Sistemas de Equações Não-Lineares

Parte I: método de Newton

Elias S. Helou Neto

Sistemas de Equações Não-Lineares

Definição

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$$

$$f_1(x_1, x_2, \ldots, x_n) = 0$$

$$f_1(x_1, x_2, \ldots, x_n) = 0$$
$$f_2(x_1, x_2, \ldots, x_n) = 0$$

Sistemas de Equações Não-Lineares

Definição

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, \ldots, x_n) &= 0 \\
f_3(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
$$

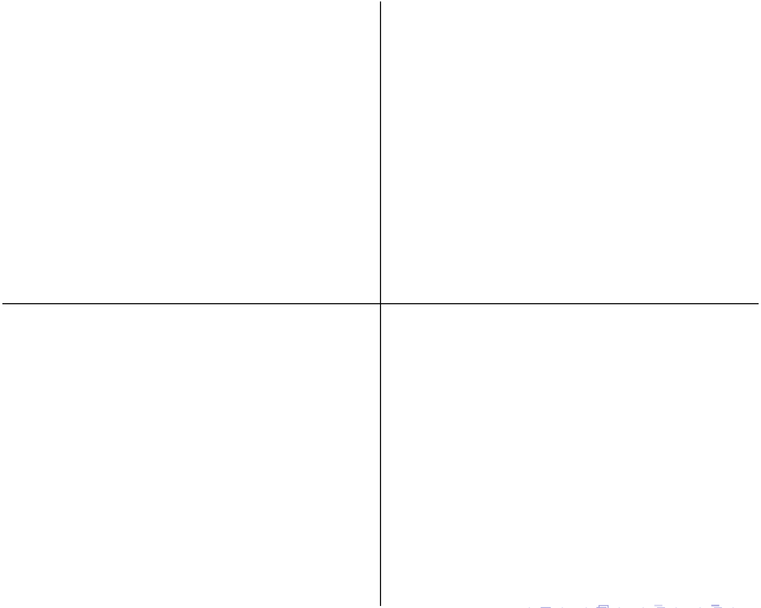
$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0\\
f_2(x_1, x_2, \ldots, x_n) &= 0\\
f_3(x_1, x_2, \ldots, x_n) &= 0\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
f_1(x_1, x_2, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, \ldots, x_n) &= 0 \\
f_3(x_1, x_2, \ldots, x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, \ldots, x_n) &= 0
\end{aligned}
$$

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$$
$$f_2(x_1, x_2) = x_1 - x_2$$

Sistemas de Equações Não-Lineares

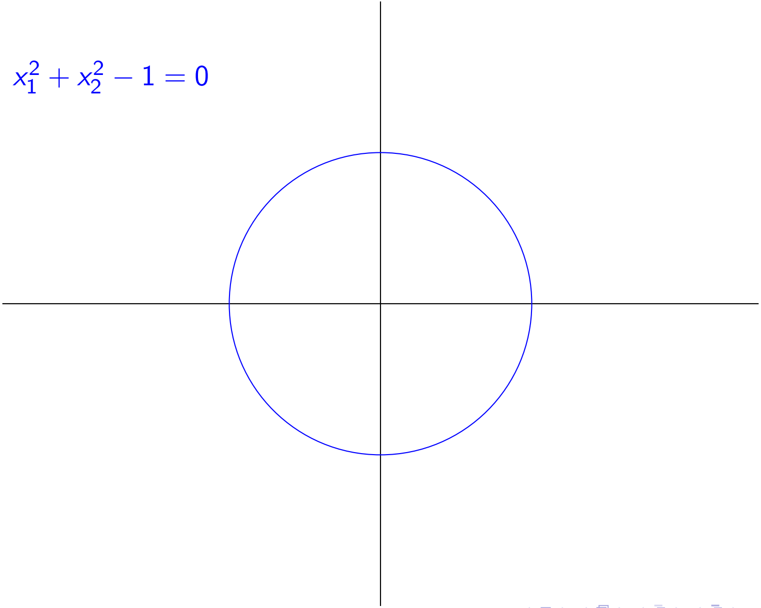
Exemplos



Sistemas de Equações Não-Lineares

Exemplos

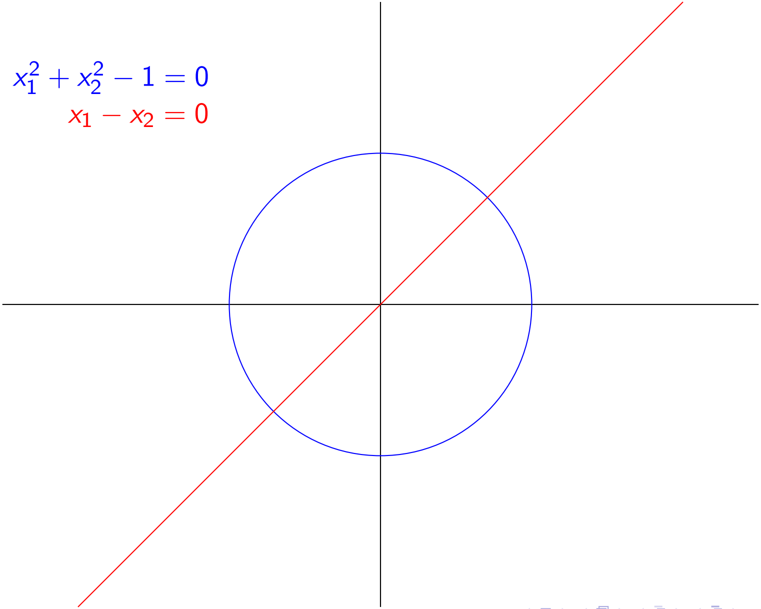
$$x_1^2 + x_2^2 - 1 = 0$$



Sistemas de Equações Não-Lineares

Exemplos

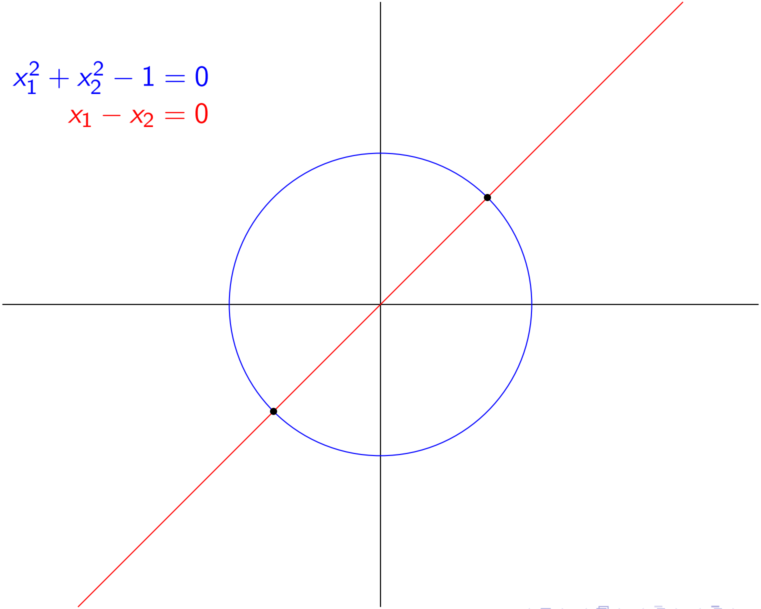
$$
\begin{aligned}
x_1^2 + x_2^2 - 1 &= 0 \\
x_1 - x_2 &= 0
\end{aligned}
$$



Sistemas de Equações Não-Lineares

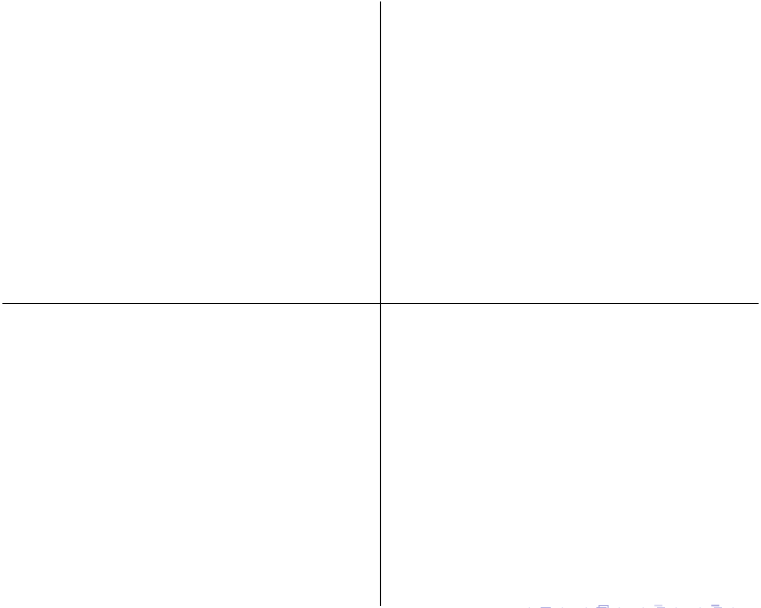
Exemplos

$$x_1^2 + x_2^2 - 1 = 0$$
$$x_1 - x_2 = 0$$



Sistemas de Equações Não-Lineares

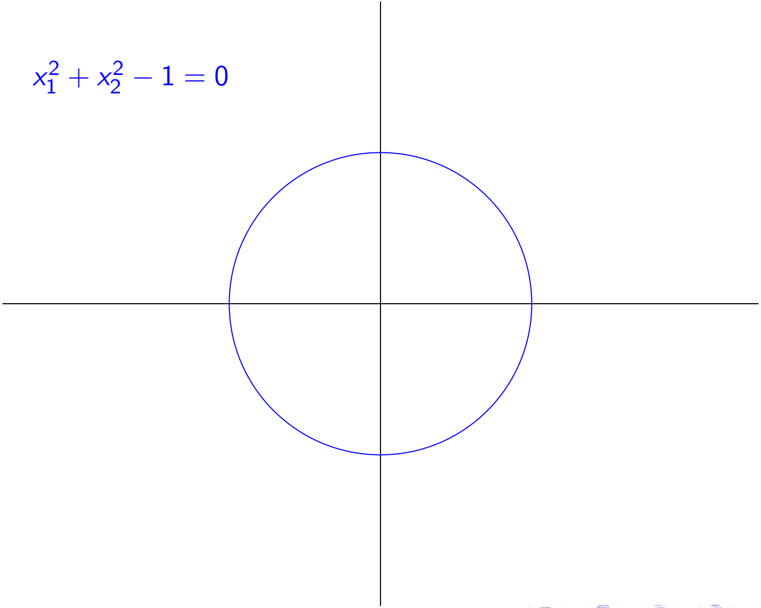
Exemplos



Sistemas de Equações Não-Lineares

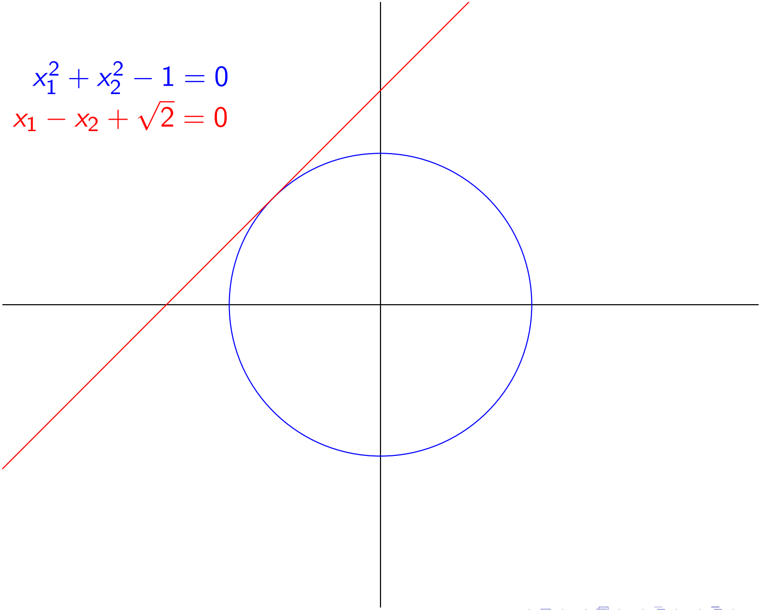
Exemplos

$$x_1^2 + x_2^2 - 1 = 0$$



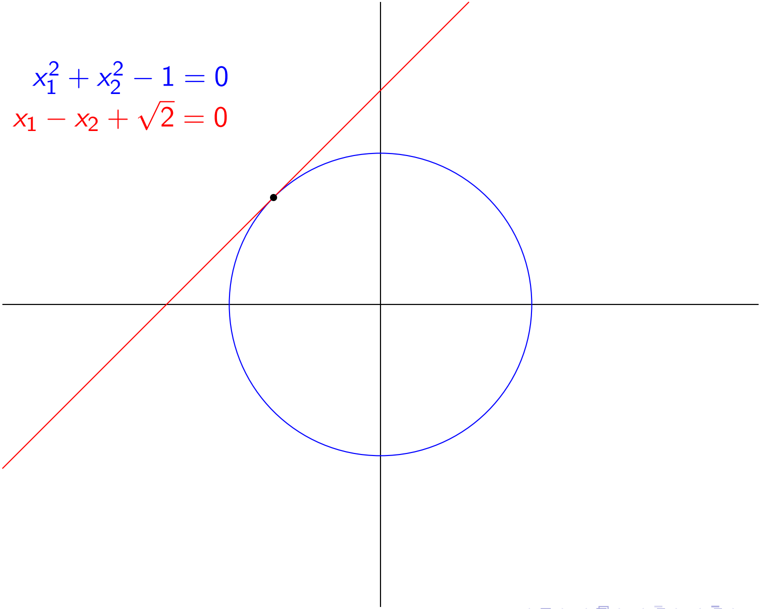
Sistemas de Equações Não-Lineares

Exemplos



Sistemas de Equações Não-Lineares

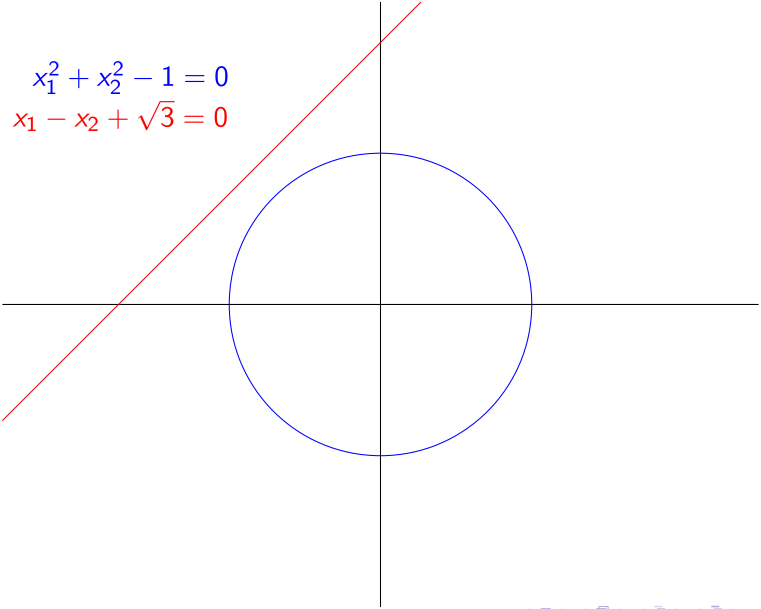
Exemplos



Sistemas de Equações Não-Lineares

Exemplos

$$x_1^2 + x_2^2 - 1 = 0$$
$$x_1 - x_2 + \sqrt{3} = 0$$



$$\boldsymbol{f}(\boldsymbol{y})$$

Sistemas de Equações Não-Lineares

Aproximação por Taylor

$$\boldsymbol{f}(\boldsymbol{y}) \approx \boldsymbol{f}(\boldsymbol{x})$$

Sistemas de Equações Não-Lineares

Aproximação por Taylor

$$\boldsymbol{f}(\boldsymbol{y}) \approx \boldsymbol{f}(\boldsymbol{x}) + J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})$$

Sistemas de Equações Não-Lineares

Aproximação por Taylor

$$\boldsymbol{f}(\boldsymbol{x}_{k+1}) \approx \boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$$

$$\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k)$$

$$\boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k) = \boldsymbol{0}$$

$$J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k) = -\boldsymbol{f}(\boldsymbol{x}_k)$$

$$\boldsymbol{x}_{k+1} - \boldsymbol{x}_k = -J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{f}(\boldsymbol{x}_k)$$

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{f}(\boldsymbol{x}_k)$$

$$J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k) = -\boldsymbol{f}(\boldsymbol{x}_k)$$

$$J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_{k+1} - \boldsymbol{x}_k) = -\boldsymbol{f}(\boldsymbol{x}_k)$$

- Resolva

$$J\boldsymbol{f}(\boldsymbol{x}_k)\boldsymbol{s}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$$

- Faça

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{s}_k$$

Método de Newton

Convergência

$$\boldsymbol{f}(\boldsymbol{y}) \approx \boldsymbol{f}(\boldsymbol{x}) + J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})$$

$$\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}) + \|\boldsymbol{y}-\boldsymbol{x}\|^2 \boldsymbol{e}(\boldsymbol{x},\boldsymbol{y})$$

Método de Newton

Convergência

$$\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) + \|\boldsymbol{y} - \boldsymbol{x}\|^2 \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y})$$

$$\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})}{\|\boldsymbol{y} - \boldsymbol{x}\|^2}$$

$$\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}) + \|\boldsymbol{y}-\boldsymbol{x}\|^2 \boldsymbol{e}(\boldsymbol{x},\boldsymbol{y})$$

$$\boldsymbol{e}(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - J\boldsymbol{f}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x})}{\|\boldsymbol{y}-\boldsymbol{x}\|^2}$$

Se $\boldsymbol{f}$ for duas vezes continuamente diferenciável, então

$$\left\|\boldsymbol{e}(\boldsymbol{x},\boldsymbol{y})\right\| \leq M.$$

$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}^* - \boldsymbol{x}_k) + \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)$$

$$\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}^* - \boldsymbol{x}_k) \\ + \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)$$

$$\begin{aligned} -\boldsymbol{f}(\boldsymbol{x}_k) + J\boldsymbol{f}(\boldsymbol{x}_k)(\boldsymbol{x}_k - \boldsymbol{x}^*) \\ = \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k) \end{aligned}$$

$$
\begin{aligned}
&-Jf(x_k)^{-1}f(x_k)+x_k-x^* \\
&\qquad = \|x^*-x_k\|^2 Jf(x_k)^{-1}e(x^*,x_k)
\end{aligned}
$$

Método de Newton

Convergência

$$
\begin{aligned}
\boldsymbol{x}_k - J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{x}^* \\
&= \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)
\end{aligned}
$$

Método de Newton

Convergência

$$
\begin{aligned}
\boldsymbol{x}_k - J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{x}^* \\
= \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)
\end{aligned}
$$

Método de Newton

Convergência

$$\mathbf{x}_{k+1} - \mathbf{x}^* = \|\mathbf{x}^* - \mathbf{x}_k\|^2 J\mathbf{f}(\mathbf{x}_k)^{-1}\mathbf{e}(\mathbf{x}^*, \mathbf{x}_k)$$

Método de Newton

Convergência

$$\boldsymbol{x}_{k+1} - \boldsymbol{x}^* = \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 J\boldsymbol{f}(\boldsymbol{x}_k)^{-1}\boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)$$

Método de Newton

Convergência

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| = \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \left\| J\boldsymbol{f}(\boldsymbol{x}_k)^{-1} \boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k) \right\|$$

Método de Newton

Convergência

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \left\| J\boldsymbol{f}(\boldsymbol{x}_k)^{-1} \right\| \|\boldsymbol{e}(\boldsymbol{x}^*, \boldsymbol{x}_k)\|$$

Método de Newton

Convergência

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \left\| J\boldsymbol{f}(\boldsymbol{x}_k)^{-1} \right\| M$$

Método de Newton

Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável

Método de Newton

Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável
- $$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$

Método de Newton

Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável
- $$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$
- $$\det\big(J\boldsymbol{f}(\boldsymbol{x}^*)\big) \neq 0$$

Método de Newton

Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável
-
$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$
-
$$\det\big(J\boldsymbol{f}(\boldsymbol{x}^*)\big) \neq 0$$
- $\boldsymbol{x}_0$ suficientemente próximo de $\boldsymbol{x}^*$

Método de Newton
Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável
- $$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$
- $$\det\big(J\boldsymbol{f}(\boldsymbol{x}^*)\big) \neq 0$$
- $\boldsymbol{x}_0$ suficientemente próximo de $\boldsymbol{x}^*$

$$\boldsymbol{x}_k \to \boldsymbol{x}^*$$

Método de Newton

Convergência

- $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ duas vezes continuamente diferenciável
- $$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$$
- $$\det\big(J\boldsymbol{f}(\boldsymbol{x}^*)\big) \neq 0$$
- $\boldsymbol{x}_0$ suficientemente próximo de $\boldsymbol{x}^*$

$$\boldsymbol{x}_k \to \boldsymbol{x}^* \quad \text{e} \quad \limsup_{k\to\infty} \frac{\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\|}{\|\boldsymbol{x}_k - \boldsymbol{x}^*\|^2} \leq c$$

Como $\det\big(J\boldsymbol{f}(\boldsymbol{x}^*)\big) \neq 0$, então existe $\epsilon > 0$ tal que

$$\|\boldsymbol{x} - \boldsymbol{x}^*\| \leq \epsilon \Rightarrow \left\|J\boldsymbol{f}(\boldsymbol{x})^{-1}\right\| < \frac{1}{2M\epsilon}$$

Método de Newton

Convergência

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}^* - \boldsymbol{x}_k\|^2 \left\| J\boldsymbol{f}(\boldsymbol{x}_k)^{-1} \right\| M$$

Método de Newton

Convergência

$$
\begin{aligned}
&\left\|\boldsymbol{x}_{k+1}-\boldsymbol{x}^{*}\right\| \\
&\quad \leq\left\|\boldsymbol{x}^{*}-\boldsymbol{x}_{k}\right\|\left\|\boldsymbol{x}^{*}-\boldsymbol{x}_{k}\right\|\left\|J \boldsymbol{f}\left(\boldsymbol{x}_{k}\right)^{-1}\right\| M
\end{aligned}
$$

Método de Newton

Convergência

Se $\|\boldsymbol{x}^* - \boldsymbol{x}_k\| \leq \epsilon$ então

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}^* - \boldsymbol{x}_k\| \left\| J\boldsymbol{f}(\boldsymbol{x}_k)^{-1} \right\| M\epsilon$$

Método de Newton

Convergência

Se $\|\boldsymbol{x}^* - \boldsymbol{x}_k\| \leq \epsilon$ então

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}^* - \boldsymbol{x}_k\| \frac{M\epsilon}{2M\epsilon}$$

Se $\|\boldsymbol{x}^* - \boldsymbol{x}_k\| \leq \epsilon$ então

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \leq \frac{1}{2}\|\boldsymbol{x}_k - \boldsymbol{x}^*\|$$

$$\boldsymbol{f}(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \boldsymbol{x}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \boldsymbol{x}$$
$$\phi(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{x}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \boldsymbol{x}$$

$$\phi(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{x}$$

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0} \Leftrightarrow \phi(\boldsymbol{x}) = \boldsymbol{x}$$

Método de Newton

Relação com Iteração de Ponto Fixo

Exercício: mostre que o método de Newton equivale à versão acelerada da iteração de ponto fixo acelerada que estudamos.