Aula 7

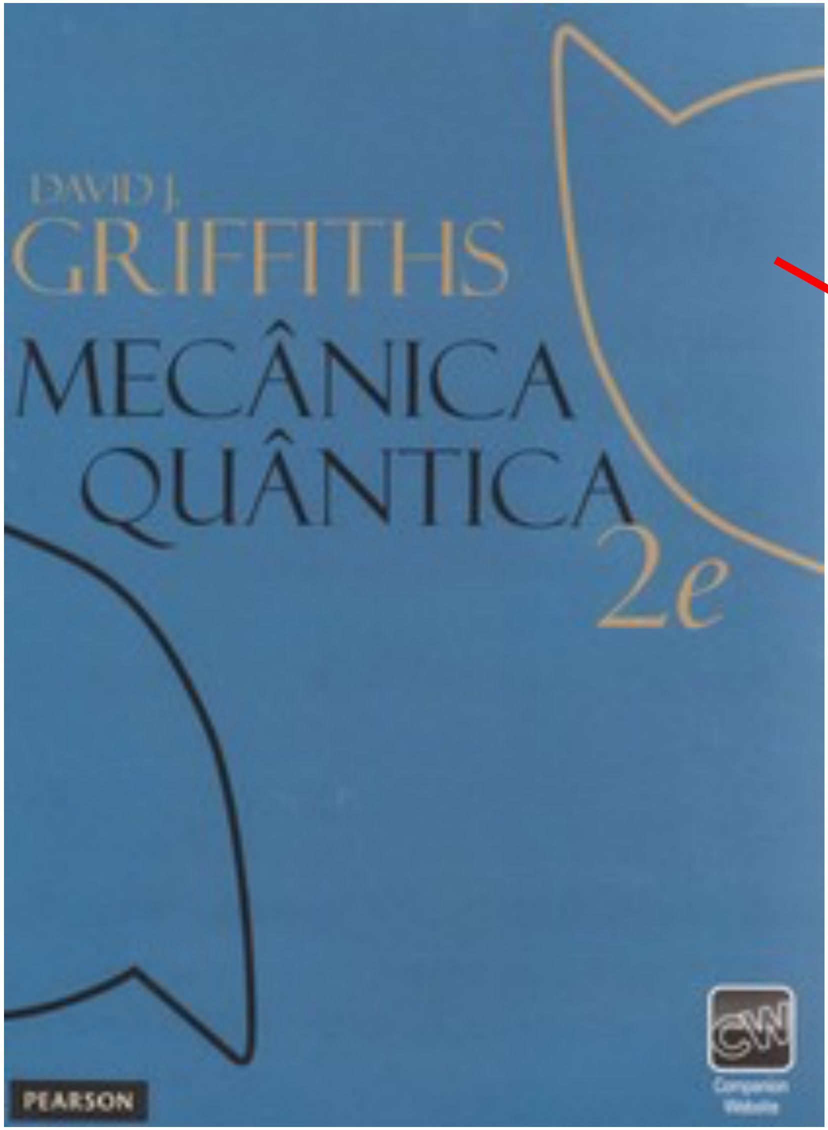
Operadores escada

Autoestados de S_y

Probabilidades e valor médio

Exemplos

Para estudar o formalismo de spin vamos usar :



gato

Operadores de spin

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

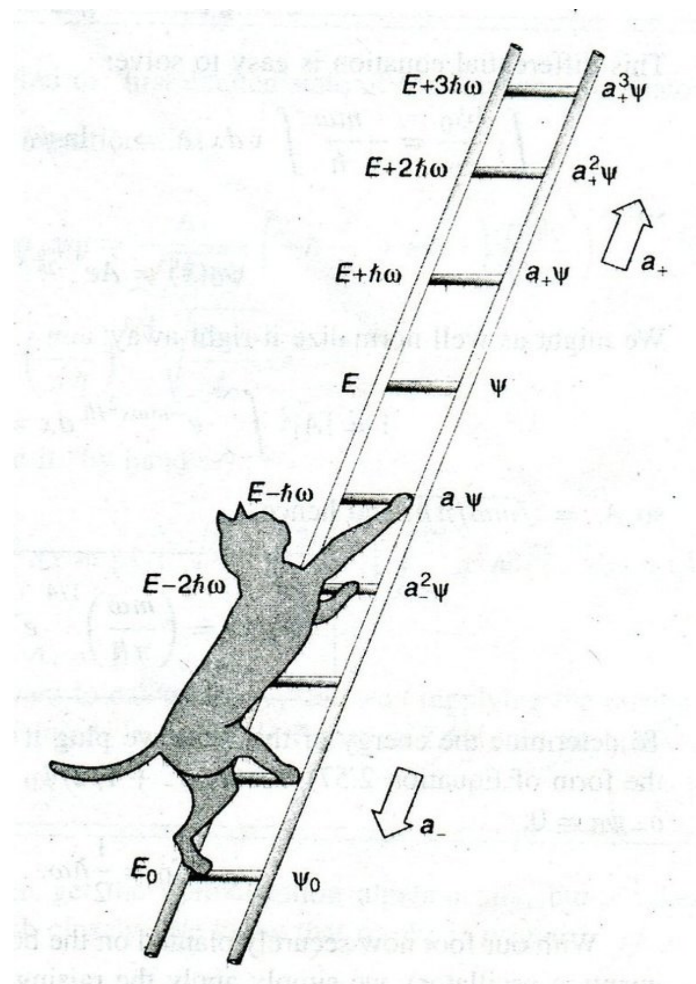
Operadores escada

$$\hat{S}_+ \;=\; \hat{S}_x + i\,\hat{S}_y$$

levantamento

$$\hat{S}_- \;=\; \hat{S}_x - i\,\hat{S}_y$$

abaixamento



$$\hat{S}_+ = \hat{S}_x + i\,\hat{S}_y$$

$$\hat{S}_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\,\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Levantamento

$$\hat{S}_+ = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

$$\hat{S}_- = \hat{S}_x - i\,\hat{S}_y$$

$$\hat{S}_- = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\,\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Abaixamento

$$\hat{S}_- = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{\hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

$$\hat{S}_+ \,|\downarrow\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boxed{\hat{S}_+ \,|\downarrow\rangle = \hbar \,|\uparrow\rangle}$$

$$\hat{S}_+ \,|\uparrow\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boxed{\hat{S}_+ \,|\uparrow\rangle = 0}$$

$$\hat{S}_- \,|\uparrow\rangle = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \boxed{\hat{S}_- \,|\uparrow\rangle = \hbar \,|\downarrow\rangle}$$

$$\hat{S}_- \,|\downarrow\rangle = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boxed{\hat{S}_- \,|\downarrow\rangle = 0}$$

Sistema de equações lineares

$$\begin{cases} A_1\,\alpha + A_2\,\beta = a\,\alpha \\ A_3\,\alpha + A_4\,\beta = a\,\beta \end{cases} \qquad \alpha = ? \qquad \beta = ?$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad A\,|\chi\rangle = a\,|\chi\rangle$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} A_1 - a & A_2 \\ A_3 & A_4 - a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{vmatrix} A_1 - a & A_2 \\ A_3 & A_4 - a \end{vmatrix} = 0$$

Para que exista solução não -trivial !

Autoestados de S_y

$$\hat{S}_y \,|\, y \,\rangle \;=\; a \,|\, y \,\rangle$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad |\, y \,\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{vmatrix} 0 - a & -i\,\frac{\hbar}{2} \\ i\,\frac{\hbar}{2} & 0 - a \end{vmatrix} = 0$$

$$a^2 = \frac{\hbar^2}{4} \longrightarrow \boxed{a = \pm\frac{\hbar}{2}}$$

- $a = +\dfrac{\hbar}{2}$

$$\hat{S}_y \,|\, y_+ \rangle = +\frac{\hbar}{2} \,|\, y_+ \rangle \qquad |\, y_+ \rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} -i\,\beta \\ i\,\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \left\{ \begin{array}{r} -\,i\,\beta = \alpha \\ i\,\alpha = \beta \end{array} \right.$$

$$\left\{ \begin{array}{r} -\,i\,\beta = \alpha \\ i\,\beta^* = \alpha^* \end{array} \right. \longrightarrow \boxed{|\beta|^2 = |\alpha|^2}$$

Normalização

$$\langle\, y_+ \,|\, y_+ \,\rangle \;=\; (\,\alpha^* \quad \beta^*\,)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \;=\; 1$$

$$\alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2 = 1 \qquad\qquad |\alpha|^2 = |\beta|^2$$

$$2\,|\alpha|^2 = 1 \quad\longrightarrow\quad \alpha = \frac{1}{\sqrt{2}}\,,\ \frac{-1}{\sqrt{2}}\,,\ \frac{i}{\sqrt{2}}\,,\ \frac{-i}{\sqrt{2}}$$

$$\alpha \;=\; \frac{1}{\sqrt{2}} \quad\longrightarrow\quad i\,\alpha = \beta \quad\longrightarrow\quad \beta \;=\; \frac{i}{\sqrt{2}}$$

$$\boxed{\;|\, y_+ \,\rangle \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\;}$$

$$\alpha = \frac{1}{\sqrt{2}}\,,\ \frac{-1}{\sqrt{2}}\,,\ \frac{i}{\sqrt{2}}\,,\ \frac{-i}{\sqrt{2}} \qquad\qquad i\,\alpha = \beta$$

$$\begin{tikzcd}
|y_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \arrow[r, "\times(-1)"] \arrow[d, "\times(i)"] & |y_+\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \arrow[d, "\times(i)"] \\
|y_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \arrow[r, "\times(-1)"] & |y_+\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix}
\end{tikzcd}$$

Todos estes estados são equivalentes !

O sinal e o "i" não fazem diferença !

Ou ainda:

$$\langle\, y_+ \,|\, y_+ \,\rangle \;=\; (\, \alpha^* \quad \beta^* \,) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \;=\; 1$$

$$\alpha^* \alpha + \beta^* \beta = |\alpha|^2 + |\beta|^2 = 1 \qquad\qquad |\alpha|^2 = |\beta|^2$$

$$2\,|\beta|^2 = 1 \qquad \longrightarrow \qquad \beta = \frac{1}{\sqrt{2}}\,,\ \frac{-1}{\sqrt{2}}\,,\ \frac{i}{\sqrt{2}}\,,\ \frac{-i}{\sqrt{2}}$$

$$\beta = \frac{1}{\sqrt{2}} \qquad \longrightarrow \qquad \alpha = -i\,\beta \qquad \longrightarrow \qquad \alpha = \frac{-i}{\sqrt{2}}$$

$$|y_+\rangle = \frac{-1}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$\beta = \frac{1}{\sqrt{2}}\,,\ \frac{-1}{\sqrt{2}}\,,\ \frac{i}{\sqrt{2}}\,,\ \frac{-i}{\sqrt{2}} \qquad\qquad \alpha = -i\,\beta$$

$$|y_+\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \xrightarrow{\times(-1)} |y_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$\downarrow \times(-i) \qquad\qquad\qquad \downarrow \times(-i)$$

$$|y_+\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \xrightarrow{\times(-1)} \boxed{|y_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

Todos estes estados são equivalentes !
O sinal e o "i" não fazem diferença !

- $$a = -\frac{\hbar}{2}$$

$$\hat{S}_y \,|\, y_- \rangle = -\frac{\hbar}{2} \,|\, y_- \rangle \qquad |\, y_- \rangle = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

$$\begin{pmatrix} -i\,\beta' \\ i\,\alpha' \end{pmatrix} = \begin{pmatrix} -\,\alpha' \\ -\,\beta' \end{pmatrix} \qquad \begin{cases} i\,\beta' = \alpha' \\ i\,\alpha' = -\,\beta' \end{cases}$$

$$\boxed{{\beta'}^2 = {\alpha'}^2}$$

Normalização

$$\langle \chi' | \chi' \rangle = \begin{pmatrix} \alpha'^* & \beta'^* \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = 1$$

$$\alpha'^* \alpha' + \beta'^* \beta' = \alpha'^2 + \beta'^2 = 1 \qquad \beta'^2 = \alpha'^2$$

$$2\,\alpha'^2 = 1 \qquad \alpha' = \pm \frac{1}{\sqrt{2}}$$

$$\alpha' = \frac{1}{\sqrt{2}} \longrightarrow i\,\alpha' = -\beta' \longrightarrow \beta' = -\frac{i}{\sqrt{2}}$$

$$\boxed{|\, y_- \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

Probabilidades

Já temos três conjuntos completos de autoestados (autovetores) :

$$|\,x_+\,\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\,x_-\,\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|\,y_+\,\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\,y_-\,\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|z_+\,\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |z_-\,\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

São três bases ! Cada uma com dois estados: $|\,\chi_+\,\rangle \quad |\,\chi_-\,\rangle$

(ortonormais)

Para um estado genérico $|\,\chi\,\rangle$

$$|\,\chi\,\rangle = a_+\,|\,\chi_+\,\rangle + a_-\,|\,\chi_-\,\rangle = \sum_i a_i\,|\,\chi_i\rangle$$

$|a_i|^2$ = probabilidade de que o estado esteja em $|\,\chi_i\rangle$

Para encontrar a_+ basta multiplicar o estado por $\langle\,\chi_+|$

$$\langle\chi_+|\,\chi\,\rangle = a_+\,\underbrace{\langle\chi_+|\,\chi_+\,\rangle}_{1} + a_-\,\underbrace{\langle\chi_+|\,\chi_-\,\rangle}_{0}$$

$$\langle\chi_+|\,\chi\,\rangle = a_+$$

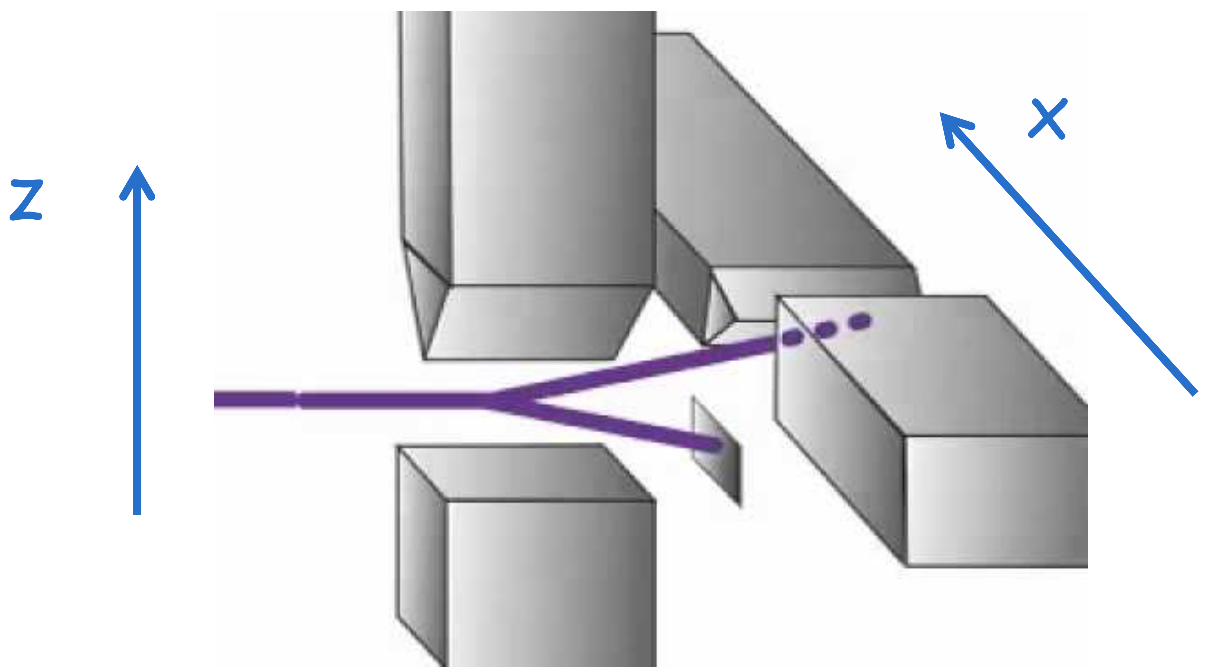
$$\boxed{P_{a_+} = |a_+|^2 = |\langle\chi_+|\,\chi\,\rangle|^2}$$

Valor médio de um operador

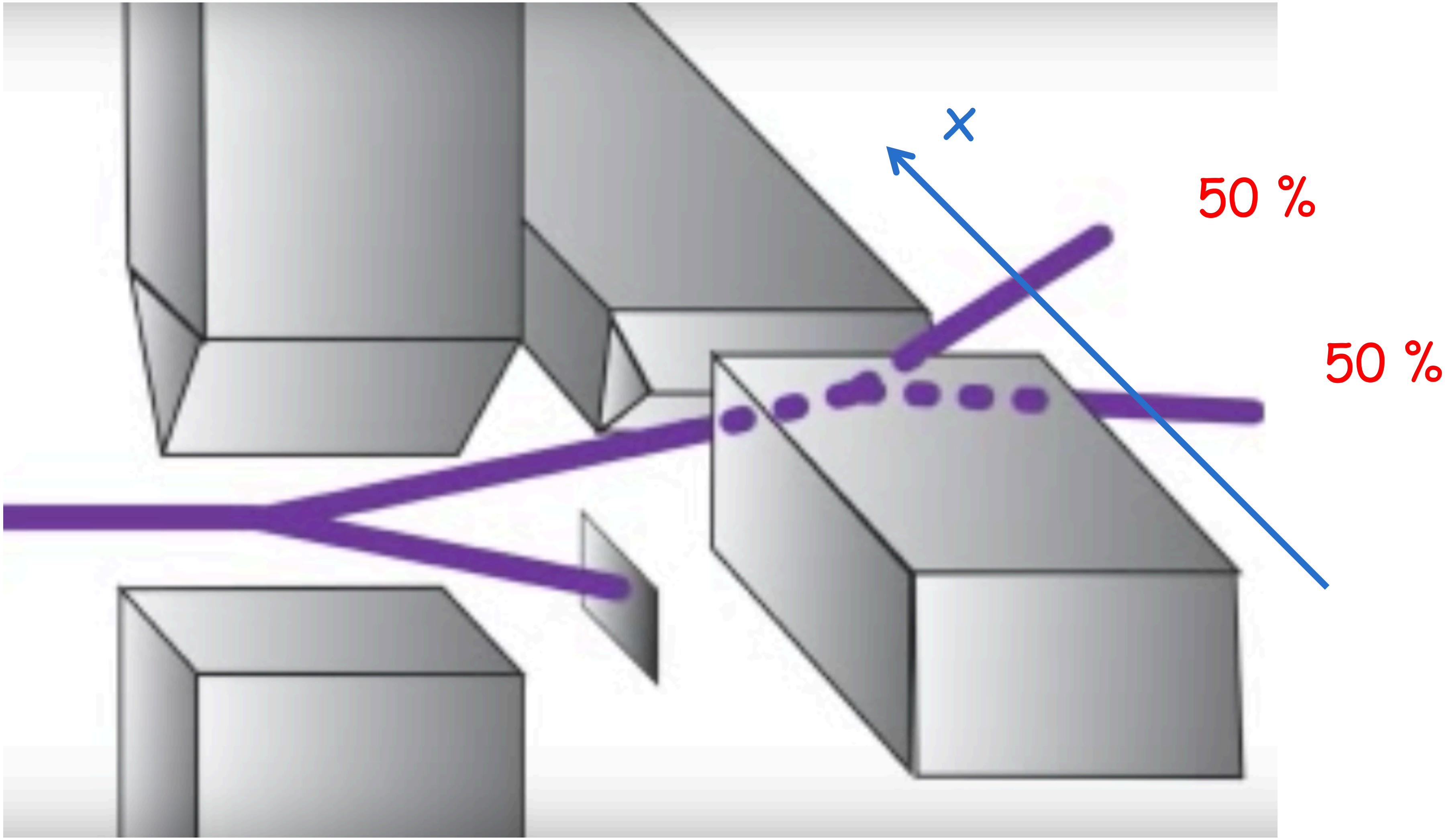
$$\langle A \rangle = \langle \chi | \hat{A} | \chi \rangle$$

Média dos valores medidos da grandeza A no estado $|\chi\rangle$
(o número de medidas deve ser infinito)

Exemplo: medidas da componente do spin na direção x quando o sistema está no estado $|\uparrow\rangle$



Na experiência vemos que :



$$\langle \hat{S}_x \rangle = \langle \uparrow | \hat{S}_x | \uparrow \rangle$$ ("sanduiche")

$$\langle \uparrow | = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad | \uparrow \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

Em metade das medidas encontramos $+\frac{\hbar}{2}$

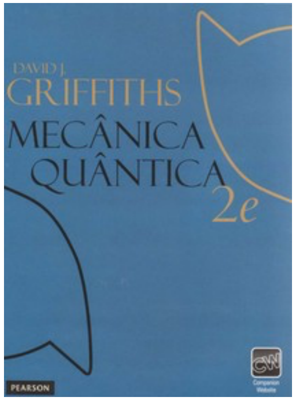
Em metade das medidas encontramos $-\frac{\hbar}{2}$

Exercício !!!

Essa vai cair no EUF !!!

Luis Raul Abramo
(presidente da comissão
de pós-graduação)

Um elétron se encontra no estado de spin dado por



$$|\chi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

Mostre que o estado está normalizado:

$$\langle\chi|\chi\rangle = \frac{1}{6}\,(1-i \quad 2) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{1}{6}(2+4) = 1$$

Probabilidade de medir S_x e encontrar $+\frac{\hbar}{2}$? (encontrar $|x_+\rangle$)

$$P_{x_+} = |\langle x_+|\chi\rangle|^2$$

$$P_{x_+} = |\langle x_+ | \chi \rangle|^2$$

$$|x_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \langle x_+ | = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\langle x_+ | \chi \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{3+i}{\sqrt{12}}$$

$$\boxed{P_{x_+} = \left| \frac{3+i}{\sqrt{12}} \right|^2 = \frac{5}{6}}$$

Probabilidade de medir S_z e encontrar $+\frac{\hbar}{2}$? (encontrar $|\uparrow\rangle$)

$$P_{z_+} = |\langle z_+|\chi\rangle|^2 = |\langle\uparrow|\chi\rangle|^2$$

$$\langle\uparrow|\chi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1+i \\ 2\end{pmatrix} = \frac{1+i}{\sqrt{6}}$$

$$\boxed{P_{z_+} = \left|\frac{1+i}{\sqrt{6}}\right|^2 = \frac{2}{6} = \frac{1}{3}}$$

Probabilidade de medir S_z e encontrar $-\frac{\hbar}{2}$?   (encontrar $|\downarrow\rangle$)

$$P_{z_-} = |\langle z_- | \chi \rangle|^2 = |\langle \downarrow | \chi \rangle|^2$$

$$\langle \downarrow | \chi \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{2}{\sqrt{6}}$$

$$\boxed{P_{z_-} = \left| \frac{2}{\sqrt{6}} \right|^2 = \frac{4}{6} = \frac{2}{3}}$$

Qual é o valor médio de S_z neste estado ?

$$\langle \hat{S}_z \rangle = \langle \chi | \hat{S}_z | \chi \rangle$$

$$\frac{1}{6} \begin{pmatrix} 1-i & 2 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{\hbar}{12}(2-4) = -\frac{\hbar}{6}$$

Cada medida individual so'
pode dar $\hbar/2$ ou $-\hbar/2$

Schroedinger