Desenvolvimento do Potencial em Multipolos I

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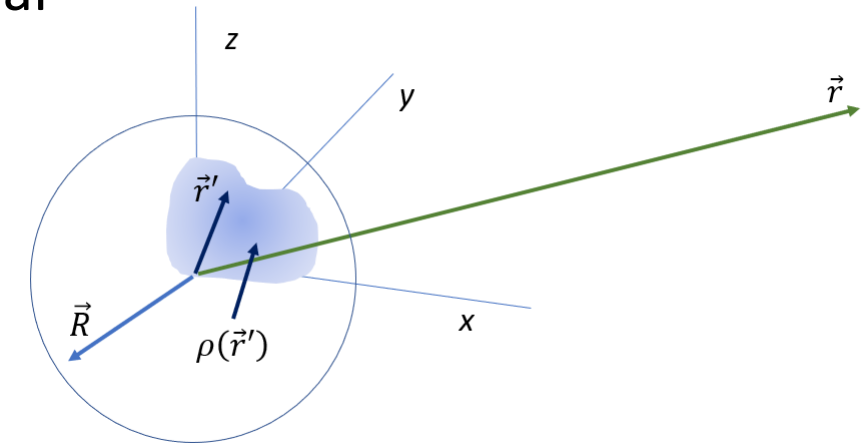
Expressão geral para o potencial em coordenadas esféricas

$$\phi(r,\theta,\varphi) = \sum_{\ell,m}\left[A_{\ell m}r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right]Y_{\ell m}(\theta,\varphi)$$

Nota: como harmônicos esféricos são complexos, coeficientes $A_{\ell m}$; $B_{\ell m}$ também complexos, em geral, para que potencial seja grandeza real

Configuração básica e método:

- Distribuição de carga concentrada dentro de uma esfera de raio R em torno de uma origem
- Expressão do campo em posições $|\vec{r}| \gg R$



Para distâncias longe da fonte localizada → $A_{\ell m} = 0$; $B_{\ell m} = \frac{1}{\epsilon_0(2\ell+1)} q_{\ell m}$;

$q_{\ell m}$ constantes a serem determinadas; então

$$\phi(r,\theta,\varphi) = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{q_{\ell m}}{r^{\ell+1}}Y_{\ell m}(\theta,\varphi)$$

Os coeficientes $q_{\ell m}$ são denominados *momentos multipolares*. Para cada valor de ℓ, existem $(2\ell+1)$ momentos multipolares.

Determinação dos momentos mutipolares: expressão para o potencial em termos da densidade de carga,

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' ; \frac{1}{|\vec{r}-\vec{r}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r'^{\ell}}{r^{\ell+1}}Y^*_{\ell m}(\theta',\varphi')Y_{\ell m}(\theta,\varphi)$$

(estamos considerando o caso $r > r'$). Substituindo na integral:

$$\phi(\vec{r}) = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{1}{r^{\ell+1}}\left[\int Y^*_{\ell m}(\theta',\varphi')r'^{\ell}\rho(\vec{r}')dV'\right]Y_{\ell m}(\theta,\varphi)$$

Comparando as duas expressões, temos

$$\boxed{q_{\ell m} = \int Y^*_{\ell m}(\theta', \varphi') r'^{\ell} \rho(\vec{r}') dV';\ \ q_{\ell;-m} = (-1)^m q^*_{\ell m}}$$

Essa segunda relação decorre da propriedade (Jackson Eq. 3.54)

$$Y_{\ell;-m}(\theta, \varphi) = (-1)^m Y^*_{\ell m}(\theta, \varphi)$$

<u>Exemplos</u>

$$\ell = 0 \rightarrow\ Y_{00} = \frac{1}{\sqrt{4\pi}} \rightarrow\ q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{r}') dV' = \frac{q}{\sqrt{4\pi}} \rightarrow \boxed{\phi_{\ell=0}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}}$$

$$\ell = 1 \rightarrow Y^*_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta\ ;\ Y^*_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{r}') r' \cos\theta'\, dV' = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{r}') z' dV' = \sqrt{\frac{3}{4\pi}} p_z;\ p_z = \int \rho(\vec{r}') z' dV'$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int \sin\theta' \,(\cos\varphi' - i\sin\varphi') r' \rho(r') dV';\; r'\sin\theta'\cos\varphi' = x'; r'\sin\theta'\sin\varphi' = y'$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\;\rho(r') dV' = -\sqrt{\frac{3}{8\pi}} (p_x - ip_y);\quad \boxed{\vec{p} = \int \vec{r}'\rho(r') dV'}$$

$$q_{1;-1} = \sqrt{\frac{3}{8\pi}} (p_x + ip_y)$$

Potencial e campo elétrico correspondente a $\ell = 1$

$$\phi_{\ell=1} = \frac{1}{\epsilon_0}\frac{1}{3r^2}\left(q_{1;-1}Y_{1;-1} + q_{10}Y_{10} + q_{11}Y_{11}\right)$$

$$\therefore \phi_{\ell=1} = \frac{1}{\epsilon_0}\frac{1}{3r^2}\left[\frac{3}{8\pi}(p_x + ip_y)\sin\theta\, e^{-i\varphi} + \frac{3}{4\pi}p_z\cos\theta + \frac{3}{8\pi}(p_x - ip_y)\sin\theta\, e^{i\varphi}\right]$$

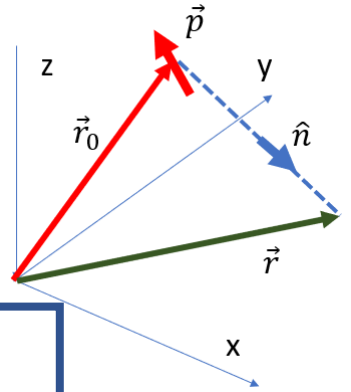
$$\therefore \phi_{\ell=1} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\left[\frac{1}{2}p_x \sin\theta\,(2\cos\varphi) + \frac{1}{2}ip_y \sin\theta\,(-2i\sin\varphi) + p_z\cos\theta\right]$$

$$\therefore \phi_{\ell=1} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\left[p_x\frac{x}{r} + p_y\frac{y}{r} + p_z\frac{z}{r}\right] \rightarrow \boxed{\phi_{\ell=1} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}}$$

Campo elétrico

$$\vec{E}_{\ell=1} = -\nabla\phi_{\ell=1} = -\frac{1}{4\pi\epsilon_0}\nabla\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\nabla(\vec{p}\cdot\vec{r}) + (\vec{p}\cdot\vec{r})\nabla\frac{1}{r^3}\right]$$

$$\nabla(\vec{p}\cdot\vec{r}) = \vec{p};\;\; \nabla\left(\frac{1}{r^3}\right) = \frac{3}{r^2}\nabla\left(\frac{1}{r}\right) = -3\frac{\vec{r}}{r^5}$$



Portanto

$$\vec{E}_{\ell=1} = \frac{1}{4\pi\epsilon_0}\left[-\frac{\vec{p}}{r^3} + \frac{3\vec{r}}{r^5}\right] \rightarrow \boxed{\vec{E}_{\ell=1} = \frac{1}{4\pi\epsilon_0}\frac{3(\vec{p}\cdot\hat{n})\hat{n} - \vec{p}}{|\vec{r} - \vec{r}_0|^3};\;\; \hat{n} = \frac{\vec{r} - \vec{r}}{|\vec{r} - \vec{r}_0|}}$$

Generalização para incluir singularidade dipolo (Jackson; Eq. 4.20) → Problema Quarta Série

Momentos de Quadrupolo

$$\ell = 2 \rightarrow Y_{20}^{*} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}(\cos\theta)^2 - \frac{1}{2}\right);\ Y_{21}^{*} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i2\varphi};$$

$$Y_{22}^{*} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(\sin\theta)^2 e^{-i2\varphi}$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int(\sin\theta')^2(\cos 2\varphi' - i\sin 2\varphi')r'^2\rho(\vec{r}')dV'$$

$$r'^2(\sin\theta')^2(\cos 2\varphi' - i\sin 2\varphi') =$$
$$= r'^2[(\sin\theta')^2(\cos\varphi')^2 - (\sin\theta')^2(\sin\varphi')^2 - 2i(\sin\theta')^2\sin\varphi'\cos\varphi'] = (x' - iy')^2$$

$$\therefore q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int(x' - iy')^2\rho(\vec{r}')dV'$$

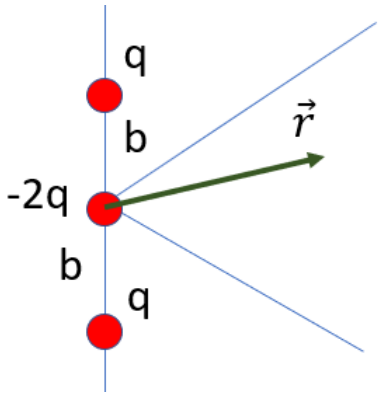
Os outros momentos quadripolares para $\ell = 2$ são calculados no livro do Jackson e os alunos podem reproduzir os cálculos para q_{20}; q_{21}. Esses momentos podem ser expressos de forma mais conveniente definido o *Tensor Momento de Quadrupolo*, de traço nulo, como

$$Q_{ij} = \int \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho(\vec{r}')\, dV'$$

em termos do qual

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{zz};\ q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(Q_{xz} - iQ_{yz}\right);\ \ q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{xx} - 2iQ_{xy} - Q_{yy}\right)$$

<u>Exemplo</u>: quadrupolo (Frenkel, seção 2.2)



$$\rho(\vec{r}) = q\delta(x)\delta(y)[\delta(z-b) - 2\delta(z) + \delta(z+b)]$$

$\rightarrow$ fazer $x' = 0;\ y' = 0$ em todos os termos

$$Q_{xx} = -q\int z'^2\,[\delta(z'-b) - 2\delta(z') + \delta(z'+b)]dz' = -2qb^2$$

$$Q_{yy} = -2qb^2;\ Q_{zz} = 4qb^2$$

Substituindo nas expressões para os q_{2m}, temos $q_{20} = 2\sqrt{\frac{5}{4\pi}}qb^2;\ q_{21} = q_{20} = 0.$

Então

$$\phi_{\ell=2} = \frac{1}{\epsilon_0}\frac{1}{5r^3}\frac{1}{2}\sqrt{\frac{5}{4\pi}}(4qb^2)\sqrt{\frac{5}{4\pi}}\frac{3(\cos\theta)^2-1}{2} = \frac{1}{4\pi\epsilon_0}\frac{2qb^2}{r^3}\frac{3(\cos\theta)^2-1}{2}$$

<u>Componentes do Campo Elétrico para (ℓ, m) fixos</u>

$$\phi_{\ell m} = \frac{1}{\epsilon_0}\frac{q_{\ell m}}{2\ell+1}\frac{1}{r^{\ell+1}}Y_{\ell m}(\theta,\varphi) \rightarrow \begin{cases} E_r = \dfrac{(\ell+1)q_{\ell m}}{(2\ell+1)\epsilon_0}\dfrac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+2}} \\ E_\theta = -\dfrac{q_{\ell m}}{(2\ell+1)\epsilon_0}\dfrac{1}{r^{\ell+2}}\dfrac{\partial}{\partial\theta}Y_{\ell m}(\theta,\varphi) \\ E_\varphi = -\dfrac{q_{\ell m}}{(2\ell+1)}\dfrac{im}{\sin\theta}\dfrac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+2}} \end{cases}$$