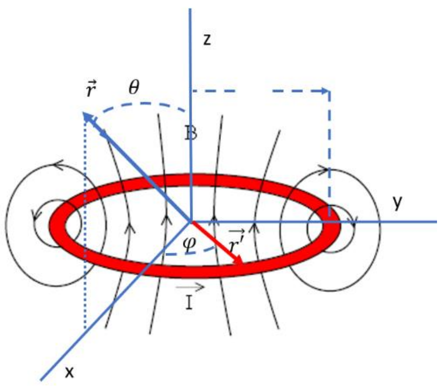
<u>Série de Exercícios 5</u>

1. Na aula "Campos Magnetostáticos I derivamos o campo de uma espira de corrente em coordenadas esféricas utilizando o potencial escalar magnético. Embora as expressões obtidas para as componentes do campo magnético sejam muito úteis para cálculos analíticos, elas não são apropriadas para cálculos precisos dos campos, sem aproximações, para aplicações práticas, utilizando métodos numéricos. Neste problema vamos derivar a expressão para o potencial vetor do campo de uma espira em coordenadas cilíndricas, que permite expressar as componentes do campo em termos de integrais elípticas. O início deste cálculo está feito na seção 5.5 do Jackson, mas referências mais úteis são A. Shadowitz; The Electromagnetic Field, Seção 5.1, e W. R. Smythe; Static and Dynamic Electricity, Cap. 7. A configuração básica é mostrada na figura. A espira está no plano (x, y) e o eixo vertical passa pelo centro da espira. Como há simetria azimutal, podemos colocar o vetor posição $\vec{r}$ no plano (x, z) para simplificar.



a) Considerando que a corrente é dada por

$$I = \int \vec{j}(\vec{r}) \cdot d\vec{S},$$

escreva a expressão para $d\vec{S}$ na direção $\hat{e}_\varphi$, em coordenadas esféricas, e mostre que a expressão para a densidade de corrente que representa o anel de corrente no plano (x, y) tem que ser dada por

$$\vec{j}(r, \theta) = I\,\frac{\delta(r - R)}{R}\,\delta\left(\theta - \frac{\pi}{2}\right)\hat{e}_\varphi$$

b) Utilizando a expressão do potencial vetor como integral da densidade de corrente, decompondo a densidade de corrente em coordenadas cartesianas, para poder fazer a integração no ângulo φ, e calculando $|\vec{r} - \vec{r}'|$, obtenha

$$\vec{A}(r, \theta) = \frac{\mu_0 I}{4\pi R}\int r'^2 dr'\, d\Omega'\,\delta\left(\theta' - \frac{\pi}{2}\right)\delta(r' - R)\left[\frac{-\sin\varphi'\,\hat{e}_x + \cos\varphi'\,\hat{e}_y}{\sqrt{R^2 + r^2 - 2rR\sin\theta\cos\varphi'}}\right]$$

c) Explique por que a integral sobre φ' cancela a componente A_x do potencial vetor e, como pela escolha dos eixos $A_y \leftrightarrow A_\varphi$, obtenha

$$A_\varphi(r, \theta) = 2\,\frac{\mu_0 I}{4\pi R}\int_0^\pi \frac{\cos\varphi'\, d\varphi'}{\sqrt{R^2 + r^2 - 2rR\sin\theta\cos\varphi'}}$$

d) Faça a transformação de variáveis $\varphi' = \pi + 2\phi'$ e obtenha

$$A_\varphi(r, \theta) = 4\,\frac{\mu_0 I}{4\pi R}\int_0^{\pi/2} \frac{(2(\sin\phi')^2 - 1)d\phi'}{\sqrt{R^2 + r^2 + 2rR\sin\theta - 4rR\sin\theta\,(\sin\phi')^2}}$$

e) Definindo

$$k^2 = \frac{4rR\sin\theta}{R^2 + r^2 + 2rR\sin\theta},$$

obtenha a expressão final para o potencial vetor (Equação 5.37 do Jackson)

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{4IR}{\sqrt{R^2 + r^2 + 2r + 2r\sin\theta}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right] \hat{e}_\varphi \,,$$

onde

$$K(k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2(\sin\alpha)^2}} \quad e \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2(\sin\alpha)^2}\, d\alpha$$

são as integrais elípticas de primeira e segunda espécies, respectivamente.

f) As componentes do campo são mais facilmente calculadas (e mais úteis para cálculos numéricos) em coordenadas cilíndricas. Fazendo a transformação para essas coordenadas ($r\cos\theta = z; r\sin\theta = \rho$), mostre que

$$k^2 = \frac{4R\rho}{(R+\rho)^2 + z^2}; \;\; \vec{A} = \frac{\mu_0 I}{2\pi} \sqrt{\frac{R}{\rho}} \left[\left(\frac{2}{k} - k \right) K(k) - \frac{2}{k} E(k) \right] \hat{e}_\varphi$$

g) Calcule as componentes do campo magnético. Para as derivadas envolvendo as integrais elípticas, utilize as seguintes relações

$$\frac{dK}{dk} = \frac{E}{k(1-k^2)} - \frac{K}{k}; \;\; \frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}$$

e obtenha

$$B_\rho(\rho, z) = \frac{\mu_0 I}{2\pi} \frac{z/\rho}{\sqrt{(R+\rho)^2 + z^2}} \left[-K(k) + \frac{R^2 + \rho^2 + z^2}{(R-\rho)^2 + z^2} E(k) \right];$$

$$B_z(\rho, z) = \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{(R+\rho)^2 + z^2}} \left[K(k) + \frac{R^2 - \rho^2 - z^2}{(R-\rho)^2 + z^2} E(k) \right]$$

Essas expressões possibilitam o cálculo numérico preciso do campo magnético de qualquer solenoide com geometria cilíndrica. As integrais elípticas são mais precisamente calculadas utilizando suas aproximações polinomiais (M. Abramowitz & I. Stegun; Handbook of Mathematical Functions; Cap. 17].

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2. Um problema clássico de Física III é o cálculo do campo magnético de um solenoide muito longo. Geralmente este cálculo é feito usando a Lei de Ampère na forma integral, supondo o campo interno na direção do eixo do solenoide e o externo nulo. Essa última hipótese não é facilmente aceita pelos alunos. No entanto, o cálculo sem aproximações baseado na Lei de Biot-Savart prova que está correta. Siga os cálculos da seção 10.2.2 do Zangwill, reproduzida abaixo, fazendo as passagens que não estejam bem detalhadas.

10.2.2 An Infinitely Long Solenoid

Figure 10.3 shows an azimuthal current $\mathbf{K}$ flowing on the surface of an infinitely long solenoid. The cross sectional shape of the solenoid is arbitrary but uniform along its length. The vector $\mathbf{R} = \mathbf{r} - \mathbf{r}_S$ in the Biot-Savart integral (10.16) is drawn for the case when the observation point P lies outside the body of the solenoid. However, the calculation to be outlined below applies equally well when P lies inside the body of the solenoid.

We exploit the fact that $\mathbf{K}d\ell = Kd\boldsymbol{\ell}$ is an azimuthal vector by factoring the surface integral in (10.16) into a z-integral and a line integral around the solenoid's perimeter:

$$\mathbf{B}(P) = \frac{\mu_0}{4\pi}\int dS\,\frac{\mathbf{K}\times\mathbf{R}}{R^3} = \frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty} dz \oint \frac{d\boldsymbol{\ell}\times\mathbf{R}}{R^3}. \tag{10.21}$$

From the geometry, $\mathbf{R} + \mathbf{R}' = -z\hat{\mathbf{z}}$ and $R^2 = R'^2 + z^2$. Using this information and $d\boldsymbol{\ell} = d\mathbf{R}'$ in (10.21) gives

$$\mathbf{B}(P) = \frac{\mu_0 K}{4\pi}\oint\left[(\mathbf{R}'\times d\mathbf{R}')\int_{-\infty}^{\infty}\frac{dz}{(R'^2+z^2)^{3/2}} + (\hat{\mathbf{z}}\times d\mathbf{R}')\int_{-\infty}^{\infty} dz\frac{z}{(R'^2+z^2)^{3/2}}\right]. \tag{10.22}$$

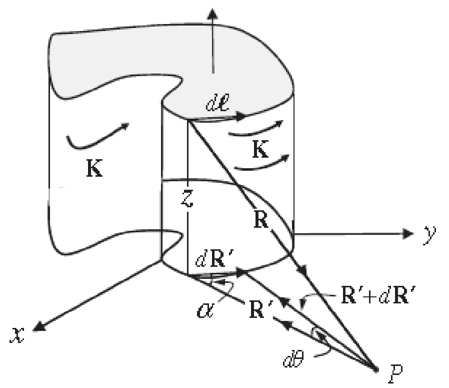

Figure 10.3: An infinitely long solenoid with a uniform cross sectional shape. The surface current density $\mathbf{K}$ has constant magnitude, but is everywhere parallel to the azimuthal vector $d\boldsymbol{\ell} = d\mathbf{R}'$ tangent to the solenoid surface.

The first integral in square brackets in (10.22) has the value $2/R'^2$. The second integral vanishes because its integrand is an odd function of z. Therefore,

$$\mathbf{B}(P) = \frac{\mu_0 K}{2\pi}\oint\frac{\mathbf{R}'\times d\mathbf{R}'}{R'^2}. \tag{10.23}$$

An important observation is that the vector $\mathbf{R}'\times d\mathbf{R}'$ points in the $-\hat{\mathbf{z}}$-direction when P lies outside the solenoid and points in the $+\hat{\mathbf{z}}$-direction when P lies inside the solenoid. Moreover,

$$|\mathbf{R}'\times d\mathbf{R}'| = R'dR'\sin(\pi-\alpha) = R'dR'\sin\alpha, \tag{10.24}$$

and the law of sines gives

$$dR'\sin\alpha = |\mathbf{R}'+d\mathbf{R}'|\sin(d\theta) \approx R'd\theta. \tag{10.25}$$

Therefore, $|\mathbf{R}'\times d\mathbf{R}'| \approx R'^2 d\theta$, and the magnitude of (10.23) is

$$B(P) = \frac{\mu_0 K}{2\pi}\oint d\theta. \tag{10.26}$$

When P lies outside the solenoid, the vector $\mathbf{R}'$ in Figure 10.3 sweeps out zero net angle θ as its tip traces out the closed circuit of the integral (10.26). When P lies inside the solenoid, $\mathbf{R}'$ sweeps out an angle 2π over the same closed circuit. Hence,

$$\mathbf{B}(P) = \begin{cases} \mu_0 K\hat{\mathbf{z}} & P \text{ inside the solenoid,} \\ 0 & P \text{ outside the solenoid.} \end{cases} \tag{10.27}$$

The magnetic field is uniform and axial everywhere inside the solenoid and vanishes everywhere outside the solenoid.

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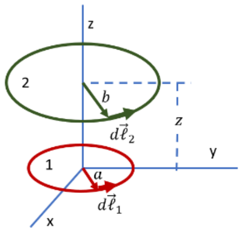
3. Considere dois anéis circulares, de raios a e b e coaxiais, separados de uma distância z entre seus planos, como mostra a figura. A indutância mútua entre esses dois anéis pode ser calculada pela Fórmula de Neumann



$$M = \frac{\mu_0}{4\pi}\oint\oint\frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$$

a) Usando coordenadas cilíndricas, com ângulos azimutais φ e φ', mostre que

$$d\vec{\ell}_1 \cdot d\vec{\ell}_2 = ab\cos(\varphi - \varphi')d\varphi d\varphi' \; ; \; |\vec{r} - \vec{r}'| = \sqrt{a^2 + b^2 - 2ab\cos(\varphi - \varphi')}$$

b) Definindo o ângulo $\alpha = \varphi - \varphi'$, mostre que a expressão para a indutância mútua fica

$$M = \frac{\mu_0 ab}{4\pi}\int_0^{2\pi} d\varphi \int_{\varphi}^{\omega - 2\pi}\frac{\cos\alpha \, d\alpha}{\sqrt{a^2 + b^2 - 2ab\cos\alpha}}$$

c) Faça as transformações $\alpha = \pi - \phi$; $\zeta = \frac{\phi}{2}$ e mostre que

$$M = -\frac{\mu_0 ab}{2\pi\sqrt{(a+b)^2 + z^2}}\int_0^{2\pi} d\varphi \int_{\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{3\pi}{2}-\frac{\varphi}{2}}\frac{1 - 2(\sin\zeta)^2}{\sqrt{1 - k^2(\sin\zeta)^2}}d\zeta \; ; \; k^2 = \frac{4ab}{(a+b)^2 + z^2}$$

d) Calcule A e B na separação de frações

$$\frac{1 - 2(\sin\zeta)^2}{\sqrt{1 - k^2(\sin\zeta)^2}} = A\sqrt{1 - k^2(\sin\zeta)^2} + \frac{B}{\sqrt{1 - k^2(\sin\zeta)^2}}$$

e argumente porque, na segunda integral, $\int_{\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{3\pi}{2}-\frac{\varphi}{2}}(\quad)d\zeta = 2\int_0^{\frac{\pi}{2}}(\quad)\,d\zeta$.

e) Obtenha, finalmente,

$$M = \mu_0\sqrt{ab}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right]$$

f) A indutância mútua é também definida como $M = \psi_{12}/I_1 = \psi_{21}/I_2$, onde ψ_{ij} é o fluxo magnético produzido pela corrente no circuito i que atravessa a área do circuito j. Usando a expressão para o potencial vetor obtido no problema 1 e $\vec{B} = \nabla \times \vec{A}$, derive a mesma expressão para a indutância mútua.

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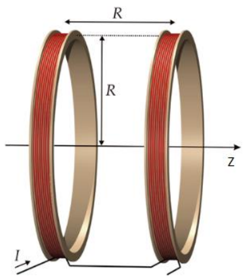
4. A Bobina de Helmholtz consiste em duas espiras circulares coaxiais, de raio R, cujos planos estão separados de uma distância igual ao raio (Zangwill, seção 10.4.2), conforme indica a figura. Utilizando as expressões para as componentes do campo produzido por uma espira de corrente deduzidas no problema 1, e os desenvolvimentos assimptóticos apropriados para as funções elípticas,



a) mostre que, ao longo do eixo da bobina, a componente B_ρ do campo se anula;

b) obtenha a expressão para o campo no ponto médio do eixo entre as espiras

$$B_{z0} = \frac{8}{5^{3/2}}\frac{\mu_0 NI}{R};$$

onde N é o número de voltas em cada espira.

c) mostre que, nesse mesmo ponto todas as três primeiras derivadas de B_z com relação à z se anulam;

d) encontre o valor aproximado do raio ρ, também no plano médio entre as espiras, em que $B_\rho \approx B_{z0}$.

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5. Considere um sistema de bobinas com corrente só na direção $\hat{e}_\varphi$, em coordenadas cilíndricas, tal que $\vec{A} = A_\varphi(\rho, z)\hat{e}_\varphi$. Mostre que, para esse sistema, as equações para as linhas de força correspondem a

$$\rho A_\varphi(\rho, z) = costante$$

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6. (Zangwill)

11.1 Magnetic Dipole Moment Practice A current distribution produces the vector potential

$$\mathbf{A}(r,\theta,\phi) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{A_0\sin\theta}{r}\exp(-\lambda r).$$

What is the magnetic moment associated with this current distribution?

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7. (Zangwill)

11.4 The Magnetic Moment of a Rotating Charged Disk A compact disk with radius R and uniform surface charge density σ rotates with angular speed ω. Find the magnetic dipole moment $\mathbf{m}$ when the axis of rotation is

(a) the symmetry axis of the disk.
(b) any diameter of the disk.

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8. Blindagem magnética. Em algumas experiências físicas, é necessário blindar campos magnéticos externos, inclusive o terrestre. Para isso, é necessário envolver a região de interesse com folhas metálicas de alto valor da permeabilidade magnéticas, denominadas "*μ-metals*". Um exemplo elucidativo de como esta blindagem ocorre é discutido na seção 5.12 do Jackson. Neste estudo dirigido os alunos deverão seguir em detalhe essa seção do livro texto, justificando adequadamente todas as passagens. Já de início, devem justificar a escolha do potencial escalar magnético fora da casca de blindagem, equação 5.117, que o Jackson simplesmente afirma que "deve ser assim". A divergência desse potencial quando $r \to \infty$ é aceitável?

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9. No artigo [L. Naser and Z. Chako; American Journal of Physics **87**, 971 (2019)], os autores discutem uma formulação alternativa para as condições de contorno para as condições de contorno em problemas magnetostáticos. Os alunos devem ler o artigo detalhadamente e reproduzir todos os cálculos das seções II e IV.

An alternative formulation of the magnetostatic boundary value problem

L. Nasser, and Z. Chacko

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An alternative formulation of the magnetostatic boundary value problem

L. Nasser[a)]
Department of Science and Mathematics, Columbia College, Chicago, Illinois 60605

Z. Chacko
Maryland Center for Fundamental Physics, Department of Physics, University of Maryland College Park, Maryland 20742

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We present an alternative formulation of the magnetostatic boundary value problem that is useful for calculating the magnetic field around a magnetic material placed in the vicinity of steady currents. The formulation differs from the standard approach in that a single-valued scalar potential plus a vector field that depends on the given currents but not on the magnetic material are used to obtain the total magnetic field instead of a magnetic vector potential. We illustrate the method with examples.
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I. INTRODUCTION

Solving for the magnetic field around linear, isotropic, magnetic materials placed in the vicinity of steady current-carrying conductors is one of the most important uses of magnetostatics. The evaluation of the fields follows from a standard manipulation of Maxwell's equations and is treated extensively in the literature.[1–4] The following description is limited to linear, isotropic materials for simplicity though the method itself is more general. The relevant Maxwell's equations are

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J}, \tag{1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2}$$

where $\vec{J}$ is a given current density and we impose continuity of the normal component of $\vec{B}$ and the tangential component of $\vec{H}$ across the surface as boundary conditions

$$\vec{B}_{\rm int} \cdot \hat{n}|_S = \vec{B}_{\rm ext} \cdot \hat{n}|_S, \tag{3}$$

$$\hat{n} \times \vec{H}_{\rm int}|_S = \hat{n} \times \vec{H}_{\rm ext}|_S, \tag{4}$$

where $\hat{n}$ is the unit normal at the surface.

In the standard procedure for solving this problem, we define the vector potential by

$$\vec{\nabla} \times \vec{A} = \vec{B}, \tag{5}$$

which then automatically satisfies Eq. (2). For convenience, we choose $\vec{\nabla} \cdot \vec{A} = 0$ and then Eq. (1) yields

$$\vec{\nabla} \times \frac{1}{\mu}(\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c}\vec{J}, \tag{6}$$

$$\nabla^2 \vec{A} = -\frac{4\mu\pi}{c}\vec{J}, \tag{7}$$

where $\mu = 1$ outside the magnetic material.

In a two-dimensional problem, a single component of $\vec{A}$ will often suffice to obtain the magnetic fields. However, in any truly three-dimensional problem solving Eq. (7) to yield all three components of $\vec{A}$ can represent a formidable task. The impracticality of applying a standard magnetic scalar potential approach to such a problem arises from the fact that the scalar potential ϕ, defined in a current-free region as

$$\vec{H} = -\vec{\nabla}\phi \tag{8}$$

is not single valued. This fact can be seen if we rewrite Eq. (1) in terms of the scalar potential using Eq. (8) and integrate over an area transverse to and including all of the current. Applying Stoke's theorem, we find

$$\Delta\phi|_{\rm closedloop} = \frac{4\pi}{c} I_{\rm enclosed}. \tag{9}$$

Hence the scalar potential picks up a contribution every time we go around a source current.

This multi-valuedness can become awkward, especially in situations where the magnetic body itself carries a free current. In this paper, we will discuss a formulation that overcomes the major shortcomings of the procedure just discussed. Essentially, the method consists of removing the rotational component of the magnetic field from the total magnetic field and evaluating the remainder as the gradient of a single-valued scalar potential. It will be shown by means of some examples that this approach offers the simplest solution to any fully three-dimensional problem. It is important to note that this procedure is well-known to engineers[5] but it appears to have eluded the notice of the general physics community. Moreover, their emphasis is in finite element methods while here we demonstrate its power in analytical calculations. There have been other publications dealing with alternate methods of calculating the magnetostatic field,[6] but to our knowledge there isn't anything in the physics literature that uses the scalar potential method described here.

II. THE NEW FORMULATION

We begin by defining a new vector $\vec{H}_0$ as

$$\vec{\nabla} \cdot \vec{H}_0 = 0, \tag{10}$$

$$\vec{\nabla} \times \vec{H}_0 = \frac{4\pi}{c}\vec{J}, \tag{11}$$

from which it is clear that $\vec{H}_0$ is the magnetic intensity in the absence of magnetic materials. We then have

$$\vec{H}_0(\vec{r}) = \frac{4\pi}{c}\int \frac{\vec{J}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3\vec{r}'. \tag{12}$$

Defining the vector $\vec{C}$ as

$$\vec{C} = \vec{H} - \vec{H}_0, \tag{13}$$

we see that

$$\vec{\nabla}\times\vec{C} = 0, \tag{14}$$

$$\vec{\nabla}\cdot(\mu\vec{C}) = -\vec{\nabla}\cdot(\mu\vec{H}_0), \tag{15}$$

since $\hat{n}\cdot\vec{H}_0$ is continuous across the surface by Eq. (10). From Eq. (14), it is clear that we may write $\vec{C} = -\vec{\nabla}\phi$, where ϕ is a single-valued scalar potential.[7] This function is continuous everywhere but is not differentiable at the boundary surface. It follows that we must distinguish between the interior potential $\phi_{\rm int}$ and the exterior potential $\phi_{\rm ext}$, and solve Laplace's equation

$$\nabla^2\phi_{\rm int} = 0, \tag{16}$$

$$\nabla^2\phi_{\rm ext} = 0, \tag{17}$$

subject to the following boundary conditions:

$$(\hat{n}\times\vec{\nabla}\phi_{\rm int})|_{\rm S} = (\vec{\nabla}\times\phi_{\rm ext})|_{\rm S}, \tag{18}$$

$$(\mu-1)\vec{H}_0\cdot\hat{n}|_{\rm S} = (\mu\vec{\nabla}\phi_{\rm int} - \vec{\nabla}\phi_{\rm ext})\cdot\hat{n}|_{\rm S}, \tag{19}$$

with $\vec{\nabla}\phi \to 0$ as $|\vec{r}| \to \infty$. Once the scalar potential is known, we obtain the magnetic field from

$$\vec{H} = -\vec{\nabla}\phi + \vec{H}_0. \tag{20}$$

Exact solutions to the equations above can sometimes be obtained by writing an expansion for ϕ in a basis appropriate to the geometry of the problem, and matching the expressions thus obtained for the interior and exterior regions at the boundary in accord with Eqs. (18) and (19). A few such examples will be evaluated shortly. In addition, for more elaborate geometries that are less prone to yielding closed-form algebraic solutions, the scalar potential ϕ may be determined numerically, often with much greater ease than the evaluation of the corresponding vector potential. The computational economy of this method is one of its most striking advantages, but we must bear in mind its generality as well; the method may be applied to any problem, irrespective of whether the magnetic material carries a source current or not.

III. ANALOGY WITH ELECTROSTATICS

Not surprisingly, this procedure has a simple analogy in electrostatics.[8,9] Consider a linear, isotropic dielectric in free space subject to an electric field $\vec{E}_0$ that may arise from charges embedded in the dielectric medium. The source charge distribution is assumed to remain unchanged, and the equations that determine the electric field are

$$\vec{\nabla}\cdot(\epsilon\vec{E}) = 0, \tag{21}$$

$$\vec{\nabla}\times\vec{E} = 0, \tag{22}$$

with boundary conditions

$$\hat{n}\times\overrightarrow{E_{\rm int}}|_{\rm S} = \hat{n}\times\overrightarrow{E_{\rm ext}}|_{\rm S}, \tag{23}$$

$$\epsilon\overrightarrow{E_{\rm int}}\cdot\hat{n}|_{\rm S} = \vec{E}_{\rm ext}\cdot\hat{n}|_{\rm S}, \tag{24}$$

and with $|\vec{E}| \to 0$ as $|\vec{r}| \to 0$. Writing

$$\vec{E}_{\rm int} = -\vec{\nabla}\phi_{\rm int} + \vec{E}_0, \tag{25}$$

$$\vec{E}_{\rm ext} = -\vec{\nabla}\phi_{\rm ext} + \vec{E}_0, \tag{26}$$

we get

$$\nabla^2\phi_{\rm int} = 0, \tag{27}$$

$$\nabla^2\phi_{\rm ext} = 0, \tag{28}$$

with boundary conditions

$$(\hat{n}\times\vec{\nabla}\phi_{\rm int})|_{\rm S} = (\vec{\nabla}\times\phi_{\rm ext})|_{\rm S}, \tag{29}$$

$$(\epsilon-1)\vec{E}_0\cdot\hat{n}|_{\rm S} = (\epsilon\vec{\nabla}\phi_{\rm int} - \vec{\nabla}\phi_{\rm ext})\cdot\hat{n}|_{\rm S}, \tag{30}$$

and with $\vec{\nabla}\phi \to 0$ as $|\vec{r}| \to \infty$. The correspondence between these equations and the previous set is clear: $(\mu-1)\vec{H}_0\cdot\hat{n}|_{\rm S}$ replaces $(\epsilon-1)\vec{E}_0\cdot\hat{n}|_{\rm S}$ as the "source term." Some examples of the use of this method are now presented.

IV. EXAMPLES OF THE METHOD

A. Current-carrying wire parallel to a cylinder of permeability μ

We begin with an infinitely long wire carrying (constant) current I and oriented parallel to the axis of an infinitely long cylinder of permeability μ. Let the radius of the cylinder be $\rho = a$ and let the distance between the wire and the cylinder be $d > a$. We take a point on the axis of the cylinder as the origin of coordinates, and note that the problem has symmetry along the cylinder's axis, which we choose to be the z-direction. Choosing the yz-plane to contain the wire, we have in polar coordinates

$$\vec{H}_0 = \frac{\lambda_0}{\rho^2 - 2\rho d\sin\theta + d^2}\left[d\cos\theta\hat{\rho} + (\rho - d\sin\theta)\hat{\theta}\right], \tag{31}$$

where λ_0 is the current per unit length in the wire. Writing the general forms of $\phi_{\rm int}(\rho, \theta)$ and $\phi_{\rm ext}(\rho, \theta)$ corresponding to solutions of $\nabla^2\phi = 0$ with no z-dependence as

$$\phi_{\rm int}(\rho,\theta) = \sum_{m=1}^{\infty}\rho^m(A_m\cos m\theta + B_m\sin m\theta), \tag{32}$$

$$\phi_{\rm ext}(\rho,\theta) = \sum_{m=1}^{\infty}\rho^{-m}(C_m\cos m\theta + D_m\sin m\theta), \tag{33}$$

the boundary conditions yield

$$A_m a^{2m} = C_m, \tag{34}$$

$$B_m a^{2m} = D_m, \tag{35}$$

and

$$(\mu-1)\vec{H}_0\cdot\hat{\rho} = \sum_{m=1}^{\infty} m(\mu+1)a^{m-1} \times (A_m\cos m\theta + B_m\sin m\theta). \tag{36}$$

To determine the coefficients A_m we note from Eq. (31) that

$$\vec{H}_0\cdot\hat{\rho} = \frac{\lambda_0 d\cos\theta}{\rho^2 - 2\rho d\sin\theta + d^2}, \tag{37}$$

and we therefore need to evaluate integrals of the form

$$I_n = \int_0^{2\pi} \frac{\cos\theta\cos n\theta}{a^2+d^2-2ad\sin\theta}d\theta, \tag{38}$$

where $n \geq 1$. We set $z = e^{i\theta}$ and switch to an integration around the unit circle $|z|=1$, giving

$$I_n = \mathrm{Re}\left\{-\frac{1}{2ad}\oint\frac{(z^2+1)z^{n-1}}{(z^2-1)-iz(a/d+d/a)}dz\right\}. \tag{39}$$

The integrand has simple poles at $z = ia/d$ and $z = id/a$, as shown in Fig. 1. Since only $z = ia/d$ lies within the unit circle, picking up the residue from that pole we find

$$I_n = \begin{cases} 0, & \text{for } n = 2m, \\ (-1)^m \dfrac{\pi}{d^2}\left(\dfrac{a}{d}\right)^{2m}, & \text{for } n = 2m+1. \end{cases} \tag{40}$$

To evaluate the B_m coefficients, we need to evaluate integrals of the form

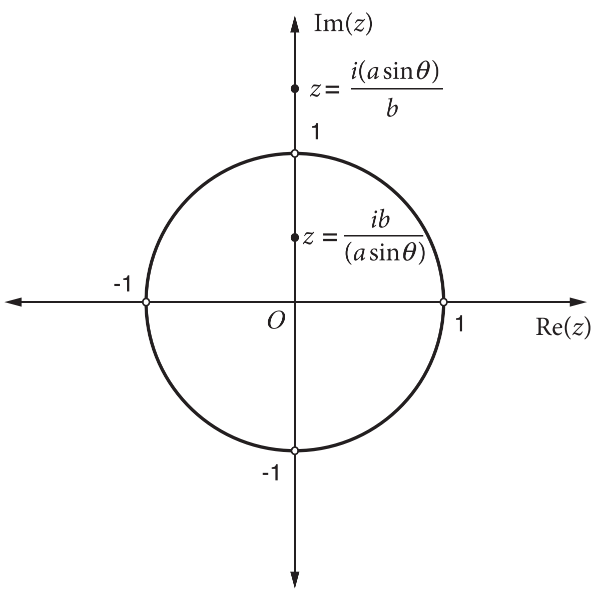

Fig. 1. Simple poles of I_n.

$$J_n = \int_0^{2\pi} \frac{\cos\theta\sin n\theta}{a^2+d^2-2ad\sin\theta}d\theta, \tag{41}$$

and following an analogous procedure, we find

$$J_n = \begin{cases} 0, & \text{for } n = 2m+1, \\ (-1)^{m+1} \dfrac{\pi}{d^2}\left(\dfrac{a}{d}\right)^{2m-1}, & \text{for } n = 2m. \end{cases} \tag{42}$$

With these results, we can now obtain

$$A_{2m} = 0, \tag{43}$$

$$A_{2m+1} = \frac{(-1)^m\Lambda}{d^{2m+1}(2m+1)}, \tag{44}$$

$$B_{2m} = -\frac{(-1)^m\Lambda}{d^{2m}2m}, \tag{45}$$

$$B_{2m+1} = 0, \tag{46}$$

where $\Lambda = \lambda_0(\mu-1)/(\mu+1)$. We then have

$$\phi_{\text{int}}(\rho,\theta) = \sum_{m=0}^{\infty}\Lambda\frac{(-1)^m}{2m+1}\frac{\cos(2m+1)\theta}{d^{2m+1}}\rho^{2m+1} - \sum_{m=1}^{\infty}\Lambda\frac{(-1)^m}{2m}\frac{\sin 2m\theta}{d^{2m}}\rho^{2m}. \tag{47}$$

Setting $y = i\rho e^{i\theta}/d$, we can recast Eq. (47) as

$$\begin{aligned}\phi_{\text{int}}(\rho,\theta) &= \mathrm{Im}\left\{\sum_{m=0}^{\infty}\frac{y^{2m+1}}{2m+1} - \sum_{m=1}^{\infty}\frac{y^{2m}}{2m}\right\} \\ &= \frac{\Lambda}{2}\arctan\left(\frac{2d\rho\cos\theta}{d^2-\rho^2}\right) \\ &\quad + \frac{\Lambda}{2}\arctan\left(\frac{\rho^2\sin 2\theta}{d^2+\rho^2\cos 2\theta}\right).\end{aligned} \tag{48}$$

Using a similar procedure, we obtain the external potential

$$\phi_{\text{ext}}(\rho,\theta) = \frac{\Lambda}{2}\arctan\left(\frac{2a^2\rho d\cos\theta}{\rho^2 d^2 - a^4}\right) + \frac{\Lambda}{2}\arctan\left(\frac{a^4\sin 2\theta}{\rho^2 d^2 + a^4\cos 2\theta}\right). \tag{49}$$

The full field is then determined from

$$\vec{B} = \mu\vec{H} = \mu(\vec{H}_0 - \vec{\nabla}\phi). \tag{50}$$

B. Sphere of constant permeability μ concentric with a ring of current

Consider a sphere of constant permeability μ and radius a, concentric with a circular ring of radius $b > a$ that carries a (constant) current I, as shown in Fig. 2. We wish to obtain the magnetic field everywhere in space. We begin by considering the case $r < b$. Using spherical coordinates, one can

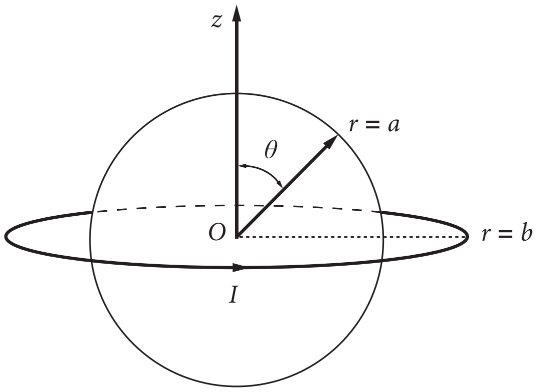

Fig. 2. Sphere of permeability μ and radius a surrounded by a concentric ring of radius b carrying current I.

show[10] that for $r < b$ the solution of Eqs. (10) and (11) for the circular ring in free space is

$$\vec{H}_0 \cdot \hat{r} = 2\lambda \sum_{n=0}^{\infty} f(n) r^{2n} P_{2n+1}(\cos\theta), \tag{51}$$

$$\vec{H}_0 \cdot \hat{\theta} = \lambda \sum_{n=0}^{\infty} g(n) r^{2n} P^1_{2n+1}(\cos\theta), \tag{52}$$

where $\lambda = \pi I/c$ and

$$f(n) = \frac{(-1)^n}{2^n b^{2n+1}} \frac{(2n+1)!!}{n!}, \tag{53}$$

$$g(n) = \frac{(-1)^n}{2^{n-1} b^{2n+1}} \frac{(2n-1)!!}{n!}, \tag{54}$$

while for $r > b$ we have

$$\vec{H}_0 \cdot \hat{r} = 2\lambda \sum_{n=0}^{\infty} \frac{a(n)}{r^{2n+3}} P_{2n+1}(\cos\theta), \tag{55}$$

$$\vec{H}_0 \cdot \hat{\theta} = \lambda \sum_{n=0}^{\infty} \frac{b(n)}{r^{2n+3}} P^1_{2n+1}(\cos\theta), \tag{56}$$

with

$$a(n) = \frac{(-1)^n (2n+1)!!\, b^{2n+2}}{2^n n!}, \tag{57}$$

$$b(n) = \frac{(-1)^n (2n+1)!!\, b^{2n+2}}{2^n (n+1)!}. \tag{58}$$

Note that since we are considering $b > a$, the boundary condition becomes

$$\begin{aligned}(\mu-1)\vec{H}_0 \cdot \hat{n}|_S &= (\mu-1)\vec{H}_0 \cdot \hat{r}|_{r=a} \\ &= 2\lambda(\mu-1)\sum_{n=0}^{\infty} f(n) a^{2n} P_{2n+1}(\cos\theta).\end{aligned} \tag{59}$$

This problem has azimuthal symmetry, so we look for solutions of Laplace's equation with this symmetry

$$\phi_{\text{int}} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta), \tag{60}$$

$$\phi_{\text{out}} = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{61}$$

From the boundary conditions (18) and (19), we obtain

$$A_\ell = B_\ell a^{2\ell+1}, \tag{62}$$

$$(\mu-1)\vec{H}_0 \cdot \hat{r}|_{r=a} = \sum_\ell A_\ell a^{\ell-1}[1 + \ell(1+\mu)] P_\ell(\cos\theta). \tag{63}$$

Using Eqs. (57)–(59) in Eq. (63) we finally obtain $A_{2n} = B_{2n} = 0$, and

$$A_{2n+1} = \frac{2\lambda(\mu-1) f(n)}{\mu(2n+1) + (2n+2)}, \tag{64}$$

$$B_{2n+1} = a^{4n+3} A_{2n+1}. \tag{65}$$

With the coefficients of the scalar potential expansion in hand, the field itself follows directly from Eq. (20).

V. CONCLUSION AND FURTHER WORK

We hope to have established that the procedure described in this paper offers a very general method of calculating the magnetic field when a source current is contained by, or placed in the vicinity of, a linear magnetic material. Moreover, this procedure is always simpler than the standard vector potential formulation for any truly three-dimensional problem. Applying the method requires nothing beyond the standard mathematical machinery needed to tackle boundary-value problems in electrostatics, and we hope that this alternative formulation will find its way into the standard curriculum on electricity and magnetism. There are many other fun problems students can solve to get practice with the method. For example, students can carry on where we left off in our second example and treat the case where the ring of current is embedded within the sphere ($b < a$). Another example students can work on is an

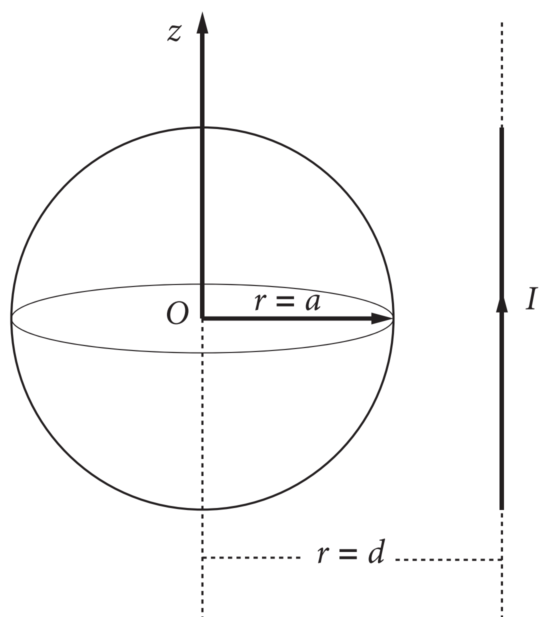

Fig. 3. Infinitely long, straight wire carrying a current I a distance d from the center of a sphere of permeability μ and radius a.

infinitely long straight wire with (constant) current I placed a distance d from the center of a sphere with permeability μ and radius a (see Fig. 3). This problem can be solved when the wire is outside the sphere ($a < d$) and also when the wire passes through the sphere ($a > d$).

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[a)]Electronic mail: lnasser@colum.edu

[1]Adams Stratton Julius, *Electromagnetic Theory*, 1st ed. (McGraw-Hill, New York and London, 1941), pp. 254–262.

[2]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (John Wiley & Sons, New York, 1974), pp. 191–194.

[3]A very interesting alternative method for calculating the magnetic field around a current loop using rotation matrices is given in Matthew I. Grivich and David P. Jackson, "The magnetic field of current-carrying polygons: An application of vector field rotations," Am. J. Phys. **68**, 469–474 (2000).

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[5]O. C. Zienkiewicz, John Lyness, and D. R. J. Owen, "Three-dimensional magnetic field determination using a scalar potential—A finite element solution," IEEE Trans. Magn. **13**(5), 1649–1656 (1977).

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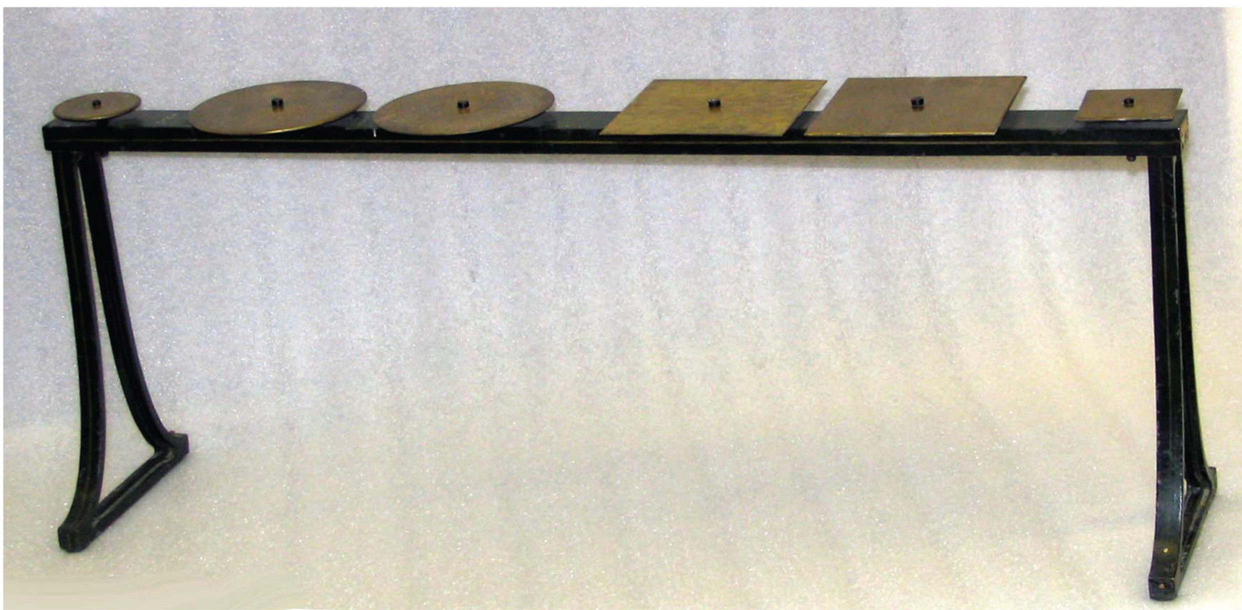
[7]This separation of the field into two parts follows from an application of the Poincare lemma: We know that for any given scalar field ψ, $\vec{\nabla} \times (\vec{\nabla}\psi) = 0$ always. The Poincare lemma then assures us that if a vector field $\vec{F}$ has the property that $\vec{\nabla} \times \vec{F} = 0$ in a given region of space, there exists a scalar function ϕ defined in that region such that $\vec{F} = \vec{\nabla}\phi$.

[8]This type of analogy is not new and has been explored in the literature. An interesting mapping between the calculation of electrostatic and magnetic fields in two dimensions has been discussed by Ying-yan Zhou, "The analogy between the calculation of a two-dimensional electrostatic field and that of a two-dimensional magnetostatic field," Am. J. Phys. **64**, 69–72 (1996).

[9]An interesting analogy between the calculation of the magnetic field of a solenoid and the electric field of a cylindrical capacitor filled with a dielectric ϵ is given by L. Lerner, "Magnetic field of a finite solenoid with a linear permeable core," Am. J. Phys. **79**, 1030–1035 (2011).

[10]See, for example, George Arfken, *Mathematical Methods for Physicists*, 2nd ed. (Academic Press, Harcourt Brace Jovanovich, San Diego, 1995), Sec. 12.5.



Chladni Plates

From the 1888 catalogue of James W. Queen & Co. of Philadelphia: "Bench, with Screw Supports for Six Plates of Brass, three round and three square. Two plates of each shape are of the same size, but one double the thickness of the other. Each pair is accompanied by a third plate of the same thickness as the first but half the diameter…$25.00" This set of Chaldni plates is at the physics department of the University of Utah, and was probably imported from the workshop of Rudolph Koenig in Paris. (Picture and text by Thomas B. Greenslade, Jr., Kenyon College)