This therefore determines the odd part of $f$. Moreover, the initial value of $\dot{y}$ is

$$\dot{y}(x,0) = cf'(x) - cf'(-x) = ch'(x). \tag{10.41}$$

(Remember that $[f(-x)]' = -f'(-x)$.) Thus the initial value of $\dot{y}$ determines the even part of $f$ (up to an irrelevant additive constant, which cancels in (10.37)).

## 10.7 Summary

The position of every part of a system may be fixed by specifying the values of a set of generalized co-ordinates. If these co-ordinates can all vary independently, the system is holonomic. This is the case in all the examples we have considered. The system is natural if the functions specifying the positions of particles in terms of the generalized co-ordinates do not involve the time explicitly. In that case, the kinetic energy is a homogeneous quadratic function of the $\dot{q}_\alpha$. For a forced system, on the other hand, $T$ may contain linear and constant terms. In either case, the equations of motion are given by Lagrange's equations. If the forces are conservative (and sometimes in other cases too), all we need is the Lagrangian function $L = T - V$. In general, for dissipative forces, the generalized forces $F_\alpha$ corresponding to the generalized co-ordinates $q_\alpha$ must be found by evaluating the work done in a small displacement.

---

## Problems

1. Masses $m$ and $2m$ are joined by a light inextensible string which runs without slipping over a uniform circular pulley of mass $2m$ and radius $a$. Using the angular position of the pulley as generalized co-ordinate, write down the Lagrangian function, and Lagrange's equation. Find the acceleration of the masses.
2. A uniform cylindrical drum of mass $M$ and radius $a$ is free to rotate about its axis, which is horizontal. A cable of negligible mass and length $l$ is wound on the drum, and carries on its free end a mass $m$. Write down the Lagrangian function in terms of an appropriate generalized co-ordinate, assuming no slipping or stretching of the cable. If the cable

is initially fully wound up, and the system is released from rest, find the angular velocity of the drum when it is fully unwound.

3. Treat the system of Problem 2 as one with two generalized co-ordinates, the angular position of the drum and the free length of cable, with an appropriate constraint. Hence find the tension in the cable. (Show that it is equal to the Lagrange multiplier.)
4. *Find the Lagrangian function for the system of Problem 2 if the cable is elastic, with elastic potential energy $\frac{1}{2}kx^2$, where $x$ is the extension of the cable. Show that the motion of the mass $m$ is a uniform acceleration at the same rate as before, with a superimposed oscillation of angular frequency given by $\omega^2 = k(M + 2m)/Mm$. Find the amplitude of this oscillation if the system is released from rest with the cable unextended.
5. Write down the kinetic energy of a particle in cylindrical polar co-ordinates in a frame rotating with angular velocity $\omega$ about the $z$-axis. Show that the terms proportional to $\omega$ and $\omega^2$ reproduce the Coriolis force and centrifugal force respectively.
6. A light inextensible string passes over a light smooth pulley, and carries a mass $4m$ on one end. The other end supports a second pulley with a string over it carrying masses $3m$ and $m$ on the two ends. Using a suitable pair of generalized co-ordinates, write down the Lagrangian function for the system, and Lagrange's equations. Find the downward accelerations of the three masses.
7. Find the tensions in the strings in Problem 6. Explain why the first pulley turns, although the total mass on each side is the same.
8. Evaluate accurately the two possible precessional angular velocities of the top described in Chapter 9, Problem 13, if the axis makes an angle of 30° with the vertical. Compare the slower value with the approximate result found earlier. Find also the minimum angular velocity $\omega_3$ for which steady precession at this angle is possible.
9. A simple pendulum of mass $m$ and length $l$ hangs from a trolley of mass $M$ running on smooth horizontal rails. The pendulum swings in a plane parallel to the rails. Using the position $x$ of the trolley and the angle of inclination $\theta$ of the pendulum as generalized co-ordinates, write down the Lagrangian function, and Lagrange's equations. Obtain an equation of motion for $\theta$ alone. If the system is released from rest with the pendulum inclined at 30° to the vertical, use energy conservation to find its angular velocity when it reaches the vertical, given that $M = 2\,\text{kg}$, $m = 1\,\text{kg}$, and $l = 2\,\text{m}$.

10. *Show that the kinetic energy of the gyroscope described in Chapter 9, Problem 21, is

$$T = \tfrac{1}{2}I_1(\Omega\sin\lambda\cos\varphi)^2 + \tfrac{1}{2}I_1(\dot\varphi + \Omega\cos\lambda)^2 + \tfrac{1}{2}I_3(\dot\psi + \Omega\sin\lambda\sin\varphi)^2.$$

From Lagrange's equations, show that the angular velocity $\omega_3$ about the axis is constant, and obtain the equation for $\varphi$ without neglecting $\Omega^2$. Show that motion with the axis pointing north becomes unstable for very small values of $\omega_3$, and find the smallest value for which it is stable. What are the stable positions when $\omega_3 = 0$? Interpret this result in terms of a non-rotating frame.

11. *Find the Lagrangian function for a symmetric top whose pivot is free to slide on a smooth horizontal table, in terms of the generalized co-ordinates $X, Y, \varphi, \theta, \psi$, and the principal moments $I_1^*, I_1^*, I_3^*$ about the centre of mass. (Note that $Z$ is related to $\theta$.) Show that the horizontal motion of the centre of mass may be completely separated from the rotational motion. What difference is there in the equation (10.15) for steady precession? Are the precessional angular velocities greater or less than in the case of a fixed pivot? Show that steady precession at a given value of $\theta$ can occur for a smaller value of $\omega_3$ than in the case of a fixed pivot.

12. *A uniform plank of length $2a$ is placed with one end on a smooth horizontal floor and the other against a smooth vertical wall. Write down the Lagrangian function, using two generalized co-ordinates, the distance $x$ of the foot of the plank from the wall, and its angle $\theta$ of inclination to the horizontal, with a suitable constraint between the two. Given that the plank is initially at rest at an inclination of $60°$, find the angle at which it loses contact with the wall. (*Hint*: First write the co-ordinates of the centre of mass in terms of $x$ and $\theta$. Note that the reaction at the wall is related to the Lagrange multiplier.)

13. Use Hamilton's principle to show that if $F$ is any function of the generalized co-ordinates, then the Lagrangian functions $L$ and $L + \mathrm{d}F/\mathrm{d}t$ must yield the same equations of motion. Hence show that the equations of motion of a charged particle in an electromagnetic field are unaffected by the 'gauge transformation' (A.42). (*Hint*: Take $F = -q\Lambda$.)

14. The stretched string of §10.6 is released from rest with its mid-point displaced a distance $a$, and each half of the string straight. Find the function $f(x)$. Describe the shape of the string after (a) a short time, (b) a time $l/2c$, and (c) a time $l/c$.

15. *Two bodies of masses $M_1$ and $M_2$ are moving in circular orbits of radii $a_1$ and $a_2$ about their centre of mass. The *restricted three-body problem* concerns the motion of a third small body of mass $m$ ($\ll M_1$ or $M_2$) in their gravitational field (*e.g.*, a spacecraft in the vicinity of the Earth–Moon system). Assuming that the third body is moving in the plane of the first two, write down the Lagrangian function of the system, using a rotating frame in which $M_1$ and $M_2$ are fixed. Find the equations of motion. (*Hint*: The identities $GM_1 = \omega^2 a^2 a_2$ and $GM_2 = \omega^2 a^2 a_1$ may be useful, with $a = a_1 + a_2$ and $\omega^2 = GM/a^3$.)
16. *For the system of Problem 15, find the equations that must be satisfied for 'equilibrium' in the rotating frame (*i.e.*, circular motion with the same angular velocity as $M_1$ and $M_2$). Consider 'equilibrium' positions on the line of centres of $M_1$ and $M_2$. By roughly sketching the effective potential energy curve, show that there are three such positions, but that all three are unstable. (*Note*: The positions are actually the solutions of a quintic equation.) Show also that there are two 'equilibrium' positions *off* the line of centres, in each of which the three bodies form an equilateral triangle. (The stability of these so-called *Lagrangian points* is the subject of Problem 12, Chapter 12. There is further consideration of this important problem in §14.4.)