let us for simplicity consider a medium containing free electrons only and which is not penetrated by a magnetic field, *i.e.*, the medium is assumed to be isotropic with no preferred direction(s) in space.⁹

⁸ If one includes also the effect of the charges on \mathbf{E} and \mathbf{B} , *i.e.*, treat ρ and \mathbf{j} as sources for fields, singularities will appear in the theory. Such so called *self-force effects* will not be treated here.

⁹ To this category belongs *unmagnetised plasma*. So do also, to a good approximation, fluid or solid metals.

Each of these electrons are assumed to be accelerated by the Lorentz force, formula (4.59) on page 61. However, if the fields are those of an electromagnetic wave one can, for reasonably high oscillation frequencies, neglect the force from the magnetic field. Of course, this is also true if there is no magnetic field present. So the equation of motion for each electron in the medium can be written

$$m \frac{d^2 \mathbf{x}}{dt^2} + m\nu \frac{d\mathbf{x}}{dt} = q\mathbf{E} \quad (9.53)$$

where m and q are the mass and charge of the electron, respectively, ν the effective *collision frequency* representing the frictional dissipative force from the surrounding medium, and \mathbf{E} the effective applied electric field sensed by the electron. For a Fourier component of the electric field $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t)$, the equation of motion becomes

$$\omega^2 q \mathbf{x}(t) - i\omega \nu q \mathbf{x}(t) = \frac{q^2}{m} \mathbf{E} \quad (9.54)$$

If the electron is at equilibrium $\mathbf{x} = \mathbf{0}$ when $\mathbf{E} = \mathbf{0}$, then its dipole moment is $\mathbf{d}(t) = q\mathbf{x}(t)$. Inserting this in equation (9.54) above, we obtain

$$\mathbf{d} = -\frac{q^2}{m(\omega^2 + i\omega\nu)} \mathbf{E} \quad (9.55)$$

This is the the lowest order contribution to the dipole moment of the medium from each electron under the influence of the assumed electric field. If $N_{\mathbf{d}}(\mathbf{x})$ electrons per unit volume can be assumed to give rise to the electric polarisation \mathbf{P} , this becomes

$$\mathbf{P} = N_{\mathbf{d}} \mathbf{d} = -\frac{N_{\mathbf{d}} q^2}{m(\omega^2 + i\omega\nu)} \mathbf{E} \quad (9.56)$$

Using this in formula (9.5) on page 170, one finds that

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon(\mathbf{x}) \mathbf{E}(t, \mathbf{x}) \quad (9.57)$$

where

$$\varepsilon(\mathbf{x}) = \varepsilon_0 - \frac{N_{\mathbf{d}}(\mathbf{x}) q^2}{m(\omega^2 + i\omega\nu)} = \varepsilon_0 \left(1 - \frac{N_{\mathbf{d}}(\mathbf{x}) q^2}{\varepsilon_0 m} \frac{1}{\omega^2 + i\omega\nu} \right) \quad (9.58)$$

The quantity

$$\omega_p(\mathbf{x}) = \sqrt{\frac{N_{\mathbf{d}}(\mathbf{x}) q^2}{\varepsilon_0 m}} \quad (9.59)$$

is called the *plasma frequency* and

$$n(\mathbf{x}) = \sqrt{\frac{\varepsilon(\mathbf{x})}{\varepsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2 + i\omega\nu}} \quad (9.60)$$

is called the *refractive index*. At points in the medium where the wave frequency ω equals this plasma frequency and the collision frequency ν vanishes, the refractive index $n = 0$, and the wave is totally reflected. In the ionised outer part of the atmosphere called the *ionosphere* this happens for radio waves of frequencies up to about 10 MHz. This is the basis for over-the-horizon radio communications.

9.4.2 Electromagnetic waves in a conducting medium

We shall now restrict ourselves to the wave equations for the electric field vector \mathbf{E} and the magnetic field vector \mathbf{B} in a electrically conductive and neutral medium, *i.e.*, a volume where there exist no net electric charge, $\rho = 0$, no dielectric effects, $\varepsilon = \varepsilon_0$, and no electromotive force, $\mathbf{E}^{\text{emf}} = \mathbf{0}$. A highly conductive metal is a good example of such a medium.

9.4.2.1 The wave equations for \mathbf{E} and \mathbf{B}

To a good approximation, metals and other conductors in free space have a conductivity σ that is not dependent on t or \mathbf{x} . The wave equations (9.51) on page 181 are then simplified to

$$\nabla^2 \mathbf{E} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \quad (9.61)$$

$$\nabla^2 \mathbf{B} - \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mathbf{0} \quad (9.62)$$

which are the *homogeneous vector wave equations* for \mathbf{E} and \mathbf{B} in a conducting medium without EMF.

We notice that for the simple propagation media considered here, the wave equation for the magnetic field \mathbf{B} has exactly the same mathematical form as the wave equation for the electric field \mathbf{E} , equation (9.61). Therefore, in this case it suffices to consider only the \mathbf{E} field, since the results for the \mathbf{B} field follow trivially. For EM waves propagating in more complicated media, containing, *e.g.*, inhomogeneities, the wave equations for \mathbf{E} and for \mathbf{B} do not have the same mathematical form.

Following the spectral component prescription leading to equation (2.38) on page 28, we obtain, in the special case under consideration, the following time-

independent wave equation

$$\nabla^2 \mathbf{E}_0 + \frac{\omega^2}{c^2} \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) \mathbf{E}_0 = \mathbf{0} \quad (9.63)$$

Multiplying by $e^{-i\omega t}$ and introducing the *relaxation time* $\tau = \varepsilon_0/\sigma$ of the medium in question, we see that the differential equation for each spectral component can be written

$$\nabla^2 \mathbf{E}(t, \mathbf{x}) + \frac{\omega^2}{c^2} \left(1 + \frac{i}{\tau \omega} \right) \mathbf{E}(t, \mathbf{x}) = \mathbf{0} \quad (9.64)$$

In the limit of long τ (low conductivity σ), (9.64) tends to

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} \mathbf{E} = \mathbf{0} \quad (9.65)$$

which is a *time-independent wave equation* for \mathbf{E} , representing undamped propagating waves. In the short τ (high conductivity σ) limit we have instead

$$\nabla^2 \mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = \mathbf{0} \quad (9.66)$$

which is a *time-independent diffusion equation* for \mathbf{E} .

For most metals $\tau \sim 10^{-14}$ s, which means that the diffusion picture is good for all frequencies lower than optical frequencies. Hence, in metallic conductors, the propagation term $\partial^2 \mathbf{E}/c^2 \partial t^2$ is negligible even for VHF, UHF, and SHF signals. Alternatively, we may say that the displacement current $\varepsilon_0 \partial \mathbf{E}/\partial t$ is negligible relative to the conduction current $\mathbf{j} = \sigma \mathbf{E}$.

If we introduce the *vacuum wave number*

$$k = \frac{\omega}{c} \quad (9.67)$$

we can write, using the fact that $c = 1/\sqrt{\varepsilon_0\mu_0}$ according to equation (1.13) on page 6,

$$\frac{1}{\tau\omega} = \frac{\sigma}{\varepsilon_0\omega} = \frac{\sigma}{\varepsilon_0} \frac{1}{c k} = \frac{\sigma}{k} \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\sigma}{k} R_0 \quad (9.68)$$

where in the last step we used the *characteristic impedance of vacuum* defined according to formula (6.3) on page 83.

9.4.2.2 Plane waves

Consider now the case where all fields depend only on the distance ζ to a given plane with unit normal $\hat{\mathbf{n}}$. Then the *del* operator becomes

$$\nabla = \hat{\mathbf{n}} \frac{\partial}{\partial \zeta} = \hat{\mathbf{n}} \nabla \quad (9.69)$$

and the microscopic Maxwell equations attain the form

$$\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{E}}{\partial \zeta} = 0 \quad (9.70a)$$

$$\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} = -\frac{\partial \mathbf{B}}{\partial t} \quad (9.70b)$$

$$\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial \zeta} = 0 \quad (9.70c)$$

$$\hat{\mathbf{n}} \times \frac{\partial \mathbf{B}}{\partial \zeta} = \mu_0 \mathbf{j}(t, \mathbf{x}) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \sigma \mathbf{E} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (9.70d)$$

Scalar multiplying (9.70d) by $\hat{\mathbf{n}}$, we find that

$$0 = \hat{\mathbf{n}} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{B}}{\partial \zeta} \right) = \hat{\mathbf{n}} \cdot \left(\mu_0 \sigma + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \right) \mathbf{E} \quad (9.71)$$

which simplifies to the first-order ordinary differential equation for the normal component E_n of the electric field

$$\frac{dE_n}{dt} + \frac{\sigma}{\varepsilon_0} E_n = 0 \quad (9.72)$$

with the solution

$$E_n = E_{n0} e^{-\sigma t / \varepsilon_0} = E_{n0} e^{-t/\tau} \quad (9.73)$$

This, together with (9.70a), shows that the *longitudinal component* of \mathbf{E} , *i.e.*, the component which is perpendicular to the plane surface is independent of ζ and has a time dependence which exhibits an exponential decay, with a decrement given by the relaxation time τ in the medium.

Scalar multiplying (9.70b) by $\hat{\mathbf{n}}$, we similarly find that

$$0 = \hat{\mathbf{n}} \cdot \left(\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} \right) = -\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} \quad (9.74)$$

or

$$\hat{\mathbf{n}} \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (9.75)$$

From this, and (9.70c), we conclude that the only longitudinal component of \mathbf{B} must be constant in both time and space. In other words, the only non-static solution must consist of *transverse components*.

9.4.2.3 Telegrapher's equation

In analogy with equation (9.61) on page 183, we can easily derive a wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \quad (9.76)$$

describing the propagation of plane waves along ζ in a conducting medium. This equation is called the *telegrapher's equation*. If the medium is an insulator so that $\sigma = 0$, then the equation takes the form of the *one-dimensional wave equation*

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \quad (9.77)$$

As is well known, each component of this equation has a solution which can be written

$$E_i = f(\zeta - ct) + g(\zeta + ct), \quad i = 1, 2, 3 \quad (9.78)$$

where f and g are arbitrary (non-pathological) functions of their respective arguments. This general solution represents perturbations which propagate along ζ , where the f perturbation propagates in the positive ζ direction and the g perturbation propagates in the negative ζ direction. In a medium, the general solution to each component of equation (9.99) on page 189 is given by

$$E_i = f(\zeta - v_\phi t) + g(\zeta + v_\phi t), \quad i = 1, 2, 3 \quad (9.79)$$

If we assume that our electromagnetic fields \mathbf{E} and \mathbf{B} are represented by a Fourier component proportional to $\exp(-i\omega t)$, the solution of equation (9.77) becomes

$$\mathbf{E} = \mathbf{E}_0 e^{-i(\omega t \pm k\zeta)} = \mathbf{E}_0 e^{i(\mp k\zeta - \omega t)} \quad (9.80)$$

By introducing the wave vector

$$\mathbf{k} = k \hat{\mathbf{n}} = \frac{\omega}{c} \hat{\mathbf{n}} = \frac{\omega}{c} \hat{\mathbf{k}} \quad (9.81)$$

this solution can be written as

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (9.82)$$

Let us consider the lower sign in front of $k\zeta$ in the exponent in (9.80). This corresponds to a wave which propagates in the direction of increasing ζ . Inserting this solution into equation (9.70b) on the previous page, gives

$$\hat{\mathbf{n}} \times \frac{\partial \mathbf{E}}{\partial \zeta} = i\omega \mathbf{B} = ik \hat{\mathbf{n}} \times \mathbf{E} \quad (9.83)$$

or, solving for \mathbf{B} ,

$$\mathbf{B} = \frac{k}{\omega} \hat{\mathbf{n}} \times \mathbf{E} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E} = \frac{1}{c} \hat{\mathbf{k}} \times \mathbf{E} = \sqrt{\epsilon_0 \mu_0} \hat{\mathbf{n}} \times \mathbf{E} \quad (9.84)$$

Hence, to each transverse component of \mathbf{E} , there exists an associated magnetic field given by equation (9.84) on the facing page. If \mathbf{E} and/or \mathbf{B} has a direction in space which is constant in time, we have a *plane wave*.

Allowing now for a finite conductivity σ in our medium, and making the spectral component Ansatz in equation (9.76) on page 185, we find that the *time-independent telegrapher's equation* can be written

$$\frac{\partial^2 \mathbf{E}}{\partial \xi^2} + \varepsilon_0 \mu_0 \omega^2 \mathbf{E} + i \mu_0 \sigma \omega \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial \xi^2} + K^2 \mathbf{E} = \mathbf{0} \quad (9.85)$$

where

$$K^2 = \varepsilon_0 \mu_0 \omega^2 \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) = \frac{\omega^2}{c^2} \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) = k^2 \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) \quad (9.86)$$

where, in the last step, equation (9.67) on page 184 was used to introduce the wave number k . Taking the square root of this expression, we obtain

$$K = k \sqrt{1 + i \frac{\sigma}{\varepsilon_0 \omega}} = \alpha + i \beta \quad (9.87)$$

Squaring, one finds that

$$k^2 \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right) = (\alpha^2 - \beta^2) + 2i\alpha\beta \quad (9.88)$$

or

$$\beta^2 = \alpha^2 - k^2 \quad (9.89)$$

$$\alpha\beta = \frac{k^2 \sigma}{2\varepsilon_0 \omega} \quad (9.90)$$

Squaring the latter and combining with the former, one obtains the second order algebraic equation (in α^2)

$$\alpha^2(\alpha^2 - k^2) = \frac{k^4 \sigma^2}{4\varepsilon_0^2 \omega^2} \quad (9.91)$$

which can be easily solved and one finds that

$$\alpha = k \sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0 \omega} \right)^2} + 1}{2}} \quad (9.92a)$$

$$\beta = k \sqrt{\frac{\sqrt{1 + \left(\frac{\sigma}{\varepsilon_0 \omega} \right)^2} - 1}{2}} \quad (9.92b)$$

As a consequence, the solution of the time-independent telegrapher's equation, equation (9.85) on the previous page, can be written

$$\mathbf{E} = \mathbf{E}_0 e^{-\beta \zeta} e^{i(\alpha \zeta - \omega t)} \quad (9.93)$$

With the aid of equation (9.84) on page 186 we can calculate the associated magnetic field, and find that it is given by

$$\mathbf{B} = \frac{1}{\omega} K \hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\omega} (\hat{\mathbf{k}} \times \mathbf{E})(\alpha + i\beta) = \frac{1}{\omega} (\hat{\mathbf{k}} \times \mathbf{E}) |A| e^{i\gamma} \quad (9.94)$$

where we have, in the last step, rewritten $\alpha + i\beta$ in the amplitude-phase form $|A| \exp(i\gamma)$. From the above, we immediately see that \mathbf{E} , and consequently also \mathbf{B} , is damped, and that \mathbf{E} and \mathbf{B} in the wave are out of phase.

In the limit $\varepsilon_0 \omega \ll \sigma$, we can approximate K as follows:

$$\begin{aligned} K &= k \left(1 + i \frac{\sigma}{\varepsilon_0 \omega} \right)^{\frac{1}{2}} = k \left[i \frac{\sigma}{\varepsilon_0 \omega} \left(1 - i \frac{\varepsilon_0 \omega}{\sigma} \right) \right]^{\frac{1}{2}} \approx k(1 + i) \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}} \\ &= \sqrt{\varepsilon_0 \mu_0 \omega} (1 + i) \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}} = (1 + i) \sqrt{\frac{\mu_0 \sigma \omega}{2}} \end{aligned} \quad (9.95)$$

In this limit we find that when the wave impinges perpendicularly upon the medium, the fields are given, *inside* the medium, by

$$\mathbf{E}' = \mathbf{E}_0 \exp \left\{ -\sqrt{\frac{\mu_0 \sigma \omega}{2}} \zeta \right\} \exp \left\{ i \left(\sqrt{\frac{\mu_0 \sigma \omega}{2}} \zeta - \omega t \right) \right\} \quad (9.96a)$$

$$\mathbf{B}' = (1 + i) \sqrt{\frac{\mu_0 \sigma}{2\omega}} (\hat{\mathbf{n}} \times \mathbf{E}') \quad (9.96b)$$

Hence, both fields fall off by a factor $1/e$ at a distance

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} \quad (9.97)$$

This distance δ is called the *skin depth*.

Assuming for simplicity that the *electric permittivity* ε and the *magnetic permeability* μ , and hence the *relative permittivity* κ_e and the *relative permeability* κ_m all have fixed values, independent on time and space, for each type of material we consider, we can derive the general *telegrapher's equation* [cf. equation (9.76) on page 185]

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta^2} - \sigma \mu \frac{\partial \mathbf{E}}{\partial t} - \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \quad (9.98)$$

describing (1D) wave propagation in a material medium.

In chapter 2 we concluded that the existence of a finite conductivity, manifesting itself in a *collisional interaction* between the charge carriers, causes the waves to decay exponentially with time and space. Let us therefore assume that in our medium $\sigma = 0$ so that the wave equation simplifies to

$$\frac{\partial^2 \mathbf{E}}{\partial \xi^2} - \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0} \quad (9.99)$$

As in the vacuum case discussed in chapter 2, assuming that \mathbf{E} is time-harmonic, *i.e.*, can be represented by a Fourier component proportional to $\exp(-i\omega t)$, the solution of equation (9.99) can be written

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (9.100)$$

where now \mathbf{k} is the wave vector *in the medium* given by equation (9.20) on page 174. With these definitions, the vacuum formula for the associated magnetic field, equation (9.84) on page 186,

$$\mathbf{B} = \sqrt{\varepsilon \mu} \hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{v_\varphi} \hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E} \quad (9.101)$$

is valid also in a material medium (assuming, as mentioned, that n has a fixed constant scalar value). A consequence of a $\kappa_e \neq 1$ is that the electric field will, in general, have a longitudinal component.

▷ Electromagnetic waves in an electrically and magnetically conducting medium ——— EXAMPLE 9.1

Derive the wave equation for the \mathbf{E} field described by the electromagnetodynamic equations (Dirac's symmetrised Maxwell equations) [*cf.* equations (1.50) on page 16]

$$\nabla \cdot \mathbf{E} = \frac{\rho^e}{\varepsilon_0} \quad (9.102a)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mu_0 \mathbf{j}^m \quad (9.102b)$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho^m \quad (9.102c)$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}^e \quad (9.102d)$$

under the assumption of vanishing net electric and magnetic charge densities and in the absence of electromotive and magnetomotive forces. Interpret this equation physically.

Assume, for symmetry reasons, that there exists a linear relation between the *magnetic* current density \mathbf{j}^m and the magnetic field \mathbf{B} (the magnetic dual of Ohm's law for *electric* currents, $\mathbf{j}^e = \sigma^e \mathbf{E}$)

$$\mathbf{j}^m = \sigma^m \mathbf{B} \quad (9.103)$$

Taking the curl of (9.102b) and using (9.102d), one finds, noting that $\varepsilon_0 \mu_0 = 1/c^2$, that

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\mu_0 \nabla \times \mathbf{j}^m - \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\
&= -\mu_0 \sigma^m \nabla \times \mathbf{B} - \frac{\partial}{\partial t} \left(\mu_0 \mathbf{j}^e + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) \\
&= -\mu_0 \sigma^m \left(\mu_0 \sigma^e \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) - \mu_0 \sigma^e \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}
\end{aligned} \tag{9.104}$$

Using the vector operator identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$, and the fact that $\nabla \cdot \mathbf{E} = 0$ for a vanishing net electric charge, we can rewrite the wave equation as

$$\nabla^2 \mathbf{E} - \mu_0 \left(\sigma^e + \frac{\sigma^m}{c^2} \right) \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0^2 \sigma^m \sigma^e \mathbf{E} = \mathbf{0} \tag{9.105}$$

This is the homogeneous electromagnetic wave equation for \mathbf{E} that we were after.

Compared to the ordinary electrodynamic wave equation for \mathbf{E} , equation (9.61) on page 183, we see that we pick up extra terms. In order to understand what these extra terms mean physically, we analyse the time-independent wave equation for a single Fourier component. Then our wave equation becomes

$$\begin{aligned}
\nabla^2 \mathbf{E} + i\omega\mu_0 \left(\sigma^e + \frac{\sigma^m}{c^2} \right) \mathbf{E} + \frac{\omega^2}{c^2} \mathbf{E} - \mu_0^2 \sigma^m \sigma^e \mathbf{E} \\
= \nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} \left[\left(1 - \frac{1}{\omega^2} \frac{\mu_0}{\varepsilon_0} \sigma^m \sigma^e \right) + i \frac{\sigma^e + \sigma^m/c^2}{\varepsilon_0 \omega} \right] \mathbf{E} = \mathbf{0}
\end{aligned} \tag{9.106}$$

Realising that, according to formula (6.3) on page 83, μ_0/ε_0 is the square of the vacuum radiation resistance R_0 , and rearranging a bit, we obtain the time-independent wave equation in Dirac's symmetrised electrodynamics

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} \left(1 - \frac{R_0^2}{\omega^2} \sigma^m \sigma^e \right) \left(1 + i \frac{\sigma^e + \sigma^m/c^2}{\varepsilon_0 \omega \left(1 - \frac{R_0^2}{\omega^2} \sigma^m \sigma^e \right)} \right) \mathbf{E} = \mathbf{0}, \tag{9.107}$$

$\omega \neq R_0 \sqrt{\sigma^m \sigma^e}$

From this equation we conclude that the existence of magnetic charges (magnetic monopoles), and non-vanishing electric and magnetic conductivities would lead to a shift in the effective wave number of the wave. Furthermore, even if the electric conductivity σ^e vanishes, the imaginary term does not necessarily vanish and the wave therefore experiences damping or growth according as σ^m is positive or negative, respectively. This would happen in a hypothetical medium which is a perfect insulator for electric currents but which can carry magnetic currents.

Finally, we note that in the particular case $\omega = R_0 \sqrt{\sigma^m \sigma^e} \stackrel{\text{def}}{=} \omega_m$, the time-independent wave equation equation (9.106) above becomes a time-independent diffusion equation

$$\nabla^2 \mathbf{E} + i\omega\mu_0 \left(\sigma^e + \frac{\sigma^m}{c^2} \right) \mathbf{E} = \mathbf{0} \tag{9.108}$$

which in time domain corresponds to the *time-dependent diffusion equation*

$$\frac{\partial \mathbf{E}}{\partial t} - D \nabla^2 \mathbf{E} = \mathbf{0} \tag{9.109}$$

with a *diffusion coefficient* given by

$$D = \frac{1}{\mu_0 \left(\sigma^e + \frac{\sigma^m}{c^2} \right)} \quad (9.110)$$

Hence, electromagnetic waves with this particular frequency do not propagate. This means that if magnetic charges (monopoles) exist in a region in the Universe, electromagnetic waves propagating through this region would, in this simplistic model, exhibit a lower cutoff at $\omega = \omega_m$. This would in fact impose a lower limit on the mass of the *photon*, the quantum of the electromagnetic field that we shall come across later.

End of example 9.1 ◀

9.5 Bibliography

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DRAFT

F

FORMULÆ

F.1 The electromagnetic field

F.1.1 The microscopic Maxwell (Lorentz) equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (\text{F.1})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{F.2})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{F.3})$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\text{F.4})$$

F.1.1.1 Constitutive relations

$$\varepsilon_0 \mu_0 = \frac{1}{c^2} \quad (\text{F.5})$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = R_0 (\approx 376.7\Omega) \quad (\text{F.6})$$

$$\mathbf{D} \approx \varepsilon \mathbf{E} \quad (\text{F.7})$$

$$\mathbf{H} \approx \frac{\mathbf{B}}{\mu} \quad (\text{F.8})$$

$$\mathbf{j} \approx \sigma \mathbf{E} \quad (\text{F.9})$$

$$\mathbf{P} \approx \varepsilon_0 \chi \mathbf{E} \quad (\text{F.10})$$

F.1.2 Fields and potentials

F.1.2.1 Vector and scalar potentials

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (\text{F.11})$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \quad (\text{F.12})$$

F.1.2.2 The Lorenz-Lorentz gauge condition in vacuum

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial\phi}{\partial t} = 0 \quad (\text{F.13})$$

F.1.3 Force and energy

F.1.3.1 The Poynting vector in vacuum

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (\text{F.14})$$

F.1.3.2 Electromagnetic field energy density in vacuum

$$u = \frac{1}{2} \varepsilon_0 (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \quad (\text{F.15})$$

F.1.3.3 Linear momentum flux tensor (the negative of the Maxwell stress tensor)

$$\mathbf{T} = \mathbf{1}_3 \frac{\varepsilon_0}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) - \varepsilon_0 (\mathbf{E}\mathbf{E} + c^2 \mathbf{B}\mathbf{B}) \quad (\text{F.16})$$

$$T_{ij} = \delta_{ij} u - \varepsilon_0 E_i E_j - \frac{1}{\mu_0} B_i B_j \quad (\text{F.17})$$

F.2 Electromagnetic radiation

F.2.1 Relationship between the field vectors in a plane wave

$$\mathbf{B} = \frac{\hat{\mathbf{k}} \times \mathbf{E}}{c} \quad (\text{F.18})$$

F.2.2 The far fields from an extended source distribution

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = \frac{-i\mu_0}{4\pi} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \int_{V'} d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{j}_\omega \times \mathbf{k} \quad (\text{F.19})$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{i}{4\pi\epsilon_0 c} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \hat{\mathbf{x}} \times \int_{V'} d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{j}_\omega \times \mathbf{k} \quad (\text{F.20})$$

F.2.3 The far fields from an electric dipole

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = -\frac{\omega\mu_0}{4\pi} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \mathbf{d}_\omega \times \mathbf{k} \quad (\text{F.21})$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} (\mathbf{d}_\omega \times \mathbf{k}) \times \mathbf{k} \quad (\text{F.22})$$

F.2.4 The far fields from a magnetic dipole

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} (\mathbf{m}_\omega \times \mathbf{k}) \times \mathbf{k} \quad (\text{F.23})$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{k}{4\pi\epsilon_0 c} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \mathbf{m}_\omega \times \mathbf{k} \quad (\text{F.24})$$

F.2.5 The far fields from an electric quadrupole

$$\mathbf{B}_\omega^{\text{far}}(\mathbf{x}) = \frac{i\mu_0\omega}{8\pi} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} (\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k} \quad (\text{F.25})$$

$$\mathbf{E}_\omega^{\text{far}}(\mathbf{x}) = \frac{i}{8\pi\epsilon_0} \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} [(\mathbf{k} \cdot \mathbf{Q}_\omega) \times \mathbf{k}] \times \mathbf{k} \quad (\text{F.26})$$

F.2.6 The fields from a point charge in arbitrary motion

$$\mathbf{E}(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0 s^3} \left[(\mathbf{x} - \mathbf{x}_0) \left(1 - \frac{v'^2}{c^2} \right) + (\mathbf{x} - \mathbf{x}') \times \frac{(\mathbf{x} - \mathbf{x}_0) \times \dot{\mathbf{v}}'}{c^2} \right] \quad (\text{F.27})$$

$$\mathbf{B}(t, \mathbf{x}) = (\mathbf{x} - \mathbf{x}') \times \frac{\mathbf{E}(t, \mathbf{x})}{c|\mathbf{x} - \mathbf{x}'|} \quad (\text{F.28})$$

$$s = |\mathbf{x} - \mathbf{x}'| - (\mathbf{x} - \mathbf{x}') \cdot \frac{\mathbf{v}'}{c} \quad (\text{F.29})$$

$$\mathbf{x} - \mathbf{x}_0 = (\mathbf{x} - \mathbf{x}') - |\mathbf{x} - \mathbf{x}'| \frac{\mathbf{v}'}{c} \quad (\text{F.30})$$

$$\left(\frac{\partial t'}{\partial t} \right)_{\mathbf{x}} = \frac{|\mathbf{x} - \mathbf{x}'|}{s} \quad (\text{F.31})$$

F.3 Special relativity

F.3.1 Metric tensor for flat 4D space

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{F.32})$$

F.3.2 Covariant and contravariant four-vectors

$$v_\mu = \eta_{\mu\nu} v^\nu \quad (\text{F.33})$$

F.3.3 Lorentz transformation of a four-vector

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad (\text{F.34})$$

$$(\Lambda^\mu_\nu) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{F.35})$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (\text{F.36})$$

$$\beta = \frac{v}{c} \quad (\text{F.37})$$

F.3.4 Invariant line element

$$ds = c \frac{dt}{\gamma} = c d\tau \quad (\text{F.38})$$

F.3.5 Four-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \mathbf{v}) \quad (\text{F.39})$$

F.3.6 Four-momentum

$$p^\mu = m_0 u^\mu = \left(\frac{E}{c}, \mathbf{p} \right) \quad (\text{F.40})$$

F.3.7 Four-current density

$$j^\mu = \rho_0 u^\mu \quad (\text{F.41})$$

F.3.8 Four-potential

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A} \right) \quad (\text{F.42})$$

F.3.9 Field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (\text{F.43})$$

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \quad (\text{F.44})$$

F.4 Vector relations

Let \mathbf{x} be the position vector (radius vector, coordinate vector) from the origin to the Cartesian coordinate point $(x_1, x_2, x_3) \equiv (x, y, z)$ in 3D Euclidean space \mathbb{R}^3 and let $|\mathbf{x}|$ denote the magnitude ('length') of \mathbf{x} . Let further $\alpha(\mathbf{x}), \beta(\mathbf{x}), \dots$, be arbitrary scalar fields, $\mathbf{a}(\mathbf{x}), \mathbf{b}(\mathbf{x}), \dots$, arbitrary vector fields, and $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}), \dots$, arbitrary rank two tensor fields in this space. Let $*$ denote *complex conjugate* and † denote *Hermitian conjugate* (transposition and, where applicable, complex conjugation).

The differential vector operator ∇ is in Cartesian coordinates given by

$$\nabla \equiv \sum_{i=1}^3 \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{=} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{=} \partial \quad (\text{F.45})$$

where $\hat{\mathbf{x}}_i$, $i = 1, 2, 3$ is the i th unit vector and $\hat{\mathbf{x}}_1 \equiv \hat{\mathbf{x}}$, $\hat{\mathbf{x}}_2 \equiv \hat{\mathbf{y}}$, and $\hat{\mathbf{x}}_3 \equiv \hat{\mathbf{z}}$. In component (tensor) notation ∇ can be written

$$\nabla_i = \partial_i = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (\text{F.46})$$

F.4.1 Spherical polar coordinates

F.4.1.1 Base vectors

$$\hat{\mathbf{r}} = \sin \theta \cos \varphi \hat{\mathbf{x}}_1 + \sin \theta \sin \varphi \hat{\mathbf{x}}_2 + \cos \theta \hat{\mathbf{x}}_3 \quad (\text{F.47a})$$

$$\hat{\boldsymbol{\theta}} = \cos \theta \cos \varphi \hat{\mathbf{x}}_1 + \cos \theta \sin \varphi \hat{\mathbf{x}}_2 - \sin \theta \hat{\mathbf{x}}_3 \quad (\text{F.47b})$$

$$\hat{\boldsymbol{\phi}} = -\sin \varphi \hat{\mathbf{x}}_1 + \cos \varphi \hat{\mathbf{x}}_2 \quad (\text{F.47c})$$

$$\hat{\mathbf{x}}_1 = \sin \theta \cos \varphi \hat{\mathbf{r}} + \cos \theta \cos \varphi \hat{\boldsymbol{\theta}} - \sin \varphi \hat{\boldsymbol{\phi}} \quad (\text{F.48a})$$

$$\hat{\mathbf{x}}_2 = \sin \theta \sin \varphi \hat{\mathbf{r}} + \cos \theta \sin \varphi \hat{\boldsymbol{\theta}} + \cos \varphi \hat{\boldsymbol{\phi}} \quad (\text{F.48b})$$

$$\hat{\mathbf{x}}_3 = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \quad (\text{F.48c})$$

F.4.1.2 Directed line element

$$dx \hat{\mathbf{x}} = d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\varphi \hat{\boldsymbol{\phi}} \quad (\text{F.49})$$

F.4.1.3 Solid angle element

$$d\Omega = \sin \theta d\theta d\varphi \quad (\text{F.50})$$

F.4.1.4 Directed area element

$$d^2x \hat{\mathbf{n}} = d\mathbf{S} = dS \hat{\mathbf{r}} = r^2 d\Omega \hat{\mathbf{r}} \quad (\text{F.51})$$

F.4.1.5 Volume element

$$d^3x = dV = dr dS = r^2 dr d\Omega \quad (\text{F.52})$$

F.4.2 Vector and tensor formulæ

In the following, $\mathbf{x} = x_i \hat{\mathbf{x}}_i$ and $\mathbf{x}' = x'_i \hat{\mathbf{x}}_i$ are position vectors, \mathbf{k} an arbitrary *constant* vector, $\mathbf{a} = \mathbf{a}(\mathbf{x})$ an arbitrary vector field, $\nabla \equiv \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i}$, $\nabla' \equiv \hat{\mathbf{x}}_i \frac{\partial}{\partial x'_i}$, and $\mathbf{1}_n$ the n -dimensional unit tensor of rank two.

$$\mathbf{1}_3 = \hat{\mathbf{x}}_i \hat{\mathbf{x}}_i \quad (\text{F.53})$$

$$(\mathbf{1}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{F.54})$$

We also introduce the vector

$$\mathbf{S} = S_i \hat{\mathbf{x}}_i \quad (\text{F.55})$$

where the components S_i are the matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{F.56})$$

F.4.2.1 General vector and tensor algebraic identities

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \delta_{ij} a_i b_j = ab \cos \theta \quad (\text{F.57})$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = \epsilon_{ijk} a_j b_k \hat{\mathbf{x}}_i = \epsilon_{jki} a_k b_i \hat{\mathbf{x}}_j = \epsilon_{kij} a_i b_j \hat{\mathbf{x}}_k \quad (\text{F.58})$$

$$(\mathbf{a} \times \mathbf{b}) = -i\mathbf{a} \cdot \mathbf{S}\mathbf{b} = -i\mathbf{a}\mathbf{S} \cdot \mathbf{b} \quad (\text{F.59})$$

$$\mathbf{A} = A_{ij} \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \quad (\text{F.60})$$

$$\mathbf{A}^\dagger = A_{ij}^* \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i \quad (\text{F.61})$$

$$A_{ij} = \hat{\mathbf{x}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{x}}_j \quad (\text{F.62})$$

$$\text{Tr}(\mathbf{A}) = A_{ii} \quad (\text{F.63})$$

$$\mathbf{a}\mathbf{b} = a_i b_j \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \quad (\text{F.64})$$

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a}\mathbf{b} + \mathbf{a}\mathbf{c} \quad (\text{F.65})$$

$$(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{a}\mathbf{c} + \mathbf{b}\mathbf{c} \quad (\text{F.66})$$

$$\mathbf{c} \cdot \mathbf{a}\mathbf{b} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} \quad (\text{F.67})$$

$$\mathbf{a}\mathbf{b} \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \quad (\text{F.68})$$

$$\mathbf{c} \times \mathbf{a}\mathbf{b} = (\mathbf{c} \times \mathbf{a})\mathbf{b} \quad (\text{F.69})$$

$$\mathbf{a}\mathbf{b} \times \mathbf{c} = \mathbf{a}(\mathbf{b} \times \mathbf{c}) \quad (\text{F.70})$$

$$\mathbf{a}\mathbf{b} \cdot \mathbf{c}\mathbf{d} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}\mathbf{d} \quad (\text{F.71})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \quad (\text{F.72})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \mathbf{a} \cdot \mathbf{c} - \mathbf{c} \mathbf{a} \cdot \mathbf{b} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \quad (\text{F.73})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0} \quad (\text{F.74})$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \quad (\text{F.75})$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d} \quad (\text{F.76})$$

F.4.2.2 Special vector and tensor algebraic identities

$$\mathbf{1}_3 \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{1}_3 = \mathbf{a} \quad (\text{F.77})$$

$$\mathbf{1}_3 \times \mathbf{a} = \mathbf{a} \times \mathbf{1}_3 \quad (\text{F.78})$$

$$(\mathbf{1}_3 \times \mathbf{a}) = (\mathbf{a} \times \mathbf{1}_3) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = -\mathbf{i} \mathbf{S} \cdot \mathbf{a} \quad (\text{F.79})$$

$$\mathbf{a} \cdot (\mathbf{1}_3 \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \quad (\text{F.80})$$

$$\mathbf{a} \cdot (\mathbf{1}_3 \times \mathbf{b}) = \mathbf{a} \times \mathbf{b} \quad (\text{F.81})$$

$$\mathbf{1}_3 \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \mathbf{a} - \mathbf{a} \mathbf{b} \quad (\text{F.82})$$

F.4.2.3 General vector and tensor analytic identities

$$\nabla \alpha = \hat{\mathbf{x}}_i \partial_i \alpha \quad (\text{F.83})$$

$$\nabla \mathbf{a} = \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \partial_i a_j \quad (\text{F.84})$$

$$\nabla \cdot \mathbf{a} = \partial_i a_i \quad (\text{F.85})$$

$$\mathbf{a} \cdot \nabla = a_i \partial_i \quad (\text{F.86})$$

$$\nabla \times \mathbf{a} = \epsilon_{ijk} \hat{\mathbf{x}}_i \partial_j a_k \quad (\text{F.87})$$

$$(\nabla \times \mathbf{a}) = -\mathbf{i} \nabla \cdot \mathbf{S} \mathbf{a} = -\mathbf{i} \nabla \mathbf{S} \cdot \mathbf{a} \quad (\text{F.88})$$

$$\mathbf{a} \times \nabla = \epsilon_{ijk} \hat{\mathbf{x}}_i a_j \partial_k \quad (\text{F.89})$$

$$\nabla(\alpha \beta) = \alpha \nabla \beta + \beta \nabla \alpha \quad (\text{F.90})$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a} \quad (\text{F.91})$$

$$= \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} + (\mathbf{a} \cdot \nabla) \mathbf{b} \quad (\text{F.92})$$

$$\nabla(\mathbf{a} \times \mathbf{b}) = (\nabla \mathbf{a}) \times \mathbf{b} - (\nabla \mathbf{b}) \times \mathbf{a} \quad (\text{F.93})$$

$$\nabla \cdot (\alpha \mathbf{a}) = \mathbf{a} \cdot \nabla \alpha + \alpha \nabla \cdot \mathbf{a} \quad (\text{F.94})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \quad (\text{F.95})$$

$$\nabla \cdot (\mathbf{a} \mathbf{b}) = (\nabla \cdot \mathbf{a}) \mathbf{b} + \mathbf{a} \cdot \nabla \mathbf{b} \quad (\text{F.96})$$

$$\nabla \times (\alpha \mathbf{a}) = \alpha \nabla \times \mathbf{a} - \mathbf{a} \times \nabla \alpha \quad (\text{F.97})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \nabla \cdot (\mathbf{b} \mathbf{a} - \mathbf{a} \mathbf{b}) \quad (\text{F.98})$$

$$\nabla \times (\mathbf{a} \mathbf{b}) = (\nabla \times \mathbf{a}) \mathbf{b} - \mathbf{a} \times \nabla \mathbf{b} \quad (\text{F.99})$$

$$\mathbf{a} \times (\nabla \times \mathbf{b}) = (\nabla \mathbf{b}) \cdot \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{b} \quad (\text{F.100})$$

$$\mathbf{a} \times (\nabla \times \mathbf{a}) = \frac{1}{2} \nabla(a^2) - \mathbf{a} \cdot \nabla \mathbf{a} \quad (\text{F.101})$$

$$(\mathbf{a} \times \nabla) \times \mathbf{b} = a_i \nabla b_i - \mathbf{a}(\nabla \cdot \mathbf{b}) \quad (\text{F.102})$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla \nabla \cdot \mathbf{a} - \nabla \cdot \nabla \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \quad (\text{F.103})$$

$$\nabla \cdot \nabla \alpha = \nabla^2 \alpha \quad (\text{F.104})$$

$$\nabla \nabla \cdot \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) = \hat{x}_i \partial_i \partial_j a_j \quad (\text{F.105})$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad (\text{F.106})$$

$$\nabla \times \nabla \alpha = \mathbf{0} \quad (\text{F.107})$$

F.4.2.4 Special vector and tensor analytic identities

$$\nabla \cdot (\mathbf{1}_3 \alpha) = \nabla \alpha \quad (\text{F.108})$$

$$\nabla \cdot (\mathbf{1}_3 \cdot \mathbf{a}) = \nabla \cdot \mathbf{a} \quad (\text{F.109})$$

$$\nabla \cdot (\mathbf{1}_3 \times \mathbf{a}) = \nabla \times \mathbf{a} \quad (\text{F.110})$$

$$\nabla \cdot \mathbf{x} = 3 \quad (\text{F.111})$$

$$\nabla \times \mathbf{x} = \mathbf{0} \quad (\text{F.112})$$

$$\nabla \mathbf{x} = \mathbf{1}_3 \quad (\text{F.113})$$

$$\nabla(\mathbf{x} \cdot \mathbf{k}) = \mathbf{k} \quad (\text{F.114})$$

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + (\mathbf{x} \times \nabla) \times \mathbf{a} \quad (\text{F.115})$$

$$\nabla |\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \quad (\text{F.116})$$

$$\nabla (|\mathbf{x} - \mathbf{x}'|) = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} = -\nabla' (|\mathbf{x} - \mathbf{x}'|) \quad (\text{F.117})$$

$$\nabla \left(\frac{1}{|\mathbf{x}|} \right) = -\frac{\mathbf{x}}{|\mathbf{x}|^3} \quad (\text{F.118})$$

$$\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \quad (\text{F.119})$$

$$\nabla \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) = -\nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) = 4\pi \delta(\mathbf{x}) \quad (\text{F.120})$$

$$\nabla \cdot \left(\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \right) = -\nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = 4\pi \delta(\mathbf{x} - \mathbf{x}') \quad (\text{F.121})$$

$$\nabla \cdot \left(\frac{\mathbf{k}}{|\mathbf{x}|} \right) = \mathbf{k} \cdot \left[\nabla \left(\frac{1}{|\mathbf{x}|} \right) \right] = -\frac{\mathbf{k} \cdot \mathbf{x}}{|\mathbf{x}|^3} \quad (\text{F.122})$$

$$\nabla \times \left[\mathbf{k} \times \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} \right) \right] = -\nabla \left(\frac{\mathbf{k} \cdot \mathbf{x}}{|\mathbf{x}|^3} \right) \text{ if } |\mathbf{x}| \neq 0 \quad (\text{F.123})$$

$$\nabla^2 \left(\frac{\mathbf{k}}{|\mathbf{x}|} \right) = \mathbf{k} \nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) = -4\pi \mathbf{k} \delta(\mathbf{x}) \quad (\text{F.124})$$

$$\nabla \times (\mathbf{k} \times \mathbf{a}) = \mathbf{k}(\nabla \cdot \mathbf{a}) + \mathbf{k} \times (\nabla \times \mathbf{a}) - \nabla(\mathbf{k} \cdot \mathbf{a}) \quad (\text{F.125})$$

F.4.2.5 Integral identities

Let $V(S)$ be the volume bounded by the closed surface $S(V)$. Denote the 3D volume element by $d^3x (\equiv dV)$ and the surface element, directed along the outward pointing surface normal unit vector $\hat{\mathbf{n}}$, by $d\mathbf{S} (\equiv d^2x \hat{\mathbf{n}})$. Then

$$\int_V d^3x \nabla \cdot \mathbf{a} = \oint_S d\mathbf{S} \cdot \mathbf{a} \quad (\text{Gauss's theorem, divergence theorem}) \quad (\text{F.126})$$

$$\int_V d^3x \nabla \alpha = \oint_S d\mathbf{S} \alpha \quad (\text{F.127})$$

$$\int_V d^3x \nabla \times \mathbf{a} = \oint_S d\mathbf{S} \times \mathbf{a} \quad (\text{F.128})$$

If $S(C)$ is an open surface bounded by the contour $C(S)$, whose line element is $d\mathbf{l}$, then

$$\oint_C d\mathbf{l} \alpha = \int_S d\mathbf{S} \times \nabla \alpha \quad (\text{F.129})$$

$$\oint_C d\mathbf{l} \cdot \mathbf{a} = \int_S d\mathbf{S} \cdot (\nabla \times \mathbf{a}) \quad (\text{Stokes' theorem, curl theorem}) \quad (\text{F.130})$$

F.5 Bibliography

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M

MATHEMATICAL METHODS

Physics is the academic discipline which systematically studies and describes the physical world, finds new fundamental laws of Nature or generalises existing ones that govern Nature's behaviour under different conditions, and—perhaps most important of all—makes new predictions, based on these physical laws. Then these predictions are put to tough tests in independent, accurately designed, carefully performed, systematic, repeatable experiments that produce objective data. Merely describing Nature and explaining physical experiments in terms of already existing laws is not physics in the true sense of the word. Had this non-creative, static view of physics been adopted by physicists since the renaissance, we would all still be doing essentially Newtonian mechanics.

Even if such a scientific giant as MICHAEL FARADAY, who started out as a lowly bookbinder apprentice and had very little formal mathematical training, was able to make truly remarkable contributions to physics (and chemistry) using practically no formal mathematics whatsoever, it is for us mere mortals most convenient to use the shorthand language of mathematics, together with the formal methods of logic, in physics. After all, mathematics was once introduced by us human beings to make it easier to quantitatively and systematically describe, understand and predict the physical world around us. Examples of this from ancient times are arithmetics and geometry. A less archaic example is differential calculus, needed by SIR ISAAC NEWTON to formulate, in a compact and unambiguous manner, the physical laws that bear his name. Another more modern example is the delta 'function' introduced by PAUL ADRIEN MAURICE DIRAC. But the opposite is also very common: the expansion and generalisation of mathematics has more than once provided excellent tools for

creating new physical ideas and to better analyse observational data. Examples of the latter include non-Euclidean geometry and group theory.

Unlike mathematics *per se*, where the criterion of logical consistency is both necessary and sufficient, a physical theory that is supposed to describe the physical reality has to fulfil the additional criterion that its predictions be empirically testable. Ultimately, as GALILEO GALILEI taught us, physical reality is defined by the outcome of experiments and observations, and not by mere Aristotelean logical reasoning, however mathematically correct and logically consistent this may be. Common sense is not enough and logic and reasoning can never ‘outsmart’ Nature. Should the outcome of repeated, carefully performed, independent experiments produce results that systematically contradict predictions based on a logically stringent theory, the only conclusion one can draw is that the theory in question is wrong.¹

On the other hand, extending existing physical theories by mathematical and logical generalisations is a very powerful way of making hypotheses and predictions of new physical phenomena. But it is not until these hypotheses and predictions have withstood empirical tests in physical experiments that they can be said to extend physics and our knowledge about Nature.

This appendix describes briefly some of the more common mathematical methods and tools that are used in Classical Electrodynamics.

¹ The theoretical physicist SIR RUDOLF PEIERLS (1907–1995) described an ideal physical theory in the following way:

‘It must firstly leave undisturbed the successes of earlier work and not upset the explanations of observations that had been used in support of earlier ideas. Secondly it must explain in a reasonable manner the new evidence which brought the previous ideas into doubt and which suggested the new hypothesis. And thirdly it must predict new phenomena or new relationships between different phenomena, which were not known or not clearly understood at the time when it was invented.’

M.1 Scalars, vectors and tensors

Every physical observable can be represented by a mathematical object. We have chosen to describe the observables in classical electrodynamics in terms of scalars, pseudoscalars, vectors, pseudovectors, tensors and pseudotensors, all of which obey certain canonical rules of transformation under a change of coordinate systems and are completely defined by these rules. Despite certain advantages (and some shortcomings), differential forms will not be exploited to any significant degree in our mathematical description of physical observables.

A *scalar*, which may or may not be constant in time and/or space, describes the scaling of a physical quantity. A *vector* describes some kind of physical motion along a curve in space due tovection. A *tensor* describes the local motion or deformation of a surface or a volume due to some form of tension and is therefore a relation between a set vectors. However, generalisations to more abstract notions of these quantities have proved useful and are therefore commonplace. The difference between a scalar, vector and tensor and a *pseudoscalar*, *pseudovector* and a *pseudotensor* is that the latter behave differently under those coordinate transformations that cannot be reduced to pure rotations.

For computational convenience, it is often useful to allow mathematical objects representing physical observables to be complex-valued, *i.e.*, to let them be analytically continued into (a domain of) the complex plane. However, since by definition our physical world is real, care must be exercised when comparing mathematical results with physical observables, *i.e.*, real-valued numbers obtained from physical measurements.

Throughout we adopt the convention that Latin indices i, j, k, l, \dots run over the range 1, 2, 3 to denote vector or tensor components in the real Euclidean three-dimensional (3D) configuration space \mathbb{R}^3 , and Greek indices $\mu, \nu, \kappa, \lambda, \dots$, which are used in four-dimensional (4D) space, run over the range 0, 1, 2, 3.

M.1.1 Vectors

Mathematically, a vector can be represented in a number of different ways. One suitable representation in a vector space of dimension N is in terms of an ordered N -tuple of real or complex numbers² (a_1, a_2, \dots, a_N) of the components along N orthogonal coordinate axes that span the vector space under consideration. Note, however, that there are many ordered N -tuples of numbers that do not comprise a vector, *i.e.*, do not exhibit vector transformation properties!

M.1.1.1 Position vector

The most basic vector, and the prototype against which all other vectors are benchmarked, is the *position vector* (*radius vector*, *coordinate vector*) which is the vector from the origin of the chosen coordinate system to the actual point of interest. Its N -tuple representation simply enumerates the coordinates of the position of this point. In this sense, the vector from the origin to a point is synonymous with the coordinates of the point itself.

In the 3D Euclidean space \mathbb{R}^3 , we have $N = 3$ and the position vector \mathbf{x} can be represented by the triplet (x_1, x_2, x_3) of its coordinates x_i , $i = 1, 2, 3$. The coordinates x_i are scalar quantities which describe the position along the unit base vectors $\hat{\mathbf{x}}_i$ which span \mathbb{R}^3 . Therefore a convenient representation of the position vector in \mathbb{R}^3 is³

$$\mathbf{x} = \sum_{i=1}^3 x_i \hat{\mathbf{x}}_i \stackrel{\text{def}}{=} x_i \hat{\mathbf{x}}_i \quad (\text{M.1})$$

where we have introduced *Einstein's summation convention* (E Σ) which states that a repeated index in a term implies summation over the range of the index in question. Whenever possible and convenient we shall in the following always assume E Σ and suppress explicit summation in our formulæ. Typographically,

² It is often very convenient to use *complex notation* in physics. This notation can simplify the mathematical treatment considerably. But since all physical observables are real, we must in the final step of our mathematical analysis of a physical problem always ensure that the results to be compared with experimental values are real-valued. In classical physics this is achieved by taking the real (or imaginary) part of the mathematical result, whereas in quantum physics one takes the absolute value.

³ We introduce the symbol $\stackrel{\text{def}}{=}$ which may be read 'is, by definition, to equal in meaning', or 'equals by definition', or, formally, $\stackrel{\text{def}}{=} \text{definiendum} \stackrel{\text{def}}{=} \text{definiens}$. Another symbol sometimes used is $:=$.

we represent a vector or vector operator in 3D Euclidean space \mathbb{R}^3 by a boldface letter or symbol in a Roman font, for instance \mathbf{a} , ∇ , \cdot , and \mathbf{x} .

Alternatively, we can describe the position vector \mathbf{x} in *component notation* as x_i where

$$x_i \stackrel{\text{def}}{=} (x_1, x_2, x_3) \quad (\text{M.2})$$

⁴ In spherical polar coordinates $(x_1, x_2, x_3) = (r, \theta, \varphi)$ and in cylindrical coordinates $(x_1, x_2, x_3) = (\rho, \varphi, z)$.

In Cartesian coordinates⁴

$$(x_1, x_2, x_3) = (x, y, z) \quad (\text{M.3})$$

This component notation is particularly useful in 4D space where we can represent the (one and the same) position vector either in its *contravariant component form*, (superscript index form) as the quartet

$$x^\mu \stackrel{\text{def}}{=} (x^0, x^1, x^2, x^3) \quad (\text{M.4})$$

or its *covariant component form* (subscript index form)

$$x_\mu \stackrel{\text{def}}{=} (x_0, x_1, x_2, x_3) \quad (\text{M.5})$$

The relation between the covariant and contravariant forms is determined by the *metric tensor* (also known as the *fundamental tensor*) whose actual form is dictated by the properties of the vector space in question. The dual representation of vectors in contravariant and covariant forms is most convenient when we work in a vector space with an indefinite *metric*. An example is *Lorentz space* \mathbb{L}^4 which is a 4D *Riemannian space* frequently employed to formulate the special theory of relativity.

M.1.2 Fields

A *field* is a physical entity that depends on one or more continuous parameters. Such a parameter can be viewed as a ‘continuous index’ which enumerates the ‘coordinates’ of the field. In particular, in a field which depends on the usual position vector \mathbf{x} of \mathbb{R}^3 , each point in this space can be considered as one degree of freedom so that a field is a representation of a physical entity which has an infinite number of degrees of freedom.

M.1.2.1 Scalar fields

We denote an arbitrary *scalar field* in \mathbb{R}^3 by

$$\alpha(\mathbf{x}) = \alpha(x_1, x_2, x_3) \stackrel{\text{def}}{=} \alpha(x_i) \quad (\text{M.6})$$

This field describes how the scalar quantity α varies continuously in 3D \mathbb{R}^3 space.

In 4D, a *four-scalar* field is denoted

$$\alpha(x^0, x^1, x^2, x^3) \stackrel{\text{def}}{=} \alpha(x^\mu) \quad (\text{M.7})$$

which indicates that the four-scalar α depends on all four coordinates spanning this space. Since a four-scalar has the same value at a given point regardless of coordinate system, it is also called an *invariant*.

M.1.2.2 Vector fields

We can represent an arbitrary 3D vector field $\mathbf{a}(\mathbf{x})$ as follows:

$$\mathbf{a}(\mathbf{x}) = a_i(\mathbf{x})\hat{\mathbf{x}}_i \in \mathbb{R}^3 \quad (\text{M.8})$$

In component notation this same vector can be represented as

$$a_i(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x})) = a_i(x_j) \quad (\text{M.9})$$

A 3D *complex vector field* $\mathbf{c}(\mathbf{x})$ is a vector in \mathbb{C}^3 (or, if we like, in \mathbb{R}^6), expressed in terms of two real vectors \mathbf{c}_R and \mathbf{c}_I in \mathbb{R}^3 in the following way

$$\mathbf{c}(\mathbf{x}) \stackrel{\text{def}}{=} \mathbf{c}_R(\mathbf{x}) + i\mathbf{c}_I(\mathbf{x}) = \mathbf{c}_R(\mathbf{x})\hat{\mathbf{c}}_R + i\mathbf{c}_I(\mathbf{x})\hat{\mathbf{c}}_I \stackrel{\text{def}}{=} c(\mathbf{x})\hat{\mathbf{c}} \in \mathbb{C}^3 \quad (\text{M.10})$$

which means that

$$\text{Re}\{\mathbf{c}\} = \mathbf{c}_R \quad (\text{M.11a})$$

$$\text{Im}\{\mathbf{c}\} = \mathbf{c}_I \quad (\text{M.11b})$$

The use of complex vectors is in many cases a very convenient and powerful technique but requires extra care since physical observables must be represented by real vectors ($\in \mathbb{R}^3$).

▷Complex representation of a rotating vector◀ EXAMPLE M.1

Let us consider the complex vector

$$\begin{aligned} \mathbf{c} &= C e^{i\omega t} (\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) = C (\cos(\omega t) + i\sin(\omega t)) (\hat{\mathbf{x}}_1 \pm i\hat{\mathbf{x}}_2) \\ &= C (\cos(\omega t)\hat{\mathbf{x}}_1 \mp \sin(\omega t)\hat{\mathbf{x}}_2) + iC (\sin(\omega t)\hat{\mathbf{x}}_1 \pm \cos(\omega t)\hat{\mathbf{x}}_2) \end{aligned} \quad (\text{M.12})$$

We see that the real part of this vector describes a rotation in positive or negative sense, depending on the sign.

◀End of example M.1▷

In 4D, an arbitrary *four-vector* field in contravariant component form can be represented as

$$a^\mu(x^\nu) = (a^0(x^\nu), a^1(x^\nu), a^2(x^\nu), a^3(x^\nu)) \quad (\text{M.13})$$

or, in *covariant* component form, as

$$a_\mu(x^\nu) = (a_0(x^\nu), a_1(x^\nu), a_2(x^\nu), a_3(x^\nu)) \quad (\text{M.14})$$

where x^ν is the *position four-vector* (*radius four-vector*, *coordinate four-vector*). Again, the relation between a^μ and a_μ is determined by the metric of the physical 4D system under consideration.

M.1.2.3 Coordinate transformations

We note that for a change of coordinates $x^\mu \rightarrow x'^\mu = x'^\mu(x^0, x^1, x^2, x^3)$, due to a transformation from one coordinate system Σ to another coordinate system Σ' , the differential position vector dx^μ transforms as

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \quad (\text{M.15})$$

This follows trivially from the rules of differentiation of x'^μ considered as a function of four variables x^ν , *i.e.*, $x'^\mu = x'^\mu(x^\nu)$. Analogous to the transformation rule for the differential dx^μ , equation (M.15), the transformation rule for the differential operator $\partial/\partial x^\mu$ under a transformation $x^\mu \rightarrow x'^\mu$ becomes

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \quad (\text{M.16})$$

which, again, follows trivially from the rules of differentiation.

Whether an arbitrary N -tuple fulfils the requirement of being an (N -dimensional) contravariant vector or not, depends on its transformation properties during a change of coordinates. For instance, in 4D an assemblage $y^\mu = (y^0, y^1, y^2, y^3)$ constitutes a *contravariant four-vector* (or the contravariant components of a four-vector) if and only if, during a transformation from a system Σ with coordinates x^μ to a system Σ' with coordinates x'^μ , it transforms to the new system according to the rule

$$y'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} y^\nu \quad (\text{M.17})$$

i.e., in the same way as the differential coordinate element dx^μ transforms according to equation (M.15) above.

The analogous requirement for a *covariant four-vector* is that it transforms, during the change from Σ to Σ' , according to the rule

$$y'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} y_\nu \quad (\text{M.18})$$

i.e., in the same way as the differential operator $\partial/\partial x^\mu$ transforms according to equation (M.16).

M.1.2.4 Tensor fields

We denote an arbitrary *tensor field* in \mathbb{R}^3 by $\mathbf{A}(\mathbf{x})$. This tensor field can be represented in a number of ways, for instance in the following *matrix representation*:⁵

$$(\mathbf{A}(\mathbf{x})) \stackrel{\text{def}}{=} (A_{ij}(x_k)) \stackrel{\text{def}}{=} \begin{pmatrix} A_{11}(\mathbf{x}) & A_{12}(\mathbf{x}) & A_{13}(\mathbf{x}) \\ A_{21}(\mathbf{x}) & A_{22}(\mathbf{x}) & A_{23}(\mathbf{x}) \\ A_{31}(\mathbf{x}) & A_{32}(\mathbf{x}) & A_{33}(\mathbf{x}) \end{pmatrix} \quad (\text{M.19})$$

⁵ When a mathematical object representing a physical observable is given in matrix representation, we indicate this by enclosing the mathematical object in question in parentheses, *i.e.*, (\dots) .

Strictly speaking, the tensor field described here is a tensor of *rank two*.

A particularly simple rank two tensor in \mathbb{R}^3 is the 3D *Kronecker delta tensor* δ_{ij} , with the following properties:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad (\text{M.20})$$

The 3D Kronecker delta tensor has the following matrix representation

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{M.21})$$

Another common and useful tensor is the *fully antisymmetric tensor* of rank three, also known as the *Levi-Civita tensor*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ 0 & \text{if at least two of } i, j, k \text{ are equal} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \end{cases} \quad (\text{M.22})$$

Clearly, this tensor fulfils the relations

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \quad (\text{M.23})$$

and

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} \quad (\text{M.24})$$

and has the following further property

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad (\text{M.25})$$

In fact, tensors may have any rank n . In this picture, a scalar is considered to be a tensor of rank $n = 0$ and a vector to be a tensor of rank $n = 1$. Consequently, the notation where a vector (tensor) is represented in its component form is called the *tensor notation*. A tensor of rank $n = 2$ may be represented by

a two-dimensional array or matrix, and a tensor of rank $n = 3$ may be represented as a vector of tensors of rank $n = 2$. Assuming that one of the indices of the Levi-Civita tensor ϵ_{ijk} , e.g., the first index $i = 1, 2, 3$, denotes the component of such a vector of tensors, these components have the matrix representations (the second and third indices, $j, k = 1, 2, 3$, are the matrix indices)

$$((\epsilon_{ijk})_{i=1}) = \begin{pmatrix} \epsilon_{111} & \epsilon_{112} & \epsilon_{113} \\ \epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\ \epsilon_{131} & \epsilon_{132} & \epsilon_{133} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = iS_1 \quad (\text{M.26a})$$

$$((\epsilon_{ijk})_{i=2}) = \begin{pmatrix} \epsilon_{211} & \epsilon_{212} & \epsilon_{213} \\ \epsilon_{221} & \epsilon_{222} & \epsilon_{223} \\ \epsilon_{231} & \epsilon_{232} & \epsilon_{233} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = iS_2 \quad (\text{M.26b})$$

$$((\epsilon_{ijk})_{i=3}) = \begin{pmatrix} \epsilon_{311} & \epsilon_{312} & \epsilon_{313} \\ \epsilon_{321} & \epsilon_{322} & \epsilon_{323} \\ \epsilon_{331} & \epsilon_{332} & \epsilon_{333} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iS_3 \quad (\text{M.26c})$$

Here we have introduced the matrix vector

$$\mathbf{S} = S_i \hat{\mathbf{x}}_i \quad (\text{M.27})$$

where the vector components S_i are the matrices⁶

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{M.28})$$

which satisfy the *angular-momentum commutation rule*

$$[S_i, S_j] = -i\epsilon_{ijk} S_k \quad (\text{M.29})$$

Tensors of rank higher than 3 are best represented in their tensor notation (component form). It is important to remember that a tensor of any rank is fully and totally characterized by its transformation properties under the change of coordinates. This is a very strict constraint.

EXAMPLE M.2 ▷Tensors in 3D space

Consider a tetrahedron-like volume element V of a solid, fluid, or gaseous body, whose atomistic structure is irrelevant for the present analysis; figure M.1 on the next page indicates how this volume may look like. Let $d\mathbf{S} = d^2x \hat{\mathbf{n}}$ be the directed surface element of this volume element and let the vector $\mathbf{T}_{\hat{\mathbf{n}}} d^2x$ be the force that matter, lying on the side of d^2x toward which the unit normal vector $\hat{\mathbf{n}}$ points, acts on matter which lies on the opposite side of d^2x . This force concept is meaningful only if the forces are short-range enough that they can be assumed to act only in the surface proper. According to Newton's third law, this surface force fulfils

$$\mathbf{T}_{-\hat{\mathbf{n}}} = -\mathbf{T}_{\hat{\mathbf{n}}} \quad (\text{M.30})$$

Using (M.30) and Newton's second law, we find that the matter of mass m , which at a given

⁶ These matrices are the *spin operators* for a (classical) 3D vector field.

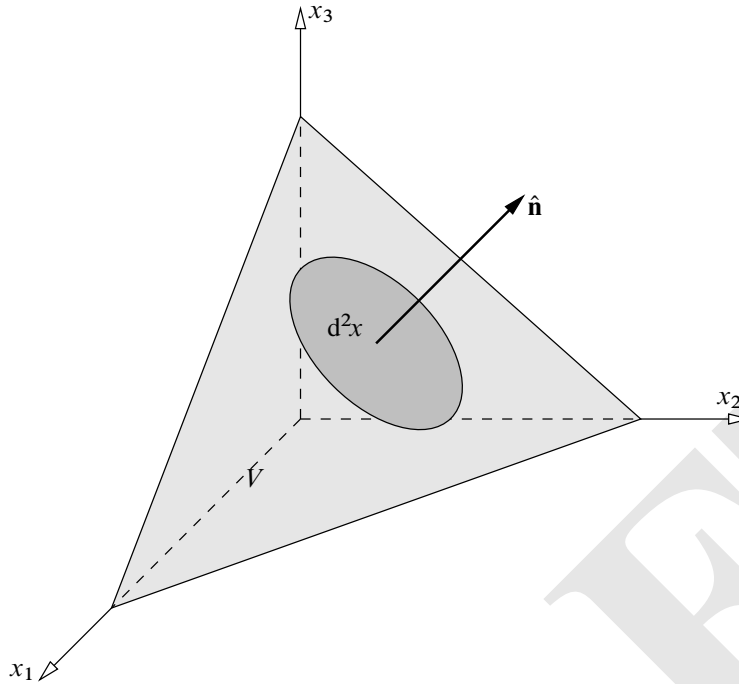


Figure M.1: Tetrahedron-like volume element V containing matter.

instant is located in V obeys the equation of motion

$$\mathbf{T}_{\hat{\mathbf{n}}} d^2x - \cos \theta_1 \mathbf{T}_{\hat{\mathbf{x}}_1} d^2x - \cos \theta_2 \mathbf{T}_{\hat{\mathbf{x}}_2} d^2x - \cos \theta_3 \mathbf{T}_{\hat{\mathbf{x}}_3} d^2x + \mathbf{F}_{\text{ext}} = m\mathbf{a} \quad (\text{M.31})$$

where \mathbf{F}_{ext} is the external force and \mathbf{a} is the acceleration of the volume element. In other words

$$\mathbf{T}_{\hat{\mathbf{n}}} = n_1 \mathbf{T}_{\hat{\mathbf{x}}_1} + n_2 \mathbf{T}_{\hat{\mathbf{x}}_2} + n_3 \mathbf{T}_{\hat{\mathbf{x}}_3} + \frac{m}{d^2x} \left(\mathbf{a} - \frac{\mathbf{F}_{\text{ext}}}{m} \right) \quad (\text{M.32})$$

Since both \mathbf{a} and $\mathbf{F}_{\text{ext}}/m$ remain finite whereas $m/d^2x \rightarrow 0$ as $V \rightarrow 0$, one finds that in this limit

$$\mathbf{T}_{\hat{\mathbf{n}}} = \sum_{i=1}^3 n_i \mathbf{T}_{\hat{\mathbf{x}}_i} \equiv n_i \mathbf{T}_{\hat{\mathbf{x}}_i} \quad (\text{M.33})$$

From the above derivation it is clear that equation (M.33) is valid not only in equilibrium but also when the matter in V is in motion.

Introducing the notation

$$T_{ij} = (\mathbf{T}_{\hat{\mathbf{x}}_i})_j \quad (\text{M.34})$$

for the j th component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, we can write equation (M.33) above in component

form as follows

$$T_{\hat{\mathbf{n}}j} = (\mathbf{T}_{\hat{\mathbf{n}}})_j = \sum_{i=1}^3 n_i T_{ij} \equiv n_i T_{ij} \quad (\text{M.35})$$

Using equation (M.35), we find that the component of the vector $\mathbf{T}_{\hat{\mathbf{n}}}$ in the direction of an arbitrary unit vector $\hat{\mathbf{m}}$ is

$$\begin{aligned} T_{\hat{\mathbf{n}}\hat{\mathbf{m}}} &= \mathbf{T}_{\hat{\mathbf{n}}} \cdot \hat{\mathbf{m}} \\ &= \sum_{j=1}^3 T_{\hat{\mathbf{n}}j} m_j = \sum_{j=1}^3 \left(\sum_{i=1}^3 n_i T_{ij} \right) m_j \equiv n_i T_{ij} m_j = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{m}} \end{aligned} \quad (\text{M.36})$$

Hence, the j th component of the vector $\mathbf{T}_{\hat{\mathbf{x}}_i}$, here denoted T_{ij} , can be interpreted as the ij th component of a tensor \mathbf{T} . Note that $T_{\hat{\mathbf{n}}\hat{\mathbf{m}}}$ is independent of the particular coordinate system used in the derivation.

We shall now show how one can use the momentum law (force equation) to derive the equation of motion for an arbitrary element of mass in the body. To this end we consider a part V of the body. If the external force density (force per unit volume) is denoted by \mathbf{f} and the velocity for a mass element dm is denoted by \mathbf{v} , we obtain

$$\frac{d}{dt} \int_V \mathbf{v} \, dm = \int_V \mathbf{f} \, d^3x + \int_S \mathbf{T}_{\hat{\mathbf{n}}} \, d^2x \quad (\text{M.37})$$

The j th component of this equation can be written

$$\int_V \frac{d}{dt} v_j \, dm = \int_V f_j \, d^3x + \int_S T_{\hat{\mathbf{n}}j} \, d^2x = \int_V f_j \, d^3x + \int_S n_i T_{ij} \, d^2x \quad (\text{M.38})$$

where, in the last step, equation (M.35) above was used. Setting $dm = \rho \, d^3x$ and using the divergence theorem on the last term, we can rewrite the result as

$$\int_V \rho \frac{d}{dt} v_j \, d^3x = \int_V f_j \, d^3x + \int_V \frac{\partial T_{ij}}{\partial x_i} \, d^3x \quad (\text{M.39})$$

Since this formula is valid for any arbitrary volume, we must require that

$$\rho \frac{d}{dt} v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \quad (\text{M.40})$$

or, equivalently

$$\rho \frac{\partial v_j}{\partial t} + \rho \mathbf{v} \cdot \nabla v_j - f_j - \frac{\partial T_{ij}}{\partial x_i} = 0 \quad (\text{M.41})$$

Note that $\partial v_j / \partial t$ is the rate of change with time of the velocity component v_j at a *fixed* point $\mathbf{x} = (x_1, x_1, x_3)$.

—End of example M.2◁

In 4D, we have three forms of *four-tensor fields* of rank n . We speak of

- a *contravariant four-tensor field*, denoted $A^{\mu_1 \mu_2 \dots \mu_n}(x^\nu)$,
- a *covariant four-tensor field*, denoted $A_{\mu_1 \mu_2 \dots \mu_n}(x^\nu)$,
- a *mixed four-tensor field*, denoted $A^{\mu_1 \mu_2 \dots \mu_k}_{\mu_{k+1} \dots \mu_n}(x^\nu)$.

The 4D *metric tensor* (*fundamental tensor*) mentioned above is a particularly important four-tensor of rank two. In covariant component form we shall denote it $g_{\mu\nu}$. This metric tensor determines the relation between an arbitrary contravariant four-vector a^μ and its covariant counterpart a_μ according to the following rule:

$$a_\mu(x^\kappa) \stackrel{\text{def}}{=} g_{\mu\nu} a^\nu(x^\kappa) \quad (\text{M.42})$$

This rule is often called *lowering of index*. The *raising of index* analogue of the index lowering rule is:

$$a^\mu(x^\kappa) \stackrel{\text{def}}{=} g^{\mu\nu} a_\nu(x^\kappa) \quad (\text{M.43})$$

More generally, the following lowering and raising rules hold for arbitrary rank n mixed tensor fields:

$$g_{\mu_k \nu_k} A^{\nu_1 \nu_2 \dots \nu_{k-1} \nu_k}_{\nu_{k+1} \nu_{k+2} \dots \nu_n}(x^\kappa) = A^{\nu_1 \nu_2 \dots \nu_{k-1}}_{\mu_k \nu_{k+1} \dots \nu_n}(x^\kappa) \quad (\text{M.44})$$

$$g^{\mu_k \nu_k} A^{\nu_1 \nu_2 \dots \nu_{k-1} \nu_k}_{\nu_{k+1} \nu_{k+2} \dots \nu_n}(x^\kappa) = A^{\nu_1 \nu_2 \dots \nu_{k-1} \mu_k}_{\nu_{k+1} \nu_{k+2} \dots \nu_n}(x^\kappa) \quad (\text{M.45})$$

Successive lowering and raising of more than one index is achieved by a repeated application of this rule. For example, a dual application of the lowering operation on a rank two tensor in its contravariant form yields

$$A_{\mu\nu} = g_{\mu\kappa} g_{\lambda\nu} A^{\kappa\lambda} \quad (\text{M.46})$$

i.e., the same rank two tensor in its covariant form.

▷ Contravariant and covariant vectors in flat Lorentz space ————— EXAMPLE M.3

The 4D Lorentz space \mathbb{L}^4 has a simple metric which can be described by the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (\text{M.47})$$

which in matrix notation becomes

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{M.48})$$

i.e., a matrix with a main diagonal that has the sign sequence, or *signature*, $\{+, -, -, -\}$.

Alternatively, one can define the metric tensor in \mathbb{L}^4 as

$$\eta_{\mu\nu} = \begin{cases} -1 & \text{if } \mu = \nu = 0 \\ 1 & \text{if } \mu = \nu = i = j = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (\text{M.49})$$

which in matrix representation becomes

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{M.50})$$

i.e., a matrix with signature $\{-, +, +, +\}$. Of course, the physics is unaffected by the choice of metric tensor.

Consider an arbitrary contravariant four-vector a^ν in this space. In component form it can be written:

$$a^\nu \stackrel{\text{def}}{=} (a^0, a^1, a^2, a^3) = (a^0, \mathbf{a}) \quad (\text{M.51})$$

According to the index lowering rule, equation (M.42) on the preceding page, we obtain the covariant version of this vector as

$$a_\mu \stackrel{\text{def}}{=} (a_0, a_1, a_2, a_3) = \eta_{\mu\nu} a^\nu \quad (\text{M.52})$$

In the $\{+, -, -, -\}$ metric we obtain

$$\mu = 0: \quad a_0 = 1 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = a^0 \quad (\text{M.53})$$

$$\mu = 1: \quad a_1 = 0 \cdot a^0 - 1 \cdot a^1 + 0 \cdot a^2 + 0 \cdot a^3 = -a^1 \quad (\text{M.54})$$

$$\mu = 2: \quad a_2 = 0 \cdot a^0 + 0 \cdot a^1 - 1 \cdot a^2 + 0 \cdot a^3 = -a^2 \quad (\text{M.55})$$

$$\mu = 3: \quad a_3 = 0 \cdot a^0 + 0 \cdot a^1 + 0 \cdot a^2 - 1 \cdot a^3 = -a^3 \quad (\text{M.56})$$

or

$$a_\mu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) = (a^0, -\mathbf{a}) \quad (\text{M.57})$$

The radius 4-vector itself in \mathbb{L}^4 and in this metric is given by

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{x}) \quad (\text{M.58})$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x^1, -x^2, -x^3) = (ct, -\mathbf{x})$$

Analogously, using the $\{-, +, +, +\}$ metric we obtain

$$a_\mu = (a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3) = (-a^0, \mathbf{a}) \quad (\text{M.59})$$

End of example M.3◀

M.1.3 Vector algebra

M.1.3.1 Scalar product

The *scalar product* (*dot product*, *inner product*) of two arbitrary 3D vectors **a** and **b** in Euclidean \mathbb{R}^3 space is the scalar number

$$\mathbf{a} \cdot \mathbf{b} = a_i \hat{\mathbf{x}}_i \cdot b_j \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j a_i b_j = \delta_{ij} a_i b_j = a_i b_i \quad (\text{M.60})$$

where we used the fact that the scalar product $\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j$ is a representation of the Kronecker delta δ_{ij} defined in equation (M.20) on page 209.⁷ The scalar product of a vector **a** in \mathbb{R}^3 with itself is

⁷ In the Russian literature, the 3D scalar product is often denoted (**ab**).

$$\mathbf{a} \cdot \mathbf{a} \stackrel{\text{def}}{=} (\mathbf{a})^2 = |\mathbf{a}|^2 = (a_i)^2 = a^2 \quad (\text{M.61})$$

and similarly for **b**. This allows us to write

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta \quad (\text{M.62})$$

where θ is the angle between **a** and **b**.

▷Scalar products in complex vector space

EXAMPLE M.4

Let **c** be a complex vector defined as in expression (M.10) on page 207.

The inner product of **c** with itself may be defined as

$$\mathbf{c}^2 \stackrel{\text{def}}{=} \mathbf{c} \cdot \mathbf{c} = \mathbf{c}_R^2 - \mathbf{c}_I^2 + 2i\mathbf{c}_R \cdot \mathbf{c}_I \stackrel{\text{def}}{=} c^2 \in \mathbb{C} \quad (\text{M.63})$$

from which we find that

$$c = \sqrt{\mathbf{c}_R^2 - \mathbf{c}_I^2 + 2i\mathbf{c}_R \cdot \mathbf{c}_I} \in \mathbb{C} \quad (\text{M.64})$$

Using this in equation (M.10) on page 207, we see that we can define the complex unit vector as being

$$\begin{aligned} \hat{\mathbf{c}} &= \frac{\mathbf{c}}{c} = \frac{\mathbf{c}_R}{\sqrt{\mathbf{c}_R^2 - \mathbf{c}_I^2 + 2i\mathbf{c}_R \cdot \mathbf{c}_I}} \hat{\mathbf{c}}_R + i \frac{\mathbf{c}_I}{\sqrt{\mathbf{c}_R^2 - \mathbf{c}_I^2 + 2i\mathbf{c}_R \cdot \mathbf{c}_I}} \hat{\mathbf{c}}_I \\ &= \frac{\mathbf{c}_R \sqrt{\mathbf{c}_R^2 - \mathbf{c}_I^2 - 2i\mathbf{c}_R \cdot \mathbf{c}_I}}{(\mathbf{c}_R^2 + \mathbf{c}_I^2) \sqrt{1 - \frac{4\mathbf{c}_R^2 \mathbf{c}_I^2 \sin^2 \theta}{(\mathbf{c}_R^2 + \mathbf{c}_I^2)^2}}} \hat{\mathbf{c}}_R + i \frac{\mathbf{c}_I \sqrt{\mathbf{c}_R^2 - \mathbf{c}_I^2 - 2i\mathbf{c}_R \cdot \mathbf{c}_I}}{(\mathbf{c}_R^2 + \mathbf{c}_I^2) \sqrt{1 - \frac{4\mathbf{c}_R^2 \mathbf{c}_I^2 \sin^2 \theta}{(\mathbf{c}_R^2 + \mathbf{c}_I^2)^2}}} \hat{\mathbf{c}}_I \in \mathbb{C}^3 \end{aligned} \quad (\text{M.65})$$

On the other hand, the definition of the scalar product in terms of the inner product of a complex vector with its own complex conjugate yields

$$|\mathbf{c}|^2 \stackrel{\text{def}}{=} \mathbf{c} \cdot \mathbf{c}^* = \mathbf{c}_R^2 + \mathbf{c}_I^2 = |c|^2 \quad (\text{M.66})$$

with the help of which we can define the unit vector as

$$\begin{aligned}\hat{\mathbf{c}} &= \frac{\mathbf{c}}{|\mathbf{c}|} = \frac{\mathbf{c}_R}{\sqrt{\mathbf{c}_R^2 + \mathbf{c}_I^2}} \hat{\mathbf{c}}_R + i \frac{\mathbf{c}_I}{\sqrt{\mathbf{c}_R^2 + \mathbf{c}_I^2}} \hat{\mathbf{c}}_I \\ &= \frac{\mathbf{c}_R \sqrt{\mathbf{c}_R^2 + \mathbf{c}_I^2}}{\mathbf{c}_R^2 + \mathbf{c}_I^2} \hat{\mathbf{c}}_R + i \frac{\mathbf{c}_I \sqrt{\mathbf{c}_R^2 + \mathbf{c}_I^2}}{\mathbf{c}_R^2 + \mathbf{c}_I^2} \hat{\mathbf{c}}_I \in \mathbb{C}^3\end{aligned}\quad (\text{M.67})$$

End of example M.4 <

In 4D space we define the scalar product of two arbitrary four-vectors a^μ and b^μ in the following way

$$a_\mu b^\mu = g_{\nu\mu} a^\nu b^\mu = a^\nu b_\nu = g^{\mu\nu} a_\mu b_\nu \quad (\text{M.68})$$

where we made use of the index lowering and raising rules (M.42) and (M.43). The result is a four-scalar, *i.e.*, an invariant which is independent of in which 4D coordinate system it is measured.

The *quadratic differential form*

$$ds^2 = g_{\mu\nu} dx^\nu dx^\mu = dx_\mu dx^\mu \quad (\text{M.69})$$

i.e., the scalar product of the differential position four-vector with itself, is an invariant called the *metric*. It is also the square of the *line element* ds which is the distance between neighbouring points with coordinates x^μ and $x^\mu + dx^\mu$.

EXAMPLE M.5 ▷ Scalar product, norm and metric in Lorentz space

In \mathbb{L}^4 the metric tensor attains a simple form [see example M.3 on page 213] and, hence, the scalar product in equation (M.68) above can be evaluated almost trivially. For the $\{+, -, -, -\}$ signature it becomes

$$a_\mu b^\mu = (a_0, -\mathbf{a}) \cdot (b^0, \mathbf{b}) = a_0 b^0 - \mathbf{a} \cdot \mathbf{b} \quad (\text{M.70})$$

The important scalar product of the \mathbb{L}^4 radius four-vector with itself becomes

$$\begin{aligned}x_\mu x^\mu &= (x_0, -\mathbf{x}) \cdot (x^0, \mathbf{x}) = (ct, -\mathbf{x}) \cdot (ct, \mathbf{x}) \\ &= (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = s^2\end{aligned}\quad (\text{M.71})$$

which is the indefinite, real *norm* of \mathbb{L}^4 . The \mathbb{L}^4 metric is the quadratic differential form

$$ds^2 = dx_\mu dx^\mu = c^2(dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (\text{M.72})$$

End of example M.5 <

M.1.3.2 Vector product

The *vector product* or *cross product* of two arbitrary 3D vectors \mathbf{a} and \mathbf{b} in ordinary \mathbb{R}^3 space is the vector⁸

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \epsilon_{ijk} \hat{\mathbf{x}}_i a_j b_k = \epsilon_{ijk} a_j b_k \hat{\mathbf{x}}_i \quad (\text{M.73})$$

⁸ Sometimes the 3D vector product of \mathbf{a} and \mathbf{b} is denoted $\mathbf{a} \wedge \mathbf{b}$ or, particularly in the Russian literature, $[\mathbf{a}\mathbf{b}]$.

Here ϵ_{ijk} is the Levi-Civita tensor defined in equation (M.22) on page 209. Alternatively,

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{e}} \quad (\text{M.74})$$

where θ is the angle between \mathbf{a} and \mathbf{b} and $\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane spanned by \mathbf{a} and \mathbf{b} .

A spatial reversal of the coordinate system, $(x'_1, x'_2, x'_3) = (-x_1, -x_2, -x_3)$, known as a *parity transformation*, changes sign of the components of the vectors \mathbf{a} and \mathbf{b} so that in the new coordinate system $\mathbf{a}' = -\mathbf{a}$ and $\mathbf{b}' = -\mathbf{b}$, which is to say that the direction of an ordinary vector is not dependent on the choice of the directions of the coordinate axes. On the other hand, as is seen from equation (M.73) above, the cross product vector \mathbf{c} does *not* change sign. Therefore \mathbf{a} (or \mathbf{b}) is an example of a ‘true’ vector, or *polar vector*, whereas \mathbf{c} is an example of an *pseudovector* or *axial vector*.

A prototype for a pseudovector is the angular momentum vector $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ and hence the attribute ‘axial’. Pseudovectors transform as ordinary vectors under translations and proper rotations, but reverse their sign relative to ordinary vectors for any coordinate change involving reflection. Tensors (of any rank) which transform analogously to pseudovectors are called *pseudotensors*. Scalars are tensors of rank zero, and zero-rank pseudotensors are therefore also called *pseudoscalars*, an example being the pseudoscalar $\hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \times \hat{\mathbf{x}}_k) = \hat{\mathbf{x}}_i \cdot (\epsilon_{ijk} \hat{\mathbf{x}}_j)$. This triple product is a representation of the ijk component of the rank three Levi-Civita pseudotensor ϵ_{ijk} .

M.1.3.3 Dyadic product

The *dyadic product* field $\mathbf{A}(\mathbf{x}) \equiv \mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})$ with two juxtaposed vector fields $\mathbf{a}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ is the *outer product* of \mathbf{a} and \mathbf{b} known as a *dyad*. Here \mathbf{a} is called the *antecedent* and \mathbf{b} the *consequent*.

The dyadic product between vectors (rank one tensors) is a special instance of the *direct product*, usually denoted \otimes , between arbitrary rank tensors. This associative but non-commuting operation is also called the *tensor product* or, when applied to matrices (matrix representations of tensors), the *Kronecker product*.

Written out in explicit form, the dyadic product $\mathbf{A}(\mathbf{x}) = \mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})$ becomes

$$\mathbf{A} = \mathbf{a}\mathbf{b} \equiv \mathbf{a} \otimes \mathbf{b} = a_1 \hat{\mathbf{x}}_1 b_1 \hat{\mathbf{x}}_1 + a_1 \hat{\mathbf{x}}_1 b_2 \hat{\mathbf{x}}_2 + \cdots + a_3 \hat{\mathbf{x}}_3 b_3 \hat{\mathbf{x}}_3 \quad (\text{M.75a})$$

$$= a_i \hat{\mathbf{x}}_i b_j \hat{\mathbf{x}}_j = a_i b_j \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i a_i b_j \hat{\mathbf{x}}_j \quad (\text{M.75b})$$

$$= \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \end{pmatrix} \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \quad (\text{M.75c})$$

In matrix representation

$$(\mathbf{A}) = (\mathbf{a}\mathbf{b}) \equiv (\mathbf{a} \otimes \mathbf{b}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \quad (\text{M.76})$$

which we identify with expression (M.19) on page 209, *viz.* a tensor in matrix representation. Hence, a dyadic of two vectors is intimately related to a rank two tensor, emphasising its vectorial characteristics.

Scalar multiplication from the right or from the left of the dyad $\mathbf{A}(\mathbf{x}) = \mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})$ by a vector \mathbf{c} , produces other vectors according to the scheme

$$\mathbf{A} \cdot \mathbf{c} \stackrel{\text{def}}{=} \mathbf{a}\mathbf{b} \cdot \mathbf{c} \stackrel{\text{def}}{=} \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = a_j b_i c_i \hat{\mathbf{x}}_j \quad (\text{M.77a})$$

$$\mathbf{c} \cdot \mathbf{A} \stackrel{\text{def}}{=} \mathbf{c} \cdot \mathbf{a}\mathbf{b} \stackrel{\text{def}}{=} (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = a_i b_j c_i \hat{\mathbf{x}}_j \quad (\text{M.77b})$$

respectively. These two vectors, proportional to \mathbf{a} and \mathbf{b} , respectively, are in general *not* identical to each other. In the first case, \mathbf{c} is known as the *postfactor*, in the second case as the *prefactor*.

Specifically, if $\mathbf{c} = \hat{\mathbf{x}}_j$, then

$$\mathbf{A} \cdot \hat{\mathbf{x}}_j = \mathbf{a}b_j = a_i b_j \hat{\mathbf{x}}_i \quad (\text{M.78a})$$

$$\hat{\mathbf{x}}_j \cdot \mathbf{A} = a_j \mathbf{b} = a_j b_i \hat{\mathbf{x}}_i \quad (\text{M.78b})$$

which means that

$$\hat{\mathbf{x}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{x}}_j = a_i b_j = A_{ij} \quad (\text{M.78c})$$

The vector product can be represented in matrix form as follows:

$$(\mathbf{c}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = -\mathbf{ia} \cdot \mathbf{Sb} = -\mathbf{iaS} \cdot \mathbf{b} \quad (\text{M.79})$$

where \mathbf{Sb} is dyadic product of the matrix vector \mathbf{S} , given by formula (M.27) on page 210, and the vector \mathbf{b} , and \mathbf{aS} is the dyadic product of the vector \mathbf{a} and the matrix vector \mathbf{S} .

▷ **Matrix representation of the vector product in \mathbb{R}^3**

EXAMPLE M.6

Prove that the matrix representation of the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is given by formula (M.79).

According to formula (M.77) on the facing page, the scalar multiplication of a dyadic product of two vectors (in our case \mathbf{S} and \mathbf{b}) from the left by a vector (in our case \mathbf{a}) is interpreted as $\mathbf{a} \cdot \mathbf{S}\mathbf{b} = (\mathbf{a} \cdot \mathbf{S})\mathbf{b}$ where

$$\mathbf{a} \cdot \mathbf{S} = a_i S_i \quad (\text{M.80})$$

and the components S_i are given by formula (M.28) on page 210. Hence

$$\begin{aligned} \mathbf{a} \cdot \mathbf{S} &= a_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= i \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_1 \\ 0 & a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & 0 \\ -a_2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -a_3 & 0 \\ a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ &= i \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \end{aligned} \quad (\text{M.81})$$

from which we find that

$$(\mathbf{a} \cdot \mathbf{S})\mathbf{b} = i \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = i \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \quad (\text{M.82})$$

In other words,

$$-i(\mathbf{a} \cdot \mathbf{S})\mathbf{b} = -i\mathbf{a} \cdot \mathbf{S}\mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = (\mathbf{a} \times \mathbf{b}) \quad (\text{M.83})$$

Likewise, $\mathbf{a}\mathbf{S} \cdot \mathbf{b} = \mathbf{a}(\mathbf{S} \cdot \mathbf{b})$ where

$$\begin{aligned} \mathbf{S} \cdot \mathbf{b} &= S_i b_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} b_1 + \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} b_2 + \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} b_3 \\ &= i \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} \end{aligned} \quad (\text{M.84})$$

and

$$\mathbf{a}(\mathbf{S} \cdot \mathbf{b}) = i \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix} = i \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \quad (\text{M.85})$$

from which follows

$$-i\mathbf{a}(\mathbf{S} \cdot \mathbf{b}) = -i\mathbf{a}\mathbf{S} \cdot \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} = (\mathbf{a} \times \mathbf{b}) \quad (\text{M.86})$$

Hence, $(\mathbf{a} \times \mathbf{b}) = -i\mathbf{a} \cdot \mathbf{S}\mathbf{b} = -i\mathbf{a}\mathbf{S} \cdot \mathbf{b}$

QED ■

—End of example M.6<

Vector multiplication from the right and from the left of the dyad \mathbf{A} by a vector is another dyad according to the scheme

$$\mathbf{A} \times \mathbf{c} = \mathbf{a} \mathbf{b} \times \mathbf{c} \stackrel{\text{def}}{=} \mathbf{a}(\mathbf{b} \times \mathbf{c}) = \epsilon_{jkl} a_i b_k c_l \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \quad (\text{M.87a})$$

$$\mathbf{c} \times \mathbf{A} = \mathbf{c} \times \mathbf{a} \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{c} \times \mathbf{a}) \mathbf{b} = \epsilon_{jkl} a_l b_i c_k \hat{\mathbf{x}}_j \hat{\mathbf{x}}_i = \epsilon_{ikl} a_l b_j c_k \hat{\mathbf{x}}_i \hat{\mathbf{x}}_j \quad (\text{M.87b})$$

respectively. In general, the two new dyads thus created are *not* identical to each other.

Specifically, if $\mathbf{A} = \mathbf{1}_3 = \hat{\mathbf{x}}_i \hat{\mathbf{x}}_i$, i.e., the *unit dyad* or the second-rank *unit tensor*, then

$$\begin{aligned} \mathbf{1}_3 \times \mathbf{c} &= -c_3 \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_2 + c_2 \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_3 \\ &\quad + c_3 \hat{\mathbf{x}}_2 \hat{\mathbf{x}}_1 - c_1 \hat{\mathbf{x}}_2 \hat{\mathbf{x}}_3 \\ &\quad - c_2 \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_3 + c_1 \hat{\mathbf{x}}_3 \hat{\mathbf{x}}_2 = \mathbf{c} \times \mathbf{1}_3 \end{aligned} \quad (\text{M.88})$$

or, in matrix representation,

$$(\mathbf{1}_3 \times \mathbf{c}) = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} = (\mathbf{c} \times \mathbf{1}_3) \quad (\text{M.89})$$

Using the matrix vector formula (M.27) on page 210, we can write this as

$$(\mathbf{1}_3 \times \mathbf{c}) = (\mathbf{c} \times \mathbf{1}_3) = -i\mathbf{S} \cdot \mathbf{c} \quad (\text{M.90})$$

One can extend the dyadic scheme and introduce \mathbf{abc} , called a *tryad*, and so on. In this vein, a vector \mathbf{a} is sometimes called a *monad*.

M.1.4 Vector analysis

M.1.4.1 The *del* operator

In \mathbb{R}^3 the *del operator* is a *differential vector operator*, denoted in *Gibbs' notation* by ∇ and defined as⁹

$$\nabla = \hat{\mathbf{x}}_i \nabla_i \stackrel{\text{def}}{=} \hat{\mathbf{x}}_i \frac{\partial}{\partial x_i} \stackrel{\text{def}}{=} \frac{\partial}{\partial \mathbf{x}} \stackrel{\text{def}}{=} \partial \quad (\text{M.91})$$

where $\hat{\mathbf{x}}_i$ is the i th unit vector in a Cartesian coordinate system. Since the operator in itself has vectorial properties, we denote it with a boldface nabla (∇).

In ‘component’ (tensor) notation the *del* operator can be written

$$\partial_i = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \quad (\text{M.92})$$

⁹ This operator was introduced by WILLIAM ROWEN HAMILTON (1805–1865) who, however, used the symbol \triangleright for it. It is therefore sometimes called the *Hamilton operator*.

In 4D, the contravariant component representation of the *four-del operator* is defined by

$$\partial^\mu = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \quad (\text{M.93})$$

whereas the covariant four-del operator is

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \quad (\text{M.94})$$

We can use this four-del operator to express the transformation properties (M.17) and (M.18) on page 208 as

$$y'^\mu = (\partial_\nu x'^\mu) y^\nu \quad (\text{M.95})$$

and

$$y'_\mu = (\partial'_\mu x^\nu) y_\nu \quad (\text{M.96})$$

respectively.

M.1.4.2 The gradient of a scalar field

The *gradient* of an \mathbb{R}^1 scalar field $\alpha(\mathbf{x})$, is an \mathbb{R}^3 vector field

$$\text{grad } \alpha(\mathbf{x}) \stackrel{\text{def}}{=} \nabla \alpha(\mathbf{x}) = \partial \alpha(\mathbf{x}) = \hat{\mathbf{x}}_i \partial_i \alpha(\mathbf{x}) \quad (\text{M.97})$$

If the scalar field depends only on one coordinate, ζ say, then

$$\nabla \alpha(\mathbf{x}) = \nabla \alpha(\zeta) = \hat{\xi} \frac{\partial \alpha(\zeta)}{\partial \zeta} = \hat{\xi} \nabla \alpha(\zeta) \quad (\text{M.98})$$

and, therefore, $\nabla = \hat{\xi} \nabla$. From this we see that the boldface notation for the gradient (∇) is very handy as it elucidates its 3D vectorial property and separates it from the nabla operator (∇) which has a scalar property.

In 4D, the *four-gradient* of a four-scalar is a covariant vector, formed as a derivative of a four-scalar field $\alpha(x^\mu)$, with the following component form:

$$\partial_\mu \alpha(x^\nu) = \frac{\partial \alpha(x^\nu)}{\partial x^\mu} \quad (\text{M.99a})$$

with the contravariant form

$$\partial^\mu \alpha(x^\nu) = \frac{\partial \alpha(x^\nu)}{\partial x_\mu} \quad (\text{M.99b})$$

EXAMPLE M.7 ▷ Gradient of a scalar product of two vector fields

The gradient of the scalar product of two vector fields \mathbf{a} and \mathbf{b} can be calculated in the following way:

$$\begin{aligned}\nabla(\mathbf{a} \cdot \mathbf{b}) &= (\hat{\mathbf{x}}_i \partial_i)(a_j \hat{\mathbf{x}}_j \cdot b_k \hat{\mathbf{x}}_k) \\ &= [(\hat{\mathbf{x}}_i \partial_i)(a_j \hat{\mathbf{x}}_j)] \cdot (b_k \hat{\mathbf{x}}_k) + (a_j \hat{\mathbf{x}}_j) \cdot [(\hat{\mathbf{x}}_i \partial_i)(b_k \hat{\mathbf{x}}_k)] \\ &= (\hat{\mathbf{x}}_i \partial_i a_j \hat{\mathbf{x}}_j) \cdot \mathbf{b} + (\hat{\mathbf{x}}_i \partial_i b_k \hat{\mathbf{x}}_k) \cdot \mathbf{a} = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a}\end{aligned}\quad (\text{M.100})$$

This is the first version of formula (F.91) on page 200.

End of example M.7◁

M.1.4.3 The divergence of a vector field

We define the 3D *divergence* of a vector field \mathbf{a} in \mathbb{R}^3 as

$$\text{div } \mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{=} \nabla \cdot \mathbf{a}(\mathbf{x}) = \frac{\partial a_i(\mathbf{x})}{\partial x_i} = \partial_i a_i(\mathbf{x}) \quad (\text{M.101})$$

which, as indicated by the notation $\alpha(\mathbf{x})$, is a *scalar* field in \mathbb{R}^3 .

The *four-divergence* of a four-vector a^μ is the four-scalar

$$\partial_\mu a^\mu(x^\nu) = \frac{\partial a^\mu(x^\nu)}{\partial x^\mu} \quad (\text{M.102})$$

M.1.4.4 The curl of a vector field

In \mathbb{R}^3 the *curl* of a vector field $\mathbf{a}(\mathbf{x})$ is another \mathbb{R}^3 vector field defined in the following way:

$$\text{curl } \mathbf{a}(\mathbf{x}) \stackrel{\text{def}}{=} \nabla \times \mathbf{a}(\mathbf{x}) = \epsilon_{ijk} \hat{\mathbf{x}}_i \partial_j a_k(\mathbf{x}) = \epsilon_{ijk} \hat{\mathbf{x}}_i \frac{\partial a_k(\mathbf{x})}{\partial x_j} \quad (\text{M.103})$$

where use was made of the Levi-Civita tensor, introduced in equation (M.22) on page 209. If \mathbf{a} is an ordinary vector (polar vector), then $\nabla \times \mathbf{a}$ is a pseudovector (axial vector) and vice versa.

Similarly to formula (M.79) on page 218, we can write the matrix representation of the curl in \mathbb{R}^3 as

$$(\nabla \times \mathbf{a}) = \begin{pmatrix} \partial_2 a_3 - \partial_3 a_2 \\ \partial_3 a_1 - \partial_1 a_3 \\ \partial_1 a_2 - \partial_2 a_1 \end{pmatrix} = -i \nabla \cdot \mathbf{S} \mathbf{a} = -i \nabla \mathbf{S} \cdot \mathbf{a} \quad (\text{M.104})$$

where \mathbf{S} is the matrix vector given by formula (M.27) on page 210.

The covariant 4D generalisation of the curl of a four-vector field $a^\mu(x^\nu)$ is the antisymmetric four-tensor field

$$A_{\mu\nu}(x^\kappa) = \partial_\mu a_\nu(x^\kappa) - \partial_\nu a_\mu(x^\kappa) = -A_{\nu\mu}(x^\kappa) \quad (\text{M.105})$$

A vector with vanishing curl is said to be *irrotational*.

M.1.4.5 The Laplacian

The 3D *Laplace operator* or *Laplacian* can be described as the divergence of the *del* operator:

$$\nabla \cdot \nabla = \nabla^2 \stackrel{\text{def}}{=} \Delta = \partial_i^2 = \frac{\partial^2}{\partial x_i^2} \equiv \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \quad (\text{M.106})$$

The symbol ∇^2 is sometimes read *del squared*. If, for a scalar field $\alpha(\mathbf{x})$, $\nabla^2 \alpha < 0$ at some point in 3D space, α has a *concentration* at that point.

Numerous vector algebra and vector analysis formulæ are given in appendix F. Those which are not found there can often be easily derived by using the component forms of the vectors and tensors, together with the Kronecker and Levi-Civita tensors and their generalisations to higher ranks and higher dimensions.

▷ The four-del operator in Lorentz space

EXAMPLE M.8

In \mathbb{L}^4 the contravariant form of the four-*del* operator can be represented as

$$\partial^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\boldsymbol{\partial} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) \quad (\text{M.107})$$

and the covariant form as

$$\partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \boldsymbol{\partial} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) \quad (\text{M.108})$$

Taking the scalar product of these two, one obtains

$$\partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 = \square^2 \quad (\text{M.109})$$

which is the *d'Alembert operator*, sometimes denoted \square , and sometimes defined with an opposite sign convention.

End of example M.8◁

EXAMPLE M.9 ▷Gradients of scalar functions of relative distances in 3D

Very often electrodynamic quantities are dependent on the relative distance in \mathbb{R}^3 between two vectors \mathbf{x} and \mathbf{x}' , *i.e.*, on $|\mathbf{x} - \mathbf{x}'|$. In analogy with equation (M.91) on page 220, we can define the primed *del* operator in the following way:

$$\nabla' = \hat{\mathbf{x}}_i \frac{\partial}{\partial x'_i} = \partial' \quad (\text{M.110})$$

Using this, the corresponding unprimed version, *viz.*, equation (M.91) on page 220, and elementary rules of differentiation, we obtain the following very useful result:

$$\begin{aligned} \nabla(|\mathbf{x} - \mathbf{x}'|) &= \hat{\mathbf{x}}_i \frac{\partial |\mathbf{x} - \mathbf{x}'|}{\partial x_i} = \hat{\mathbf{x}}_i \frac{\partial \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}}{\partial x_i} \\ &= \hat{\mathbf{x}}_i \frac{(x_i - x'_i)}{|\mathbf{x} - \mathbf{x}'|} = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} = -\hat{\mathbf{x}}_i \frac{\partial |\mathbf{x} - \mathbf{x}'|}{\partial x'_i} = -\nabla'(|\mathbf{x} - \mathbf{x}'|) \end{aligned} \quad (\text{M.111})$$

Likewise

$$\nabla\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla'\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) \quad (\text{M.112})$$

End of example M.9◁

EXAMPLE M.10 ▷Divergence in 3D

For an arbitrary \mathbb{R}^3 vector field $\mathbf{a}(\mathbf{x}')$, the following relation holds:

$$\nabla' \cdot \left(\frac{\mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{\nabla' \cdot \mathbf{a}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \mathbf{a}(\mathbf{x}') \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \quad (\text{M.113})$$

which demonstrates how the primed divergence, defined in terms of the primed *del* operator in equation (M.110), works.

End of example M.10◁

EXAMPLE M.11 ▷The Laplacian and the Dirac delta

A very useful formula in 3D \mathbb{R}^3 is

$$\nabla \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi \delta(\mathbf{x} - \mathbf{x}') \quad (\text{M.114})$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the 3D *Dirac delta* ‘function’. This formula follows directly from the fact that

$$\int_V d^3x \nabla \cdot \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \int_V d^3x \nabla \cdot \left(-\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \right) = \oint_S d^2x \hat{\mathbf{n}} \cdot \left(-\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \right) \quad (\text{M.115})$$

equals -4π if the integration volume $V(S)$, enclosed by the surface $S(V)$, includes $\mathbf{x} = \mathbf{x}'$, and equals 0 otherwise.

End of example M.11◁

▷The curl of a gradient

EXAMPLE M.12

Using the definition of the \mathbb{R}^3 curl, equation (M.103) on page 222, and the gradient, equation (M.97) on page 221, we see that

$$\nabla \times [\nabla \alpha(\mathbf{x})] = \epsilon_{ijk} \hat{\mathbf{x}}_i \partial_j \partial_k \alpha(\mathbf{x}) \quad (\text{M.116})$$

which, due to the assumed well-behavedness of $\alpha(\mathbf{x})$, vanishes:

$$\begin{aligned} \epsilon_{ijk} \hat{\mathbf{x}}_i \partial_j \partial_k \alpha(\mathbf{x}) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \alpha(\mathbf{x}) \hat{\mathbf{x}}_i \\ &= \left(\frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_3 \partial x_2} \right) \alpha(\mathbf{x}) \hat{\mathbf{x}}_1 \\ &\quad + \left(\frac{\partial^2}{\partial x_3 \partial x_1} - \frac{\partial^2}{\partial x_1 \partial x_3} \right) \alpha(\mathbf{x}) \hat{\mathbf{x}}_2 \\ &\quad + \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_1} \right) \alpha(\mathbf{x}) \hat{\mathbf{x}}_3 \\ &\equiv \mathbf{0} \end{aligned} \quad (\text{M.117})$$

We thus find that

$$\nabla \times [\nabla \alpha(\mathbf{x})] \equiv \mathbf{0} \quad (\text{M.118})$$

for any arbitrary, well-behaved \mathbb{R}^3 scalar field $\alpha(\mathbf{x})$.

In 4D we note that for any well-behaved four-scalar field $\alpha(x^\kappa)$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \alpha(x^\kappa) \equiv 0 \quad (\text{M.119})$$

so that the four-curl of a four-gradient vanishes just as does a curl of a gradient in \mathbb{R}^3 .

Hence, a gradient is always irrotational.

End of example M.12◁

▷The divergence of a curl

EXAMPLE M.13

With the use of the definitions of the divergence (M.101) and the curl, equation (M.103) on page 222, we find that

$$\nabla \cdot [\nabla \times \mathbf{a}(\mathbf{x})] = \partial_i [\nabla \times \mathbf{a}(\mathbf{x})]_i = \epsilon_{ijk} \partial_i \partial_j a_k(\mathbf{x}) \quad (\text{M.120})$$

Using the definition for the Levi-Civita symbol, defined by equation (M.22) on page 209, we find that, due to the assumed well-behavedness of $\mathbf{a}(\mathbf{x})$,

$$\begin{aligned} \partial_i \epsilon_{ijk} \partial_j a_k(\mathbf{x}) &= \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} a_k \\ &= \left(\frac{\partial^2}{\partial x_2 \partial x_3} - \frac{\partial^2}{\partial x_3 \partial x_2} \right) a_1(\mathbf{x}) \\ &\quad + \left(\frac{\partial^2}{\partial x_3 \partial x_1} - \frac{\partial^2}{\partial x_1 \partial x_3} \right) a_2(\mathbf{x}) \\ &\quad + \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2 \partial x_1} \right) a_3(\mathbf{x}) \\ &\equiv 0 \end{aligned} \quad (\text{M.121})$$

i.e., that

$$\nabla \cdot [\nabla \times \mathbf{a}(\mathbf{x})] \equiv 0 \quad (\text{M.122})$$

for any arbitrary, well-behaved \mathbb{R}^3 vector field $\mathbf{a}(\mathbf{x})$.

In 4D, the four-divergence of the four-curl is *not* zero, for

$$\partial^\nu A_\nu^\mu = \partial^\mu \partial_\nu a^\nu(x^\kappa) - \square^2 a^\mu(x^\kappa) \neq 0 \quad (\text{M.123})$$

End of example M.13◁

EXAMPLE M.14 ▷A non-trivial vector analytic triple product formula◁

When differential operators appear inside multiple-product vector formulae, one has to ensure that the range of the operator is taken into account in a proper and correct manner. The following calculation of the triple product $(\mathbf{a} \times \nabla) \times \mathbf{b}$, where \mathbf{a} and \mathbf{b} are 3D vector fields, illustrates how this situation can be handled.

$$\begin{aligned} (\mathbf{a} \times \nabla) \times \mathbf{b} &= (\epsilon_{ijk} \hat{\mathbf{x}}_i a_j \partial_k) \times \mathbf{b} = (\epsilon_{ijk} a_j \partial_k) (\hat{\mathbf{x}}_i \times \mathbf{b}) \\ &= (\epsilon_{ijk} a_j \partial_k) (\epsilon_{lmn} \hat{\mathbf{x}}_l \delta_{im} b_n) = \epsilon_{ijk} \epsilon_{lmn} \delta_{im} \hat{\mathbf{x}}_l a_j \partial_k b_n \\ &= \epsilon_{ijk} \epsilon_{lin} \hat{\mathbf{x}}_l a_j \partial_k b_n = -\epsilon_{ijk} \epsilon_{iln} \hat{\mathbf{x}}_l a_j \partial_k b_n \end{aligned} \quad (\text{M.124a})$$

where dummy summation indices were changed. Using formula (M.25) on page 209 with $m = n$, we can rewrite this as

$$\begin{aligned} (\mathbf{a} \times \nabla) \times \mathbf{b} &= -(\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl}) \hat{\mathbf{x}}_l a_j \partial_k b_n \\ &= \hat{\mathbf{x}}_k a_j \partial_k b_j - \hat{\mathbf{x}}_j a_j \partial_k b_k \\ &= a_i \nabla b_i - \mathbf{a}(\nabla \cdot \mathbf{b}) \end{aligned} \quad (\text{M.124b})$$

This is formula (F.102) on page 201.

End of example M.14◁

M.2 Analytical mechanics

M.2.1 Lagrange's equations

As is well known from elementary analytical mechanics, the *Lagrange function* or *Lagrangian* L is given by

$$L(q_i, \dot{q}_i, t) = L\left(q_i, \frac{dq_i}{dt}, t\right) = T - V \quad (\text{M.125})$$

where q_i is the *generalised coordinate*, T the *kinetic energy* and V the *potential energy* of a mechanical system, Using the action

$$S = \int_{t_1}^{t_2} dt L(q_i, \dot{q}_i, t) \quad (\text{M.126})$$

and the *variational principle* with fixed endpoints t_1 and t_2 ,

$$\delta S = 0 \quad (\text{M.127})$$

one finds that the Lagrangian satisfies the *Euler-Lagrange equations*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (\text{M.128})$$

To the generalised coordinate q_i one defines a *canonically conjugate momentum* p_i according to

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (\text{M.129})$$

and note from equation (M.128) above that

$$\frac{\partial L}{\partial q_i} = \dot{p}_i \quad (\text{M.130})$$

If we introduce an arbitrary, differentiable function $\alpha = \alpha(q_i, t)$ and a new Lagrangian L' related to L in the following way

$$L' = L + \frac{d\alpha}{dt} = L + \dot{q}_i \frac{\partial \alpha}{\partial q_i} + \frac{\partial \alpha}{\partial t} \quad (\text{M.131})$$

then

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial \alpha}{\partial q_i} \quad (\text{M.132a})$$

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{d}{dt} \frac{\partial \alpha}{\partial q_i} \quad (\text{M.132b})$$

Or, in other words,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \quad (\text{M.133})$$

where

$$p'_i = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial \alpha}{\partial \dot{q}_i} = p_i + \frac{\partial \alpha}{\partial \dot{q}_i} \quad (\text{M.134a})$$

and

$$q'_i = -\frac{\partial L'}{\partial \dot{p}_i} = -\frac{\partial L}{\partial \dot{p}_i} = q_i \quad (\text{M.134b})$$

M.2.2 Hamilton's equations

From L , the *Hamiltonian* (*Hamilton function*) H can be defined via the *Legendre transformation*

$$H(p_i, q_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \quad (\text{M.135})$$

After differentiating the left and right hand sides of this definition and setting them equal we obtain

$$\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \quad (\text{M.136})$$

According to the definition of p_i , equation (M.129) on the preceding page, the second and fourth terms on the right hand side cancel. Furthermore, noting that according to equation (M.130) on the previous page the third term on the right hand side of equation (M.136) is equal to $-\dot{p}_i dq_i$ and identifying terms, we obtain the *Hamilton equations*:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i = \frac{dq_i}{dt} \quad (\text{M.137a})$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -\frac{dp_i}{dt} \quad (\text{M.137b})$$

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