11.7 Summary

Near a position of equilibrium of any natural, conservative system, the kinetic energy may be taken to be a homogeneous quadratic function of the $\dot{q}_\alpha$, with constant coefficients, and the potential energy to be a homogeneous quadratic function of the q_α. We can always find a set of orthogonal co-ordinates, in terms of which T is reduced to a sum of squares. Lagrange's equations then take on a simple form. To find the normal modes of oscillation, we substitute solutions of the form $q_\alpha = A_\alpha e^{i\omega t}$, and obtain a set of simultaneous linear equations for the coefficients. The condition for consistency of these equations is the characteristic equation, which determines the frequencies of the normal modes. The stability condition is that all the roots of this equation for ω^2 should be positive.

The problem of finding the normal modes is equivalent to that of finding normal co-ordinates, which reduce not only T but also V to a sum of squares. In terms of normal co-ordinates, the system is reduced to a set of uncoupled harmonic oscillators, whose frequencies are the characteristic frequencies of the system. The general solution to the equations of motion is a superposition of all the normal modes. In it, each normal co-ordinate is oscillating at its own frequency, and with amplitude and phase determined by the initial conditions.

The linearized analysis of small amplitude oscillations near to a position of stable equilibrium in the form of normal modes is a technique which is applicable generally. For some systems, which are special but important, the idea of a normal mode may be generalized. Such systems may then be analyzed as a combination of 'nonlinear' normal modes, where no small amplitude approximation needs to be made — see §14.1.

Problems

1. A double pendulum, consisting of a pair, each of mass m and length l, is released from rest with the pendulums displaced but in a straight line. Find the displacements of the pendulums as functions of time.
2. Find the normal modes of a pair of coupled pendulums (like those of Fig. 11.2) if the two are of different masses M and m, but still the same length l. Given that the pendulum of mass M is started oscillating with amplitude a, find the maximum amplitude of the other pendulum in

the subsequent motion. Does the amplitude of the first pendulum ever fall to zero?

3. A spring of negligible mass, and spring constant (force/extension) k, supports a mass m, and beneath it a second, identical spring, carrying a second, identical mass. Using the vertical displacements x and y of the masses from their positions with the springs unextended as generalized co-ordinates, write down the Lagrangian function. Find the position of equilibrium, and the normal modes and frequencies of vertical oscillations.
4. Three identical pendulums are coupled, as in Fig. 11.2, with springs between the first and second and between the second and third. Find the frequencies of the normal modes, and the ratios of the amplitudes.
5. The first of the three pendulums of Problem 4 is initially displaced a distance a, while the other two are vertical. The system is released from rest. Find the maximum amplitudes of the second and third pendulums in the subsequent motion.
6. Three identical springs, of negligible mass, spring constant k, and natural length a are attached end-to-end, and a pair of particles, each of mass m, are fixed to the points where they meet. The system is stretched between fixed points a distance $3l$ apart ($l > a$). Find the frequencies of normal modes of (a) longitudinal, and (b) transverse oscillations.
7. *A bead of mass m slides on a smooth circular hoop of mass M and radius a, which is pivoted at a point on its rim so that it can swing freely in its plane. Write down the Lagrangian in terms of the angle of inclination θ of the diameter through the pivot and the angular position φ of the bead relative to a fixed point on the hoop. Find the frequencies of the normal modes, and sketch the configuration of hoop and bead at the extreme point of each.
8. *The system of Problem 7 is released from rest with the centre of the hoop vertically below the pivot and the bead displaced by a small angle φ_0. Given that $M = 8m$ and that $2a$ is the length of a simple pendulum of period 1 s, find the angular displacement θ of the hoop as a function of time. Determine the maximum value of θ in the subsequent motion, and the time at which it first occurs.
9. A simple pendulum of mass m, whose period when suspended from a rigid support is 1 s, hangs from a supporting block of mass $2m$ which can move along a horizontal line (in the plane of the pendulum), and is restricted by a harmonic-oscillator restoring force. The period of the

oscillator (with the pendulum removed) is 0.1 s. Find the periods of the two normal modes. When the pendulum bob is swinging in the slower mode with amplitude 100 mm, what is the amplitude of the motion of the supporting block?

10. *The system of Problem 9 is initially at rest, and the pendulum bob is given an impulsive blow which starts it moving with velocity $0.5\,\mathrm{m\,s^{-1}}$. Find the position of the support as a function of time in the subsequent motion.
11. *A particle of charge q and mass m is free to slide on a smooth horizontal table. Two fixed charges q are placed at $\pm a\boldsymbol{j}$, and two fixed charges $12q$ at $\pm 2a\boldsymbol{i}$. Find the electrostatic potential near the origin (see §6.2). Show that this is a position of stable equilibrium, and find the frequencies of the normal modes of oscillation near it.
12. *A rigid rod of length $2a$ is suspended by two light, inextensible strings of length l joining its ends to supports also a distance $2a$ apart and level with each other. Using the longitudinal displacement x of the centre of the rod, and the transverse displacements y_1, y_2 of its ends, as generalized co-ordinates, find the Lagrangian function (for small x, y_1, y_2). Determine the normal modes and frequencies. (*Hint*: First find the height by which each end is raised, the co-ordinates of the centre of mass and the angle through which the rod is turned.)
13. *Each of the pendulums in Fig 11.2 is subjected to a damping force, of magnitude $\alpha\dot{x}$ and $\alpha\dot{y}$ respectively, while there is a damping force $\beta(\dot{x}-\dot{y})$ in the spring. Show that the equations for the normal co-ordinates q_1 and q_2 are still uncoupled. Find the amplitudes of the forced oscillations obtained by applying a periodic force to one pendulum. Given that the forcing frequency is that of the uncoupled pendulums, and that β is negligible, find the range of values of α for which the amplitude of the second pendulum is less than half that of the first.
14. *Show that a stretched string is equivalent mathematically to an infinite number of uncoupled oscillators, described by the co-ordinates
$$q_n(t) = \sqrt{\frac{2}{l}}\int_0^l y(x,t)\sin\frac{n\pi x}{l}\,\mathrm{d}x.$$
Determine the amplitudes of the various normal modes in the motion described in Chapter 10, Problem 14. Why, physically, are the modes for even values of n not excited?
15. Show that a typical equation of the set (11.33) may be satisfied by setting $A_\alpha = \sin\alpha k$ $(\alpha = 1, 2, \ldots, n)$, provided that $\omega = 2\omega_0\sin\frac{1}{2}k$.

Hence show by considering the required condition when $\alpha = n + 1$ that the frequencies of the normal modes are $\omega_r = 2\omega_0 \sin[r\pi/2(n+1)]$, with $r = 1, 2, \dots, n$. Why may we ignore values of r greater than $n+1$? Show that, in the limit of large n, the frequency of the rth normal mode tends to the corresponding frequency of the continuous string with the same total length and mass.

16. A particle moves under a conservative force with potential energy $V(\boldsymbol{r})$. The point $\boldsymbol{r} = \mathbf{0}$ is a position of equilibrium, and the axes are so chosen that x, y, z are normal co-ordinates. Show that, if V satisfies Laplace's equation, $\boldsymbol{\nabla}^2 V = 0$ (see §6.7), then the equilibrium is necessarily unstable, and hence that stable equilibrium under purely gravitational and electrostatic forces is impossible. (Of course, *dynamic* equilibrium — stable periodic motion — can occur. Note also that the two-dimensional stable equilibrium of Problem 11 does not contradict this result because there is another force imposed, confining the charge to the horizontal plane.)

of some particular co-ordinate q_α, then the corresponding generalized momentum p_α is conserved. In such a case, the number of degrees of freedom is effectively reduced by one.

More generally, we have seen that any symmetry property of the system leads to a corresponding conservation law. This can be of great importance in practice, since the amount of labour involved in solving a complicated problem can be greatly reduced by making full use of all the available symmetries. If there is a sufficient number of symmetries, then the system is 'integrable' (in the sense of Liouville) and the conservation laws may then be exploited to produce (in principle) the complete solution to the problem.

The Hamiltonian function is also of great importance in quantum mechanics, and many of the features of our discussion carry over to that case. We have seen that the variables appear in pairs. To each co-ordinate q_α there corresponds a momentum p_α. Such pairs are called *canonically conjugate*. This relationship between pairs of variables is of central importance in quantum mechanics, where there is an 'uncertainty principle' according to which it is impossible to measure both members of such a pair simultaneously with arbitrary accuracy.

The relationship between symmetries and conservation laws also applies to quantum mechanics. In relativity, the transformations we consider are slightly different (Lorentz transformations rather than Galilean), but the same principles apply, and lead to very similar conservation laws.

The relationship between the relativity principle and the familiar conservation laws (including the 'conservation law' $\boldsymbol{P} = M\dot{\boldsymbol{R}}$) is of the greatest importance for the whole of physics. It is the basic reason for the universal character of these laws, which were originally derived as rather special consequences of Newton's laws, but can now be seen as having a far more fundamental role.

Problems

1. A particle of mass m slides on the inside of a smooth cone of semi-vertical angle α, whose axis points vertically upwards. Obtain the Hamiltonian function, using the distance r from the vertex, and the azimuth angle φ as generalized co-ordinates. Show that stable circular motion is possible for any value of r, and determine the corresponding angular velocity, ω. Find the angle α if the frequency of small oscillations about this circular motion is also ω.

2. Find the Hamiltonian function for the forced pendulum considered in §10.4, and verify that it is equal to $T' + V'$. Determine the frequency of small oscillations about the stable 'equilibrium' position when $\omega^2 > g/l$.
3. A light, inextensible string passes over a small pulley and carries a mass $2m$ on one end. On the other end is a mass m, and beneath it, supported by a spring with spring constant k, a second mass m. Find the Hamiltonian function, using the distance x of the first mass beneath the pulley, and the extension y in the spring, as generalized co-ordinates. Show that x is ignorable. To what symmetry property does this correspond? (In other words, what operation can be performed on the system without changing its energy?) If the system is released from rest with the spring unextended, find the positions of the particles at any later time.
4. A particle of mass m moves in three dimensions under the action of a central, conservative force with potential energy $V(r)$. Find the Hamiltonian function in terms of spherical polar co-ordinates, and show that φ, but not θ, is ignorable. Express the quantity $\boldsymbol{J}^2 = m^2 r^4(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)$ in terms of the generalized momenta, and show that it is a second constant of the motion.
5. *Find the Hamiltonian for the pendulum hanging from a trolley described in Chapter 10, Problem 9. Show that x is ignorable. To what symmetry does this correspond?
6. *Obtain the Hamiltonian function for the top with freely sliding pivot described in Chapter 10, Problem 11. Find whether the minimum angular velocity required for stable vertical rotation is greater or less than in the case of a fixed pivot. Can you explain this result physically?
7. *To prove that the effective potential energy function $U(\theta)$ of the symmetric top (see §12.4) has only a single minimum, show that the equation $U(\theta) = E$ can be written as a cubic equation in the variable $z = \cos\theta$, with three roots in general. Show, however, that $f(z)$ has the same sign at both $z = \pm 1$, and hence that there are either two roots or none between these points: for every E there are at most two values of θ for which $U(\theta) = E$.
8. Find the Hamiltonian for a charged particle in electric and magnetic fields in cylindrical polars, starting from the Lagrangian function (10.29). Show that in the case of an axially symmetric, static magnetic field, described by the single component $A_\varphi(\rho, z)$ of the vector

potential, it takes the form

$$H = \frac{1}{2m}\left(p_z^2 + p_\rho^2 + \frac{(p_\varphi - q\rho A_\varphi)^2}{\rho^2}\right).$$

(*Note*: Remember that the subscripts φ on the generalized momentum p_φ and on the component A_φ mean different things.)

9. A particle of mass m and charge q is moving around a fixed point charge $-q'$ ($qq' > 0$), and in a uniform magnetic field $\boldsymbol{B}$. The motion is confined to the plane perpendicular to $\boldsymbol{B}$. Write down the Lagrangian function in polar co-ordinates rotating with the Larmor angular velocity $\omega_{\mathrm{L}} = -qB/2m$ (see §5.5). Hence find the Hamiltonian function. Show that φ is ignorable, and interpret the conservation law. (Note that J_z is *not* a constant of the motion.)
10. Consider a system like that of Problem 9, but with a charge $+q'$ at the origin. By examining the effective radial potential energy function, find the radius of a stable circular orbit with angular velocity ω_{L}, and determine the angular frequency of small oscillations about it.
11. *A particle of mass m and charge q is moving in the equatorial plane $z = 0$ of a magnetic dipole of moment μ, described (see Appendix A, Problem 12) by a vector potential with the single non-zero component $A_\varphi = \mu_0\mu \sin\theta/4\pi r^2$. Show that it will continue to move in this plane. Initially, it is approaching from a great distance with velocity v and impact parameter b, whose sign is defined to be that of p_φ. Show that v and p_φ are constants of the motion, and that the distance of closest approach to the dipole is $\frac{1}{2}(\sqrt{b^2 \mp a^2} \pm b)$, according as $b > a$ or $b < a$, where $a^2 = \mu_0 q\mu/\pi m v$. (Here $q\mu$ is assumed positive.) Find also the range of values of b for which the velocity can become purely radial, and the distances at which it does so. Describe qualitatively the appearance of the orbits for different values of b. (*Hint*: It may be useful to sketch the effective radial potential energy function.)
12. *Find the Hamiltonian for the restricted three-body problem described in Chapter 10, Problems 15 and 16. Investigate the stability of one of the Lagrangian 'equilibrium' positions off the line of centres by assuming a solution where $x - x_0, y - y_0, p_x + m\omega y_0$ and $p_y - m\omega x_0$ are all small quantities proportional to e^{pt}, with p constant. Show that the possible values for p are given by

$$p^4 + \omega^2 p^2 + \frac{27 M_1 M_2 \omega^4}{4(M_1 + M_2)^2} = 0,$$

and hence that the points are stable provided that the masses M_1 and M_2 are sufficiently different. Specifically, given that $M_1 > M_2$ show that the minimum possible ratio for stability is slightly less than 25.

13. The stability condition of Problem 12 is well satisfied for the case of the Sun and Jupiter, for which $M_1/M_2 = 1047$. Indeed, in that case these positions are occupied by the so-called *Trojan* asteroids, whose orbital periods are the same as Jupiter's, 11.86 years. Find for this case the periods of small oscillations about the 'equilibrium' points (in the plane of the orbit).
14. *The magnetic field in a particle accelerator is axially symmetric (as in Problem 8), and in the plane $z = 0$ has only a z component. Defining $J = p_\varphi - q\rho A_\varphi$, show, using (A.40) and (A.55), that $\partial J/\partial\rho = -q\rho B_z$, and $\partial J/\partial z = q\rho B_\rho$. What is the relation between $\dot{\varphi}$ and J? Treat the third term of the Hamiltonian in Problem 8 as an effective potential energy function $U(\rho, z) = J^2/2m\rho^2$, compute its derivatives, and write down the 'equilibrium' conditions $\partial U/\partial\rho = \partial U/\partial z = 0$. Hence show that a particle of mass m and charge q can move in a circle of any given radius a in the plane $z = 0$ with angular velocity equal to the cyclotron frequency for the field at that radius (see §5.2).
15. *To investigate the stability of the motion described in the preceding question, evaluate the second derivatives of U at $\rho = a, z = 0$, and show that they may be written

$$\frac{\partial^2 U}{\partial\rho^2} = \frac{q^2}{m}\left[B_z\left(B_z + \rho\frac{\partial B_z}{\partial\rho}\right)\right]_{\rho=a,z=0},$$

$$\frac{\partial^2 U}{\partial\rho\,\partial z} = 0, \qquad \frac{\partial^2 U}{\partial z^2} = -\frac{q^2}{m}\left[B_z\rho\frac{\partial B_z}{\partial\rho}\right]_{\rho=a,z=0}.$$

(*Hint*: You will need to use the φ component of the equation $\boldsymbol{\nabla}\wedge\boldsymbol{B} = \boldsymbol{0}$, and the fact that, since $B_\rho = 0$ for all ρ, $\partial B_\rho/\partial\rho = 0$ also.) Given that the dependence of B_z on ρ near the equilibrium orbit is described by $B_z \propto (a/\rho)^n$, show that the orbit is stable if $0 < n < 1$.
16. Show that the Poisson brackets of the components of angular momentum are

$$[J_x, J_y] = J_z$$

(together with two other relations obtained by cyclic permutation of x, y, z). Interpret this result in terms of the transformation of one component generated by another.

17. *Show that the condition that Hamilton's equations remain unchanged under the transformation generated by G is $\mathrm{d}G/\mathrm{d}t = 0$ even in the case when G has an explicit time-dependence, in addition to its dependence via $q(t)$ and $p(t)$. Proceed as follows. The first set of Hamilton's equations, (12.6), will be unchanged provided that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_\alpha) = \delta\left(\frac{\partial H}{\partial p_\alpha}\right).$$

Write both sides of this equation in terms of G and use (12.33) applied both to $\partial G/\partial p_\alpha$ and to G itself to show that it is equivalent to the condition

$$\frac{\partial}{\partial p_\alpha}\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right) = 0.$$

Thus $\mathrm{d}G/\mathrm{d}t$ is independent of each p_α. Similarly, by using the other set of Hamilton's equations, (12.7), show that it is independent of each q_α. Thus $\mathrm{d}G/\mathrm{d}t$ must be a function of t alone. But since we can always add to G any function of t alone without affecting the transformation it generates, this means we can choose it so that $\mathrm{d}G/\mathrm{d}t = 0$.