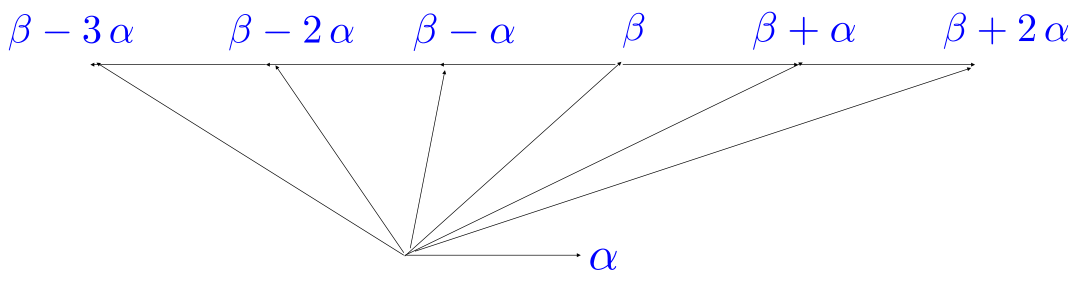
# Root Strings

We have shown in theorem 2.5 that if $\alpha$ and $\beta$ are non proportional roots then $\alpha + \beta$ is a root whenever $\alpha.\beta < 0$, and $\alpha - \beta$ is a root whenever $\alpha.\beta > 0$. We can use this result further to see if $\alpha + m\beta$ or $\alpha - n\beta$ (for $m, n$ integers) are roots. In this way we can obtain a set of roots forming a string. We then come to the concept of the *$\alpha$-root string through $\beta$*. Let $p$ be the largest positive integer for which $\beta + p\alpha$ is a root, and let $q$ be largest positive integer for which $\beta - q\alpha$ is a root. We will show that the set of vectors

$$\beta + p\alpha \; ; \; \beta + (p-1)\alpha \; ; \; ... \; \beta + \alpha \; ; \; \beta \; ; \; \beta - \alpha \; ; \; ... \; \beta - q\alpha \qquad (2.181)$$

are all roots. They constitute the $\alpha$-root string through $\beta$.



$$\beta + s\,\alpha \qquad \text{no roots} \qquad \beta + (r+1)\,\alpha \ldots \beta + p\,\alpha$$

Suppose that $\beta + p\alpha$ and $\beta - q\alpha$ are roots and that the string is broken, let us say, on the positive side. That is, there exist positive integers $r$ and $s$ with $p > r > s$ such that

1. $\beta + (r+1)\alpha$ is a root but $\beta + r\alpha$ is not a root

2. $\beta + (s+1)\alpha$ is not a root but $\beta + s\alpha$ is a root

According to theorem 2.5, since $\beta + r\alpha$ is not a root then we must have

$$\alpha.\,(\beta + (r+1)\alpha) \leq 0 \tag{2.182}$$

For the same reason, since $\beta + (s+1)\alpha$ is not a root we have

$$\alpha.\,(\beta + s\alpha) \geq 0 \tag{2.183}$$

Therefore we get that

$$((r+1) - s)\,\alpha^2 \leq 0 \tag{2.184}$$

and since $\alpha^2 > 0$

$$s - r \geq 1 \tag{2.185}$$

But this is a contradiction with our assumption that $r > s > 0$. So this proves that the string can not be broken on the positive side. The proof that the string is not broken on the negative side is similar.

Notice that the action of the Weyl reflection $\sigma_\alpha$ on a given root is to add or subtract a multiple of the root $\alpha$. Since all roots of the form $\beta + n\alpha$ are contained in the $\alpha$-root string through $\beta$, we conclude that this root string is invariant under $\sigma_\alpha$. In fact $\sigma_\alpha$ reverses the $\alpha$-root string. Clearly the image of $\beta + p\alpha$ under $\sigma_\alpha$ has to be $\beta - q\alpha$, and vice versa, since they are the roots that are most distant from the hyperplane perpendicular to $\alpha$. We then have

$$\sigma_\alpha(\beta - q\alpha) = \beta - q\alpha - \frac{2\alpha.(\beta - q\alpha)}{\alpha^2}\alpha = \beta + p\alpha \tag{2.186}$$

and since the only possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are 0, $\pm 1$, $\pm 2$ and $\pm 3$ we get that

$$q - p = \frac{2\alpha.\beta}{\alpha^2} = 0,\ \pm 1,\ \pm 2,\ \pm 3 \tag{2.187}$$

Denoting $\beta - q\alpha$ by $\gamma$ we see that for the $\alpha$-root string through $\gamma$ we have $q = 0$ and therefore the possible values of $p$ are 0, 1, 2 and 3. Consequently the number of roots in any string can not exceed 4.

For a simply laced Lie algebra the only possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are 0 and $\pm 1$. Therefore the root strings, in this case, can have at most two roots.

Notice that if $\alpha$ and $\beta$ are distinct simple roots, we necessarily have $q = 0$, since $\beta - \alpha$ is never a root in this case. So

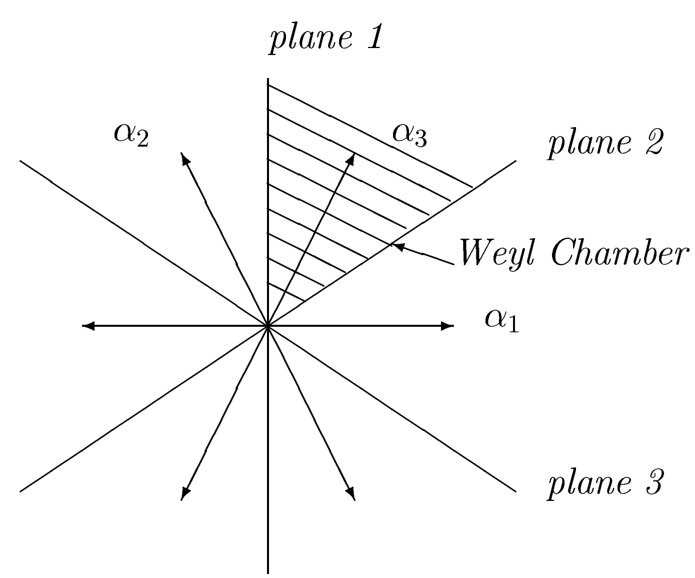
$$[E_{-\alpha}, E_{\beta}] = [E_{\alpha}, E_{-\beta}] = 0 \tag{2.188}$$

If, in addition, $\alpha.\beta = 0$ we get from (2.187) that $p = 0$ and consequently $\alpha + \beta$ is not a root either. For a semisimple Lie algebra, since if $\alpha$ is a root then $-\alpha$ is also a root, it follows that

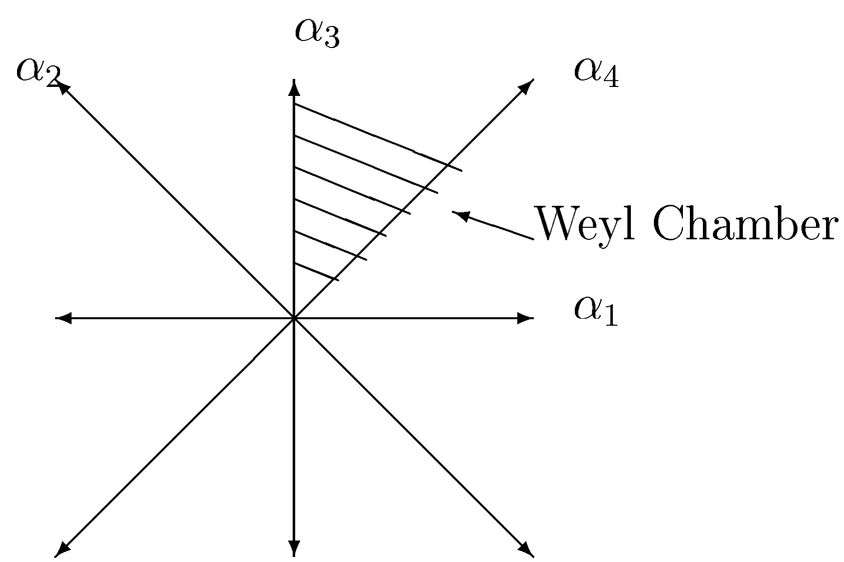
$$[E_{\alpha}, E_{\beta}] = [E_{-\alpha}, E_{-\beta}] = 0 \tag{2.189}$$

for $\alpha$ and $\beta$ simple roots and $\alpha.\beta = 0$. We can read this result from the Dynkin diagram since, if two points are not linked then the corresponding simple roots are orthogonal.

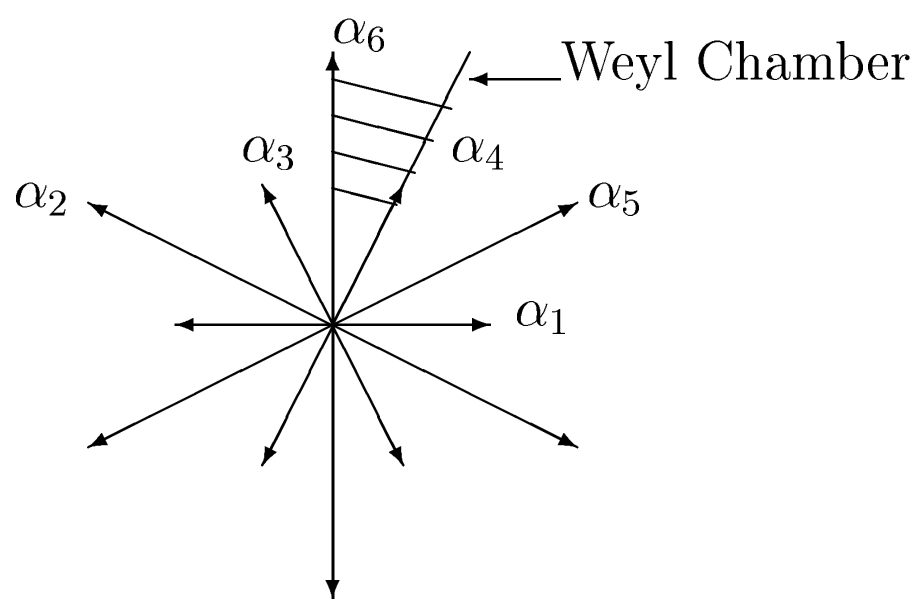
**Example 2.17** *For the algebra of $SU(3)$ we see from the diagram shown in figure 2.6 that the $\alpha_1$-root string through $\alpha_2$ contains only two roots namely 2 and 3= 2+1.*



**Example 2.18** *From the root diagram shown in figure 2.7 we see that, for the algebra of* $SO(5)$, *the* $\alpha_1$*-root string through* $\alpha_2$ *contains thre roots* $\alpha_2$, $\alpha_3 = \alpha_1 + \alpha_2$, *and* $\alpha_4 = \alpha_2 + 2\alpha_1$.



**Example 2.19** *The algebra $G_2$ is the only simple Lie algebra which can have root strings with four roots. From the diagram shown in figure 2.8 we see that the $\alpha_1$-root string through $\alpha_2$ contains the roots $\alpha_2$, $\alpha_3 = \alpha_2+\alpha_1$ , $\alpha_5 = \alpha_2+2\alpha_1$ and $\alpha_6 = 2\alpha_2 + 3\alpha_1$.*



## Height of a root

$$\alpha = \sum_{a=1}^{r} n_a \, \alpha_a \longrightarrow h(\alpha) = \sum_{a=1}^{r} n_a$$

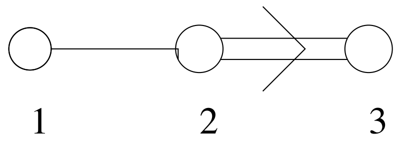
## Highest root $\psi$

$$\psi = \sum_{a=1}^{r} m_a \, \alpha_a \qquad h(\psi) \text{ is maximum}$$

**Coxeter Number** $\qquad \text{coxeter} = h(\psi) + 1$

**Curiosity** $\qquad \dim \mathcal{G} = (\text{coxeter} + 1)\text{rank}\, \mathcal{G}$

# Cartan Matrix from Dynkin Diagram




*We see that the simple root* $3$ *(according to the rules of section 2.11 ) has a length smaller than that of the other two. So we have* $K_{23} = -2$ *and* $K_{32} = -1$. *Since the roots* $1$ *and* $2$ *have the same length we have* $K_{12} = K_{21} = -1$. $K_{13}$ *and* $K_{31}$ *are zero because there are no links between the roots* $1$ *and* $3$. *Therefore*

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{2.191}$$

$SO(7)$

# Roots from Cartan Matrix

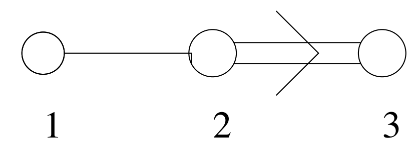
1. The roots of height 1 are just the simple roots.

2. We have seen in (2.189) that if two simple roots are orthogonal then their sum is not a root. On the other hand if they are not orthogonal then their sum is necessarily a root. From theorem 2.6 one has $\alpha.\beta \leq 0$ for $\alpha$ and $\beta$ simple, and therefore from theorem 2.5 one gets their sum is a root (if they are not orthogonal). Consequently to obtain the roots of height 2 one just look at the Dynkin diagram. The sum of pairs of simple roots which corresponding points are linked, by one or more lines, are roots. These are the only roots of height 2.

3. The procedure to obtain the roots of height 3 or greater is the following: suppose $\alpha^{(l)} = \sum_{a=1}^{rank\mathcal{G}} n_a \alpha_a$ is a root o height $l$, i.e. $\sum_{a=1}^{rank\mathcal{G}} n_a = l$. Using the Cartan matrix one evaluates
$$\frac{2\alpha^{(l)}.\alpha_b}{\alpha_b^2} = \sum_{a=1}^{rank\mathcal{G}} n_a K_{ab} \tag{2.192}$$
where $\alpha_b$ is a simple root. If this quantity is negative one gets from theorem 2.5 that $\alpha^{(l)} + \alpha_b$ is a root of height $l+1$. If it is zero or positive on uses (2.187) to write
$$p = q - \sum_{a=1}^{rank\mathcal{G}} n_a K_{ab} \tag{2.193}$$
where $p$ and $q$ are the highest positive integers such that $\alpha^{(l)} + p\alpha_b$ and $\alpha^{(l)} - q\alpha_b$ are roots. The integer $q$ can be determined by looking at the set of roots of height smaller than $l$ (which have already been determined) and checking what is the root of smallest height of the form $\alpha^{(l)} - m\alpha_b$. One then finds $p$ from (2.193). If $p$ does not vanish, $\alpha^{(l)} + \alpha_b$ is a root. Notice that if $p \geq 2$ one also determines roots of height greater than $l+1$. By applying this procedure using all simple roots and all roots of height $l$ one determines all roots of height $l+1$.

4. The process finishes when no roots of a given height $l+1$ is found. That is because there can not exists roots of height $l+2$ if there are no roots of height $l+1$.

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$




**Example 2.21** *In example 2.20 we have determined the Cartan matrix of* $SO(7)$ *from its Dynkin diagram. We now determine its root system following the procedure described above. The dimension of* $SO(7)$ *is* $21$ *and its rank is* $3$*. So, the number of positive roots is* $9$*. The first three are the simple roots* $\alpha_1$ *,* $\alpha_2$ *and* $\alpha_3$ *. Looking at the Dynkin diagram in figure 2.10 we see that* $\alpha_1 + \alpha_2$ *and* $\alpha_2 + \alpha_3$ *are the only roots of height* $2$*, since* $\alpha_1$ *and* $\alpha_3$ *are orthogonal. We have* $\frac{2(\alpha_1+\alpha_2).\alpha_a}{\alpha_a^2} = K_{1a} + K_{2a}$ *which, from (2.191), is equal to* $1$ *for* $a = 1, 2$ *and* $-2$ *for* $a = 3$*. Therefore, from (2.193), we get that* $2\alpha_1 + \alpha_2$ *and* $\alpha_1 + 2\alpha_2$ *are not roots but* $\alpha_1 + \alpha_2 + \alpha_3$ *and* $\alpha_1 + \alpha_2 + 2\alpha_3$ *are roots. Analogously we have* $\frac{2(\alpha_2+\alpha_3).\alpha_a}{\alpha_a^2} = K_{2a} + K_{3a}$ *which is equal to* $-1$ *for* $a = 1$*,* $1$ *for* $a = 2$ *and* $0$ *for* $a = 3$*. Therefore the only new root we obtain is* $\alpha_2 + 2\alpha_3$*. This exhausts the roots of height* $3$*. One can check that the only root of height* $4$ *is* $\alpha_1 + \alpha_2 + 2\alpha_3$ *which we have obtained before. Now* $\frac{2(\alpha_1+\alpha_2+2\alpha_3).\alpha_a}{\alpha_a^2} = K_{1a} + K_{2a} + 2K_{3a}$ *which is equal to* $1$*,* $-1$ *and* $2$ *for* $a = 1, 2, 3$ *respectively. Since it is negative for* $a = 2$ *we get that* $\alpha_1 + 2\alpha_2 + 2\alpha_3$ *is a root. This is the only root of height* $5$*, and it is in fact the highest root of* $SO(7)$*. So the Coxeter number of* $SO(7)$ *is* $6$*. Summarizing we have that the positive roots of* $SO(7)$ *are*

*roots of height* $1$ $\quad \alpha_1$*;* $\alpha_2$*;* $\alpha_3$
*roots of height* $2$ $\quad (\alpha_1 + \alpha_2)$*;* $(\alpha_2 + \alpha_3)$
*roots of height* $3$ $\quad (\alpha_1 + \alpha_2 + \alpha_3)$*;* $(\alpha_2 + 2\alpha_3)$
*roots of height* $4$ $\quad (\alpha_1 + \alpha_2 + 2\alpha_3)$
*roots of height* $5$ $\quad (\alpha_1 + 2\alpha_2 + 2\alpha_3)$

*These could also be determined starting from the simple roots and using Weyl reflections.*

# Chevalley Basis

$$H_a \equiv \frac{2\alpha_a.H}{\alpha_a^2} \tag{2.194}$$

where $\alpha_a$ ($a = 1, 2, ...$ rank $\mathcal{G}$) are the simple roots and $\alpha_a.H = \alpha_a^i H^i$ , where $H_i$ are the Cartan subalgebra generators in the Cartan-Weyl basis and $\alpha_a^i$ are the components of the simple root $\alpha_a$ in that basis, i.e. $[H_i, E_{\alpha_a}] = \alpha_a^i E_{\alpha_a}$. The generators $H_a$ are not orthonormal like the $H_i$ . From (2.134) and (2.170) we have that

$$Tr(H_a H_b) = \frac{4\alpha_a.\alpha_b}{\alpha_a^2 \alpha_b^2} = \frac{2}{\alpha_a^2} K_{ab} \tag{2.195}$$

The generators $H_a$ obviously commute among themselves

$$[H_a, H_b] = 0 \tag{2.196}$$

The commutation relations between $H_a$ and step operators are given by (see (2.124))

$$[H_a, E_\alpha] = \frac{2\alpha.\alpha_a}{\alpha_a^2} E_\alpha = K_{\alpha a} E_\alpha \tag{2.197}$$

where we have defined $K_{\alpha a} \equiv \frac{2\alpha.\alpha_a}{\alpha_a^2}$. Since $\alpha$ can be written as in (2.167) we see that $K_{\alpha a}$ is a linear combination with integer coefficients, all of the same sign, of the $a$-columm of the Cartan matrix

$$K_{\alpha a} = \frac{2\alpha.\alpha_a}{\alpha_a^2} = \sum_{b=1}^{rank \mathcal{G}} n_b K_{ba} \tag{2.198}$$

where $\alpha = \sum_{b=1}^{rank \mathcal{G}} n_b \alpha_b$. Notice that the factor multiplying $E_\alpha$ on the r.h.s of (2.197) is an integer. In fact this is a property of the Chevalley basis. All the structure constants of the algebra in this basis are integer numbers. The commutation relations (2.197) are determined once one knows the root system of the algebra.

We now consider the commutation relations between step operators. From (2.125)

$$[E_\alpha, E_\beta] \;=\; \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha = m_a H_a & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.199}$$

where $m_a$ are integers in the expansion $\frac{\alpha}{\alpha^2} = \sum_{a=1}^{rank\mathcal{G}} m_a \frac{\alpha_a}{\alpha_a^2}$. The structure constants $N_{\alpha\beta}$, in the Chevalley basis, are integers and can be determined from the root system of the algebra and also from the Jacobi identity . Let us explain now how to do that.

Notice that from the antisymmetry of the Lie bracket

$$N_{\alpha\beta} = -N_{\beta\alpha} \tag{2.200}$$

for any pair of roots $\alpha$ and $\beta$. The structure constants $N_{\alpha\beta}$ are defined up to rescaling of the step operators. If we make the transformation

$$E_\alpha \rightarrow \rho_\alpha E_\alpha \tag{2.201}$$

keeping the Cartan subalgebra generators unchanged, then from (2.199) the structure constants $N_{\alpha\beta}$ must transform as

$$N_{\alpha\beta} \rightarrow \frac{\rho_\alpha \rho_\beta}{\rho_{\alpha+\beta}} N_{\alpha\beta} \tag{2.202}$$

and

$$\rho_\alpha \rho{-\alpha} = 1 \tag{2.203}$$

As we have said in section 2.9, any symmetry of the root diagram can be elevated to an automorphism of the corresponding Lie algebra. In any semisimple Lie algebra the transformation $\alpha \rightarrow -\alpha$ is a symmetry of the root diagram since if $\alpha$ is a root so is $-\alpha$. We then define the transformation $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ as

$$\sigma(H_a) = -H_a \; ; \quad \sigma(E_\alpha) = \eta_\alpha E_{-\alpha} \tag{2.204}$$

and $\sigma^2 = 1$. From the commutation relations (2.196), (2.197) and (2.199) one sees that such transformation is an automorphism if

$$\begin{aligned} \eta_\alpha \eta_{-\alpha} &= 1 \\ N_{\alpha\beta} &= \frac{\eta_\alpha \eta_\beta}{\eta_{\alpha+\beta}} N_{-\alpha,-\beta} \end{aligned} \tag{2.205}$$

Using the freedom to rescale the step operators as in (2.202) one sees that it is possible to satisfy (2.205) and make (2.204) an automorphism. In particular it is possible to choose all $\eta_\alpha$ equals to $-1$ and therefore

$$N_{\alpha\beta} = -N_{-\alpha,-\beta} \tag{2.206}$$

$$\beta + p\alpha \; ; \; \beta + (p-1)\alpha \; ; \; ... \; \beta + \alpha \; ; \; \beta \; ; \; \beta - \alpha \; ; \; ... \; \beta - q\alpha$$

$$[[E_{\beta+n\,\alpha}\, , \, E_{-\alpha}] \, , \, E_{\alpha}] - [[E_{\beta+n\,\alpha}\, , \, E_{\alpha}] \, , \, E_{-\alpha}] = [[E_{\alpha}\, , \, E_{-\alpha}] \, , \, E_{\beta+n\,\alpha}]$$

$$\begin{array}{lll}
N_{\beta+p\,\alpha\,,\,-\alpha}\, N_{\beta+(p-1)\,\alpha\,,\,\alpha} & & = 2\dfrac{\alpha\cdot(\beta+p\,\alpha)}{\alpha^2} \\[2ex]
N_{\beta+(p-1)\,\alpha\,,\,-\alpha}\, N_{\beta+(p-2)\,\alpha\,,\,\alpha} & -N_{\beta+(p-1)\,\alpha\,,\,\alpha}\, N_{\beta+p\,\alpha\,,\,-\alpha} & = 2\dfrac{\alpha\cdot(\beta+(p-1)\,\alpha)}{\alpha^2} \\[2ex]
N_{\beta+(p-2)\,\alpha\,,\,-\alpha}\, N_{\beta+(p-3)\,\alpha\,,\,\alpha} & -N_{\beta+(p-2)\,\alpha\,,\,\alpha}\, N_{\beta+(p-1)\,\alpha\,,\,-\alpha} & = 2\dfrac{\alpha\cdot(\beta+(p-2)\,\alpha)}{\alpha^2} \\
\vdots & \vdots & \vdots \\
N_{\beta+\alpha\,,\,-\alpha}\, N_{\beta\,,\,\alpha} & -N_{\beta+\,\alpha\,,\,\alpha}\, N_{\beta+2\,\alpha\,,\,-\alpha} & = 2\dfrac{\alpha\cdot(\beta+\alpha)}{\alpha^2}
\end{array}$$

$$\begin{aligned}
N_{\beta+\alpha,-\alpha} N_{\beta\alpha} &= \frac{2\alpha.\beta}{\alpha^2} p + 2\,(p + (p-1) + (p-2) + ... + 1) \\
&= p(q+1)
\end{aligned}$$

From the fact that the Killing form is invariant under the adjoint representation (see (2.48) it follows that it is invariant under inner automorphisms, i.e. $Tr(\sigma(T)\sigma(T')) = Tr(TT')$ with $\sigma(T) = gTg^{-1}$. However one can show that the Killing form is invariant any automorphism (inner or outer). Using this fact for the automorphism (2.204) (with $\eta_\alpha = -1$), the invariance property (2.46) and the normalization (2.134) one gets

$$\begin{aligned} Tr([E_\alpha, E_\beta]E_{-\alpha-\beta}) &= N_{\alpha\beta}\frac{2}{(\alpha+\beta)^2} \\ &= -Tr([E_{-\alpha}, E_{-\beta}]E_{\alpha+\beta}) \\ &= -Tr([E_{\alpha+\beta}, E_{-\alpha}]E_{-\beta}) \\ &= -N_{\alpha+\beta,-\alpha}\frac{2}{\beta^2} \end{aligned} \tag{2.209}$$

Consequently

$$N_{\alpha+\beta,-\alpha} = -\frac{\beta^2}{(\alpha+\beta)^2}N_{\alpha\beta} \tag{2.210}$$

Substituting this into (2.208) we get

$$N_{\alpha\beta}^2 = \frac{(\alpha+\beta)^2}{\beta^2}p(q+1) \tag{2.211}$$

Therefore, up to a sign, the structure constants $N_{\alpha\beta}$ defined in (2.199) can be determined from the root system of the algebra.

Using the Jacobi identity for the step operators $E_\alpha$, $E_\alpha$ and $E_{\beta-n\alpha}$, with $n$ varying from 1 to $q$ where $q$ is the highest integer such that $\beta - q\alpha$ is a root, and doing similar calculations we obtain that

$$N_{\beta,-\alpha}^2 = \frac{(\beta-\alpha)^2}{\beta^2}q(p+1) \tag{2.212}$$

$$\begin{aligned} N_{\beta+\alpha,-\alpha}N_{\beta\alpha} &= \frac{2\alpha.\beta}{\alpha^2}p + 2\,(p + (p-1) + ( \\ &= p(q+1) \end{aligned}$$

The relation (2.211) can be put in a simpler form. From (2.187) we have that (see section 25.1 of [HUM 72])

$$
\begin{aligned}
(q+1) - p\frac{(\alpha+\beta)^2}{\beta^2} &= p + \frac{2\alpha.\beta}{\alpha^2} + 1 - p\frac{(\alpha+\beta)^2}{\beta^2} \\
&= \frac{2\alpha.\beta}{\alpha^2} + 1 - p\frac{\alpha^2}{\beta^2} - p\frac{2\alpha.\beta}{\beta^2} \\
&= \left(\frac{2\alpha.\beta}{\alpha^2} + 1\right)\left(1 - p\frac{\alpha^2}{\beta^2}\right) \qquad (2.213)
\end{aligned}
$$

We want to show the r.h.s of this relation is zero. We distinguish two cases:

1. In the case where $\alpha^2 \geq \beta^2$ we have $\mid \frac{2\alpha.\beta}{\alpha^2} \mid \leq \mid \frac{2\alpha.\beta}{\beta^2} \mid$. From table 2.2 we see that the possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are $-1$, 0 or 1. In the first case we get that the first factor on the r.h.s of (2.213) vanishes. On the other two cases we have that $\alpha.\beta \geq 0$ and then $(\alpha+\beta)^2$ is strictly larger than both, $\alpha^2$ and $\beta^2$ . Since we are assuming $\alpha+\beta$ is a root and since, as we have said at the end of section 2.8, there can be no more than two different root lengths in each component of a root system, we conclude that $\alpha^2 = \beta^2$ . For the same reason $\beta + 2\alpha$ can not be a root since $(\beta+2\alpha)^2 > (\beta+\alpha)^2$ and therefore $p = 1$. But this implies that the second factor on the r.h.s of (2.213) vanishes.

2. For the case of $\alpha^2 < \beta^2$ we have that $(\alpha+\beta)^2 = \alpha^2$ or $\beta^2$, since otherwise we would have three different root lengths. This forces $\alpha.\beta$ to be strictly negative. Therefore we have $(\beta-\alpha)^2 > \beta^2 > \alpha^2$ and consequently $\beta-\alpha$ is not a root and so $q = 0$. But $\mid \frac{2\alpha.\beta}{\beta^2} \mid < \mid \frac{2\alpha.\beta}{\alpha^2} \mid$ and therefore $\frac{2\alpha.\beta}{\beta^2} = -1, 0$ or 1. Since $\alpha.\beta < 0$ we have $\frac{2\alpha.\beta}{\beta^2} = -1$. Then from (2.187) we have $p = -\frac{2\alpha.\beta}{\beta^2}\frac{\beta^2}{2\alpha.\beta} = \frac{\beta^2}{\alpha^2}$. Therefore the second factor on the r.h.s of (2.213) vanishes.

$$p = -\frac{2\,\alpha\cdot\beta}{\beta^2}\,\frac{\beta^2}{\alpha^2} = \frac{\beta^2}{\alpha^2}$$

Then, we have shown that

$$q + 1 = p\frac{(\alpha+\beta)^2}{\beta^2} \qquad (2.214)$$

and from (2.211)

$$N_{\alpha\beta}^2 = (q+1)^2 \qquad (2.215)$$

$$
\begin{aligned}
[H_a, H_b] &= 0 && (2.216)\\
[H_a, E_\alpha] &= \frac{2\alpha.\alpha_a}{\alpha_a^2} E_\alpha = K_{\alpha a} E_\alpha && (2.217)\\
[E_\alpha, E_\beta] &= \begin{cases} (q+1)\varepsilon(\alpha,\beta) E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha = \frac{2\alpha.H}{\alpha^2} = m_a H_a & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} && (2.218)
\end{aligned}
$$

where we have denoted $\varepsilon(\alpha,\beta)$ the sign of the structure constant $N_{\alpha\beta}$, i.e. $N_{\alpha\beta} = (q+1)\varepsilon(\alpha,\beta)$. These signs, also called cocycles, are determined through the Jacobi identity as explained in section 2.14. As we have said before $q$ is the highest positive integer such that $\beta - q\alpha$ is a root. However when $\alpha+\beta$ is a root, which is the case we are interested in (2.218), it is true that $q$ is also the highest positive integer such that $\alpha - q\beta$ is a root. The reason is the following: in a semisimple Lie algebra the roots always appear in pairs ($\alpha$ and $-\alpha$). Therefore if $\beta - \alpha$ is a root so is $\alpha - \beta$. In addition we have seen in section 2.12 that the root strings are unbroken and they can have at most four roots. Therefore, since we are assuming that $\alpha+\beta$ is a root, the only possible way of not satisfying what we said before is to have, let us say, the $\alpha$-root string through $\beta$ as $\beta - 2\alpha$, $\beta - \alpha$, $\beta$, $\beta + \alpha$; and the $\beta$-root string through $\alpha$ as $\alpha - \beta$, $\alpha$, $\alpha + \beta$ or $\alpha - \beta$, $\alpha$, $\alpha + \beta$, $\alpha + 2\beta$. But from (2.187) we have

$$
\frac{2\alpha.\beta}{\alpha^2} = 1 \qquad (2.219)
$$

and

$$
\frac{2\alpha.\beta}{\beta^2} = 0 \;\; or \;\; -1 \qquad (2.220)
$$

which are clearly incompatible.

We have said in section 2.12 that for a simply laced Lie algebra there can be at most two roots in a root string. Therefore if $\alpha+\beta$ is a root $\alpha-\beta$ is not, and therefore $q = 0$. Consequently the structure constants $N_{\alpha\beta}$ are always $\pm 1$ for a simply laced algebra.

## 2.14 Finding the cocycles $\varepsilon(\alpha, \beta)$

As we have seen the Dynkin diagram of an algebra contains all the necessary information to construct the commutation relations (2.216)-( 2.218). However that information is not enough to determine the cocycles $\varepsilon(\alpha, \beta)$ defined in ( 2.218). For that we need the Jacobi identity. We now explain how to use such identities to determine the cocycles. We will show that the consistency conditions imposed on the cocycles are such that they can be split into a number of sets equal to the number of positive non simple roots. The sign of a cocycle in a given set completly determines the signs of all other cocycles of that set, but has no influence in the determination of the cocycles in the other sets. Therefore the cocycles $\varepsilon(\alpha, \beta)$ are determined by the Jacobi identities up to such "gauge freedom" in fixing independently the signs of the cocycles of different sets.

From the antisymmetry of the Lie bracket the cocycles have to satisfy

$$\varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha) \tag{2.221}$$

In addition, from the choice made in (2.206) one has

$$\varepsilon(\alpha, \beta) = -\varepsilon(-\alpha, -\beta) \tag{2.222}$$

$$\sigma(H_a) = -H_a \; ; \quad \sigma(E_\alpha) = \eta_\alpha E_{-\alpha} \qquad\qquad N_{\alpha\beta} \;=\; \frac{\eta_\alpha \eta_\beta}{\eta_{\alpha+\beta}} N_{-\alpha,-\beta} \qquad\qquad N_{\alpha\beta} = -N_{-\alpha,-\beta}$$

Consider three roots $\alpha$, $\beta$ and $\gamma$ such that their sum vanishes. The Jacobi identity for their corresponding step operators yields, using (2.216) - (2.218)

$$
\begin{aligned}
0 &= [[E_\alpha, E_\beta], E_\gamma] + [[E_\gamma, E_\alpha], E_\beta] + [[E_\beta, E_\gamma], E_\alpha] \\
&= -((q_{\alpha\beta}+1)\varepsilon(\alpha,\beta)\frac{2\gamma.H}{\gamma^2} + (q_{\gamma\alpha}+1)\varepsilon(\gamma,\alpha)\frac{2\beta.H}{\beta^2} \\
&\quad +(q_{\beta\gamma}+1)\varepsilon(\beta,\gamma)\frac{2\alpha.H}{\alpha^2}) \\
&= -(((q_{\beta\gamma}+1)\varepsilon(\beta,\gamma) - \frac{\alpha^2}{\gamma^2}(q_{\alpha\beta}+1)\varepsilon(\alpha,\beta))\frac{2\alpha.H}{\alpha^2} \\
&\quad +((q_{\gamma\alpha}+1)\varepsilon(\gamma,\alpha) - \frac{\beta^2}{\gamma^2}(q_{\alpha\beta}+1)\varepsilon(\alpha,\beta))\frac{2\beta.H}{\beta^2})
\end{aligned}
\tag{2.223}
$$

Since the integers $q's$ are non negative we get

$$\varepsilon(\alpha,\beta) = \varepsilon(\beta,\gamma) = \varepsilon(\gamma,\alpha) \tag{2.224}$$

and also

$$\frac{1}{\gamma^2}(q_{\alpha\beta}+1) = \frac{1}{\alpha^2}(q_{\beta\gamma}+1) = \frac{1}{\beta^2}(q_{\gamma\alpha}+1) \tag{2.225}$$

Further relations are found by considering Jacobi identities for three step operators corresponding to roots adding up to a fourth root. Now such identities yield relations involving products of two cocycles. However, in many situations there are only two non vanishing terms in the Jacobi identity. Consider three roots $\alpha$, $\beta$ and $\gamma$ such that $\alpha+\beta$, $\beta+\gamma$ and $\alpha+\beta+\gamma$ are roots but $\alpha+\gamma$ is not a root. Then the Jacobi identity for the corresponding step operators yields

$$\begin{aligned} 0 &= [[E_\alpha, E_\beta], E_\gamma] + [[E_\gamma, E_\alpha], E_\beta] + [[E_\beta, E_\gamma], E_\alpha] \\ &= (q_{\alpha\beta}+1)(q_{\alpha+\beta,\gamma}+1)\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) \\ &\quad +(q_{\beta\gamma}+1)(q_{\beta+\gamma,\alpha}+1)\varepsilon(\beta,\gamma)\varepsilon(\beta+\gamma,\alpha) \end{aligned} \tag{2.226}$$

Therefore one gets

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma) \tag{2.227}$$

and

$$(q_{\alpha\beta}+1)(q_{\alpha+\beta,\gamma}+1) = (q_{\beta\gamma}+1)(q_{\beta+\gamma,\alpha}+1) \tag{2.228}$$

There remains to consider the cases where the three terms in the Jacobi identity for three step operators do not vanish. Such thing happens when we have three roots $\alpha$, $\beta$ and $\gamma$ such that $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$ and $\alpha+\beta+\gamma$ are roots as well. We now classify all cases where that happens. We shall denote long roots by $\mu$, $\nu$, $\rho$, ... and short roots by $e$, $f$, $g$, ... . From the properties of roots discussed in section 2.8 one gets that $\frac{2\mu.\nu}{\mu^2}$, $\frac{2\mu.e}{\mu^2}$, $\frac{2e.f}{e^2} = 0, \pm 1$. Let us consider the possible cases:

1. *All three roots are long.* If $\mu+\nu$ is a root then $\frac{(\mu+\nu)^2}{\mu^2} = 2 + \frac{2\mu.\nu}{\mu^2}$. Since $\mu+\nu$ can not be a longer than $\mu$ one gets $\frac{2\mu.\nu}{\mu^2} = -1$. So $\mu+\nu$ is a long root and if $\mu+\nu+\rho$ is also a root one gets by the same argument that $\frac{2(\mu+\nu).\rho}{\mu^2} = -1$. Therefore $\mu+\rho$ and $\nu+\rho$ can not be roots simultaneously since that would imply, by the same arguments, $\frac{2\mu.\rho}{\mu^2} = \frac{2\nu.\rho}{\mu^2} = -1$ which is a contradiction with the result above.

2. *Two roots are long and one short.* If $\mu+e$ is a root then $\frac{(\mu+e)^2}{\mu^2} = 1+\frac{e^2}{\mu^2}+\frac{2\mu.e}{\mu^2}$. Since $\mu+e$ can not be longer than $\mu$ it follows that $\frac{2\mu.e}{\mu^2} = -1$. Therefore $\mu+e$ is a short root since $(\mu+e)^2 = e^2$. So, if $\mu+e+\nu$ is a root then $\frac{(\mu+e+\nu)^2}{\nu^2} = 1+\frac{(\mu+e)^2}{\mu^2}+\frac{2(\mu+e).\nu}{\nu^2}$ and therefore $\frac{2(\mu+e).\nu}{\nu^2} = -1$. Consequently $\mu+\nu$ and $\nu+e$ can not be roots simultaneously since that would imply, by the same arguments, $\frac{2\mu.\nu}{\nu^2} = \frac{2\nu.e}{\nu^2} = -1$.

3. *Two roots are short and one long.* Analogously if $e+f$ and $\mu+e+f$ are roots one gets $\frac{2(e+f).\mu}{\mu^2} = -1$ independently of $e+f$ being shost or long. So, it is impossible for $\mu+e$ and $\mu+f$ to be both roots since one would get $\frac{2\mu.e}{\mu^2} = \frac{2\mu.f}{\mu^2} = -1$.

4. *All three roots are short.* If $e+f$ is a root then $\frac{(e+f)^2}{e^2} = 2+\frac{2e.f}{e^2}$ and there exists three possibilities:
   (a) $\frac{2e.f}{e^2} = -1$ and $e+f$ is a short root.
   (b) $\frac{2e.f}{e^2} = 1$ and $\frac{(e+f)^2}{e^2} = 3$ (can only happen in $G_2$).
   (c) $\frac{2e.f}{e^2} = 0$ and $\frac{(e+f)^2}{e^2} = 2$ (can only happen in $B_n$, $C_n$ and $F_4$).

We then conclude that the only possibility for the ocurrence of three short roots $e$, $f$ and $g$ such that the sum of any two of them and $e+f+g$ are all roots is that two of them are ortoghonal, let us say $e.f = 0$ and $\frac{2e.g}{g^2} = \frac{2f.g}{g^2} = -1$. This can only happen in the algebras $C_n$ or $F_4$. Therefore none of the three terms in the Jacobi identity for the corresponding step operators will vanish. We have

$$\begin{aligned} 0 &= [[E_e, E_f], E_g] + [[E_g, E_e], E_f] + [[E_f, E_g], E_e] \\ &= (q_{ef} + 1)(q_{e+f,g} + 1)\varepsilon(e, f)\varepsilon(e + f, g) \\ &\quad +(q_{ge} + 1)(q_{g+e,f} + 1)\varepsilon(g, e)\varepsilon(g + e, f) \\ &\quad +(q_{fg} + 1)(q_{f+g,e} + 1)\varepsilon(f, g)\varepsilon(f + g, e) \end{aligned} \tag{2.229}$$

According to the discussion in section 2.12 any root string in an algebra where the ratio of the squared lengths of roots is 1 or 2 can have at most 3 roots. From (2.187) we see that $q_{ef} = 1$ and $q_{ge} = q_{fg} = q_{e+f,g} = q_{g+e,f} = q_{f+g,e} = 0$. Therefore

$$\varepsilon(e, f)\varepsilon(e + f, g) = \varepsilon(g, e)\varepsilon(f, g + e) = \varepsilon(f, g)\varepsilon(e, f + g) \tag{2.230}$$

We can then determine the cocycles using the following algorithm:

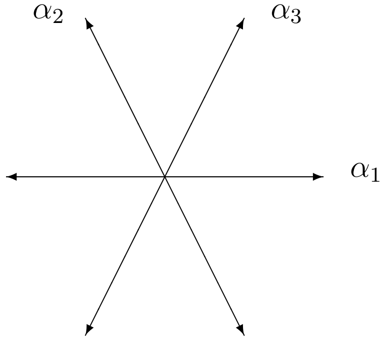
1. The cocycles involving two negative roots, $\varepsilon(-\alpha, -\beta)$ with $\alpha$ and $\beta$ both positive, is determined from those involving two positive roots through the relation (2.222).

2. The cocycles involving one positive and one negative root, $\varepsilon(-\alpha, \beta)$ with both $\alpha$ and $\beta$ both positive, are also determined from those involving two positive roots through the relations (2.224) and (2.222). Indeed, if $-\alpha + \beta$ is a positive root we write $-\alpha + \beta = \gamma$ and if it is negative we write $-\alpha+\beta = -\gamma$ with $\gamma$ positive in both cases. Therefore from (2.224) and (2.222) it follows $\varepsilon(-\alpha, \beta) = \varepsilon(-\gamma, -\alpha) = -\varepsilon(\gamma, \alpha)$ in the first case, and $\varepsilon(-\alpha, \beta) = \varepsilon(\beta, \gamma)$ in the second case.

3. Let $\rho$ be a positive non simple root which can be written as $\rho = \alpha + \beta = \gamma + \delta$ with $\alpha$, $\beta$, $\gamma$ and $\delta$ all positive roots. Then the cocycles $\varepsilon(\alpha, \beta)$ and $\varepsilon(\gamma, \delta)$ can be related to each other by using combinations of the relations (2.227)

Using such algorithm one can then verify that there will be one cocycle to be chosen freely, for each positive non-simple root of the algebra. Once those cocycles are chosen, all the other are uniquely determined.

# Example $SU(3)$




$$\frac{2\alpha_1 \cdot \alpha_2}{\alpha_2^2} = \frac{2\alpha_1 \cdot \alpha_2}{\alpha_1^2} = -1$$

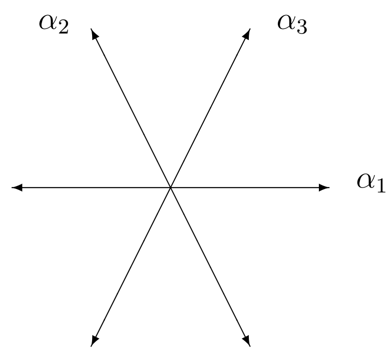
$$\frac{2\alpha_1 \cdot \alpha_3}{\alpha_3^2} = \frac{2\alpha_2 \cdot \alpha_3}{\alpha_3^2} = 1$$

$$[H_1\, ,\, H_2] = 0 \qquad\qquad [H_a\, ,\, E_{\alpha_b}] = \frac{2\alpha_a \cdot \alpha_b}{\alpha_a^2}\, E_{\alpha_b}$$

$$[H_1\, ,\, E_{\alpha_1}] = 2\, E_{\alpha_1} \qquad\qquad [H_2\, ,\, E_{\alpha_1}] = -\, E_{\alpha_1}$$

$$[H_1\, ,\, E_{\alpha_2}] = -\, E_{\alpha_2} \qquad\qquad [H_2\, ,\, E_{\alpha_2}] = 2\, E_{\alpha_2}$$

$$[H_1\, ,\, E_{\alpha_3}] = E_{\alpha_3} \qquad\qquad [H_2\, ,\, E_{\alpha_3}] = E_{\alpha_3}$$




$$[E_{\alpha_1}\,,\,E_{\alpha_2}] = \varepsilon(1\,,\,2)\,E_{\alpha_3}$$

$$[E_{\alpha_1}\,,\,E_{-\alpha_3}] = \varepsilon(1\,,\,-3)\,E_{-\alpha_2}$$

$$[E_{\alpha_2}\,,\,E_{-\alpha_3}] = \varepsilon(2\,,\,-3)\,E_{-\alpha_1}$$

$$\varepsilon(a\,,\,b) = -\varepsilon(-a\,,\,-b) \longrightarrow \varepsilon(-1\,,\,-2) = -\varepsilon(1\,,\,2)$$

$$\varepsilon(-1\,,\,3) = -\varepsilon(1\,,\,-3)$$

$$\varepsilon(-2\,,\,3) = -\varepsilon(2\,,\,-3)$$

$$\alpha_1 + \alpha_2 + (-\alpha_3) = 0$$

$$\varepsilon(1\,,\,2) = \varepsilon(2\,,\,-3) = \varepsilon(-3\,,\,1)$$