PRO 5859

Statistical Process Monitoring

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Outline

Multivariate process monitoring

Multivariate process monitoring - Introduction

- Simultaneous Monitoring or control of two or more related quality characteristics
- The use of separate control chart for each characteristic may be misleading
- α^*=type I error for the joint control procedure:
  - p statistically independent quality characteristics and α is the type I error for each $\overline{X}$, then $\alpha^* = 1 - (1 - \alpha)^p$)
  - But if p s are not independent, the above equation does not hold.

About multivariate normal distribution

- Consider p variables, given by $\mathbf{X}' = (X_1 \ X_2 \ \dots \ X_p)$
- With its respective means $\boldsymbol{\mu}' = (\mu_1 \ \mu_2 \ \dots \ \mu_p)$
- And their variances and covariances described by a matrix $\boldsymbol{\Sigma}_{\mathbf{p}\times\mathbf{p}}$
- The multivariate normal probability density function is

$$f(\mathbf{X}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} \exp^{-1/2(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}$$

About multivariate normal distribution - random sample

- A random sample of size n: $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$
- Sample mean vector

$$\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i = \left[\ \overline{X}_1\ \overline{X}_2\ \ldots\ \overline{X}_p\right]'$$

χ^2 control chart

- When $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are known
- Monitored statistic

$$\chi_0^2 = n(\overline{\mathbf{X}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu})$$

- Upper control limit: UCL$=\chi^2_{\alpha,p}$

T^2 Control Chart

- In practice, it is usually necessary to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$
- Assuming the process is in-control, take m samples of size n
- Obtain

$$\bar{x}_{jk} = \frac{1}{n}\sum_{i=1}^{n} x_{ijk}, \; S_{jk}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ijk} - \bar{x}_{jk})^2$$

$$j = 1, \ldots, p; k = 1, \ldots, m$$

- and the covariance between quality characteristics j and h in the k-th sample

$$S_{jhk} = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ijk} - \bar{x}_{jk})(x_{ihk} - \bar{x}_{hk}), \; k = 1, \ldots, m; \; j \neq h$$

T^2 Control Chart

- Estimates of mean, variance and covariance are respectively given as:

$$\bar{\bar{x}}_j = \frac{1}{m}\sum_{k=1}^{m}\bar{x}_{jk};\ \bar{S}_j^2 = \frac{1}{m}\sum_{k=1}^{m}S_{jk}^2;\ j = 1,\ldots,p$$

$$\bar{S}_{jh} = \frac{1}{m}\sum_{k=1}^{m}S_{jhk},\ j \neq h$$

- $\bar{\bar{x}}_j$ is the j-th element of the vector $\bar{\bar{\mathbf{x}}}$, an unbiased estimator of the vector $\boldsymbol{\mu}$
- $\bar{S}_j^2$ is the j-th element of diagonal of the matrix $\mathbf{S}$ and $\bar{S}_{jh}$ is the jh-th element of the same matrix. Matrix $\mathbf{S}$ is an unbiased estimator of $\boldsymbol{\Sigma}$

T^2 Control Chart

- This procedure is called Hotelling T^2 control chart
- The monitored statistics is

$$T^2 = n(\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}})'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}})$$

T^2 Control Chart

- Careful selection of the control limit must be taken for T^2 statistic
- It depends on the phases of control chart usage
- Phase 1 - use of charts for establishing control; that is, testing whether the process was in control when the m preliminary subgroups were drawn and the estimates computed - called retrospective analysis

$$UCL = \frac{p(m-1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

T^2 Control Chart

- Phase 2 - the chart is used for monitoring future production

$$UCL = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

T^2 Control chart: interpreting out-of-control signals

- One difficult in any multivariate control chart - practical interpretation of the signals
- Which of p variable is responsible for the signal?
- Standard practices:
- Alt (1985): plot univariate $\overline{X}$ charts on the individual variables with Bonferroni-type control limits (use $z_{\alpha/(2p)}$ in place of $z_{\alpha/2}$)

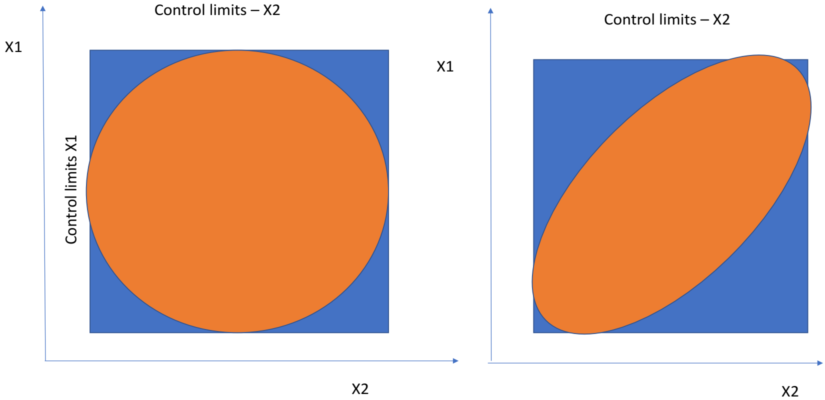


Figure 1: Two no-correlated and correlated variables- ellipsoid contours

T^2 Control chart: interpreting out-of-control signals

- One difficult in any multivariate control chart - practical interpretation of the signals
- Which of p variable is responsible for the signal?
- Standard practices:
  - Runger et al. (1996): Decomposition of T^2 into components that reflects the contribution of each individual variable:

    $$d_i = T^2 - T^2_{(i)};\ T^2_{(i)}$$

    is the statistic for all variables except the ith one, i=1,. . . , p

T^2 control chart: Questions for seminars

- Describe the procedures: Case 1 - Haridy et al. (2014)- a procedure to build for exact simultaneous confidence intervals
- Case 2 - Jackson (1980): use of control charts based on p principal components
- Case 3: Murphy (1987); Case 4: Chua & Montgomery (1992)- , Case 5-Tracy et al. (1996) Mason et al. (1995, 1996)
- How is the performance of all these methods?
- Find other related contributions in the literature

T^2 control chart for individual observation

- Some industrial settings the subgroup size n=1 like chemical process
- m samples, each of size $n = 1$ are available
- Let $\overline{\mathbf{X}}$ and the matrix $\mathbf{S}$ the sample mean vector and covariance matrix of these observations

$$T^2 = (\mathbf{x} - \overline{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \overline{\mathbf{x}})$$

- Phase 2 control limit:

$$UCL = \frac{p(m+1)(m-1)}{m^2 - mp} F_{\alpha,p,m-p} \text{ or } \chi^2_{\alpha,p} \text{ if m} > 100$$

- Phase 1 control limit

$$UCL = \frac{(m-1)^2}{m} \beta_{\alpha,p/2,(m-p-1)/2}$$

$\beta_{\alpha,p/2,(m-p-1)/2}$ is the upper α percentage of a Beta distribution with parameters p/2,(m-p-1)/2

Multivariate control chart (MCC) to monitor space-time count series

- Vectors of the deviance residuals (after fitting a STARMA model) used to build MCUSUM and MEWMA control charts to monitor multivariate space-time count series.
- Chart parameters estimated by simulation to meet a desired in-control average run length and to minimize out-of-control average run length

MCC to monitor space-time count series

- A complementary simulation study is performed to measure the impact of the omission of the spatial dependencies on the performance of the control charts.
- Results highlight that false alarms will be signaled much earlier on average.

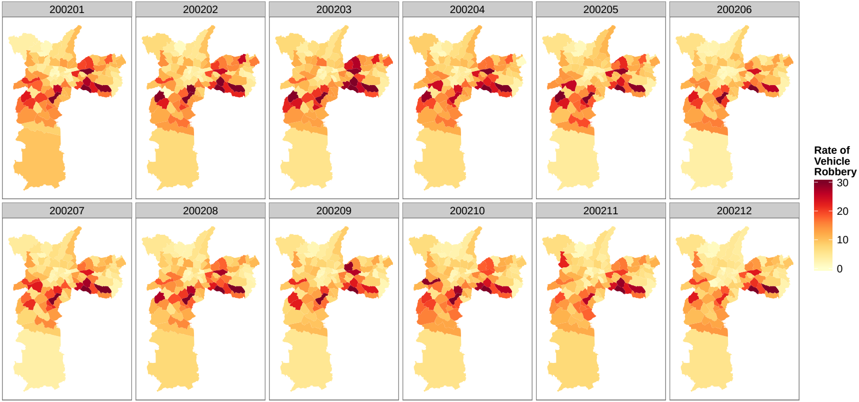
MCC to monitor space-time count series

- For illustrative purposes, consider the data set of the monthly rates of vehicle robberies registered in 93 police districts located in São Paulo City (Brazil).
- Data from January 2001 to December 2013 are used to fit the STARMA model. Chart parameters are searched by simulation
- Observed vehicle robberies from January 2014 to April 2016 and used to draw the control charts.

MCC to monitor space-time count series

- The control charts signal as out-of-control for almost all months of 2014 and the beginning of 2015.
- An exploratory analysis is used to identify which districts are responsible for these signals.
- In the case of omission of spatial dependencies, in the current application, these control charts will give false alarms on average 2.5 (MEWMA) and 7 months (MCUSUM) earlier due to this fault.

MCC to monitor space-time count series



MCC to monitor space-time count series

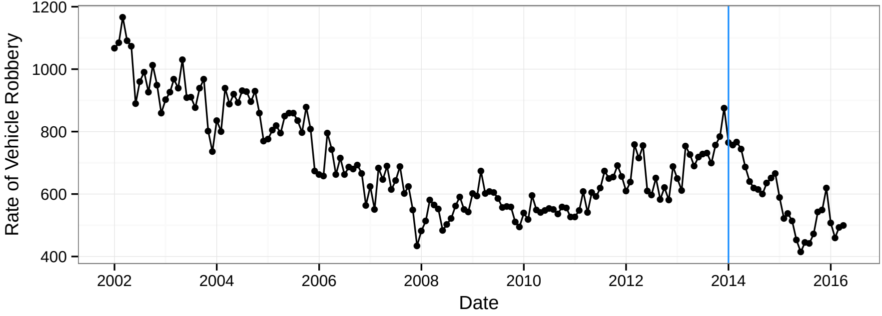


Figure 3: Vehicle robbery rate per one thousand vehicles.

MCC to monitor space-time count series

- STARMA(p, λ_p, q, δ_q) model

$$\mathbf{Z}_t^* = \sum_{k=1}^{p}\sum_{j=0}^{\lambda_k} \phi_{k,j}\mathbf{W}_j\mathbf{Z}_{t-k}^* - \sum_{k=1}^{q}\sum_{j=0}^{\delta_k} \theta_{k,j}\mathbf{W}_j\varepsilon_{t-k} + \varepsilon_t, \quad (1)$$

where

- p and q are the lags of autoregressive and moving average components, respectively;
- λ_k is the degree of spatial dependency within the k-th autoregressive lag component;
- δ_k is the degree of spatial dependency within the k-th moving average lag component;
- $\phi_{k,j}$ are the parameters of the autoregressive components; and
- $\theta_{k,j}$ are the parameters of the moving average component.

MCC to monitor space-time count series

- $\varepsilon_t = \{\varepsilon_{1,t}, ..., \varepsilon_{N,t}\}$ in (1) are normally distributed with $\boldsymbol{\mu} = \mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_N$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix.
- The standardized matrix $\mathbf{W}_j$, with dimension $N \times N$, is used to describe the spatial neighborhood relationship of order j among N locations.
- For an order j, the elements $w_{im_j} > 0$ indicate the neighborhood relationship strength between i-th and m-th locations with $w_{ii_j} = 0$ and $\sum_{m=1}^{N} w_{im_j} = 1$, where $i = 1, ..., N$.
- For $j = 0$, the matrix $\mathbf{W}_0 = \mathbf{I}$. So $\phi_{k,0}$ and $\theta_{k,0}$ are, respectively, the "pure" temporal components of the autoregressive and the moving average in the STARMA model.

Frame Title

- Order 1: Matrix $\mathbf{W}_1$—for the police districts whose minimum distance δ is $\leq$ 0.5 km;
- Order 2: Matrix $\mathbf{W}_2$—for the police districts whose minimum distance $\delta \in]0.5km;\ 3km]$;
- Order 3: Matrix $\mathbf{W}_3$—for the police districts whose minimum distance $\delta \in]3km;\ 6km]$

MCC to monitor space-time count series

- $X_{i,t}$ be the monthly number of vehicle robberies at the i-th location at time t.
- It is assumed that $X_{i,t}$ follows a Negative Binomial distribution (as overdispersion is also observed,
- $Z^*_{i,t} = \frac{X_{i,t}}{fl_t} \times 10^6$ is the respective rate of vehicle robberies, where fl_t is the fleet of registered vehicles at time t and $i = 1, \ldots, N$.
- For N (here $N = 93$) locations, $\mathbf{Z}^*_t = (Z^*_{1,t}, Z^*_{2,t}, \cdots, Z^*_{N,t})$ at instant t.

Frame Title

- In the checking stage of the STARMA model, normality assumption was not considered reasonable,
- some possible transformations in the response variable that would allow to consider the normality assumption as satisfied are evaluated.
- After evaluating several possibilities, we observed that deviance residuals yield the symmetrical variable presented the best results in the sense of not rejecting the hypotheses of the model.

MCC to monitor space-time count series

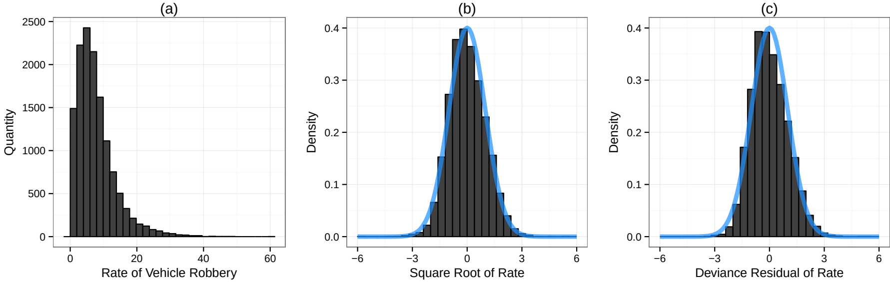


Figure 4: Histogram of the rate of vehicle robbery and its transformations.

MCC to monitor space-time count series

- $$Z_{i,t}^{DR} = sign(Z_{i,t}^{*} - \mu_i)\sqrt{g_{i,t}^2}, \quad (2)$$

  where $g_{i,t}^2$ is calculated as follows:

  $$g_{i,t}^2 = \begin{cases} 2\gamma_i \ln\left(1 + \mu_i/\gamma_i\right) & \text{if } Z_{i,t}^{*} = 0 \\ 2Z_{i,t}^{*} \ln\left(Z_{i,t}^{*}/\mu_i\right) - 2\gamma_i(1 + Z_{i,t}^{*}/\gamma_i) \ln\left(\frac{1+Z_{i,t}^{*}/\gamma_i}{1+\mu_i/\gamma_i}\right) & \text{if } Z_{i,t}^{*} > 0 \end{cases} \quad (3)$$

  μ_i in (2) and (3) is replaced by the sample mean $\overline{Z}_i^{*}$ for any t,
- and parameter γ_i is estimated by satisfying $Var(Z_{i,t}^{*}) = \gamma_i \pi_i/(1 - \pi_i)^2$, with $\pi_i = \mu_i/(\mu_i + \gamma_i)$.
- These equalities are due to the assumption that the original random count variable $X_{i,t}$ follows a Negative Binomial Distribution

MCC to monitor space-time count series

$$\begin{aligned}\hat{\mathbf{Z}}_t = &+ 0.80409\ \mathbf{Z}_{t-1} + 0.04687\ \mathbf{Z}_{t-12} \\ &+ 0.22849\mathbf{W}_1\mathbf{Z}_{t-1} - 0.10124\mathbf{W}_1\mathbf{Z}_{t-2} \\ &- 0.51406\ \hat{\varepsilon}_{t-1}\end{aligned} \tag{4}$$

MCC to monitor space-time count series

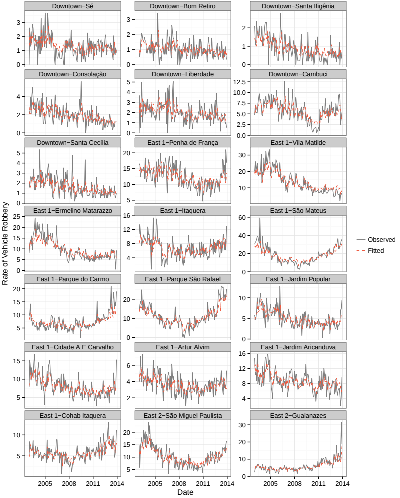


Figure 5: Observed versus fitted theft rates

MCC to monitor space-time count series

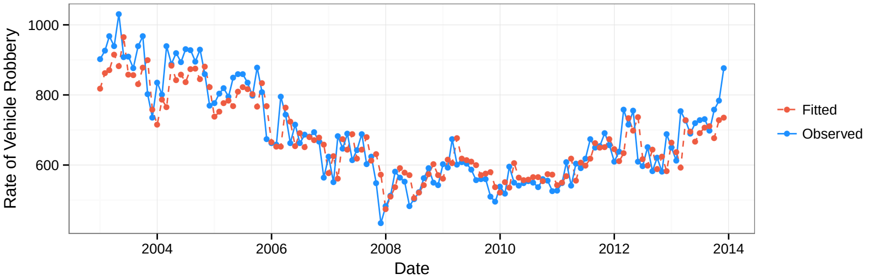


Figure 6: Aggregate rate - observed versus fitted

MCC to monitor space-time count series

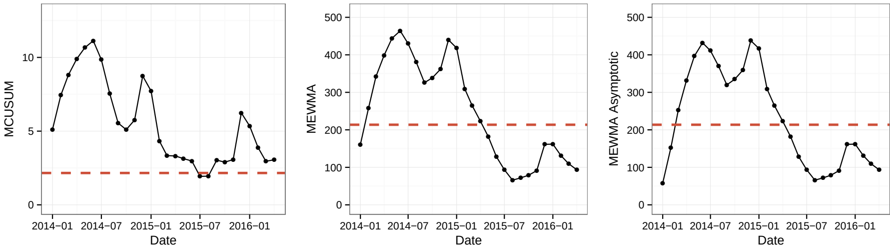
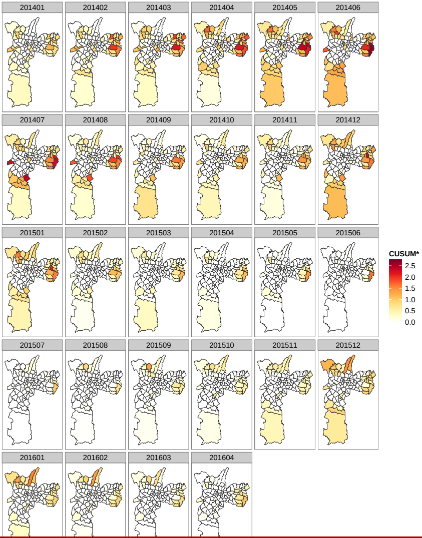


Figure 7: MCUSUM and MEWMA control chart.

MCC to monitor space-time count series



Monitoring bivariate means by attribute charts

- Some contributions are found in the literature like:
  - np_{xy} and np_w proposed by Ho & Costa (2015) and
  - *Max D* by Melo et al. (2017*b*)
- Like other attribute charts for monitoring a variable quality characteristic, the items are classified using some device.
- In bivariate processes, the classifications are made on the dimensions X and Y.
- What differs among the proposals is the statistics used to monitoring.

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Some assumptions: the values of the dimensions X and Y are standardized
- Only upper discriminating limit (UDL) is used and equal for the (standardized) dimensions X and Y.
- Items are classified as first, second or third class according to the UDL
  - First class: if (X $<$UDL) and (Y$<$ UDL)
  - Thirs class: if (X $>$ UDL) and (Y$>$ UDL)
  - Otherwise results: the item is classified as

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Let $p_1 =$ P[(X $<$UDL) and (Y$<$ UDL)] - the probability of an item be of the first class
- p_3 =P[(X $>$ UDL) and (Y$>$ UDL)] - the probability of an item be of the third class
- And $p_2 = 1 - p_1 - p_3 =$ probability of the item be of the second class
- After classification: n_1, n_2, and n_3 items classified as the first, second and third class
- n_1, n_2, n_3 follows a trinomial distribution with parameters: n, p_1, p_2, p_3

np_{xy} and np_w charts proposed by Ho & Costa (2015)

- Control chart np_{xy}: the monitored statistic is $M = n_2 + n_3$
  - The process is declared out of control whenever $M > UCL_{xy}$
  - M follows a binomial distribution with parameters: $n; (1 - p_0)$
- Control chart np_w: the monitored statistic is $W = n_2 + 2n_3$
  - The process is declared out of control whenever $W > UCL_w$

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table I. *ARL* values for $n = 6$ and $\rho = 0.8$

k_x	k_y	np_{xy} chart			np_w chart		
0.0	0.00	370.31	370.35	370.40	370.40	370.40	370.40
0.00	0.25	149.06	**148.41***	170.88	**164.82**	167.94	166.03
0.00	0.50	**53.51***	53.85	81.89	**79.78**	84.99	84.54
0.00	0.75	**19.59***	20.05	**40.82**	41.68	46.53	48.27
0.00	1.00	**8.06***	8.40	**20.91**	23.15	27.25	30.30
0.25	0.25	83.14	**81.87**	85.22	77.13	76.32	**75.60***
0.25	0.50	38.85	**38.52***	43.86	39.18	39.46	**39.03**
0.25	0.75	**16.84***	17.02	23.47	**21.52**	22.50	22.53
0.25	1.00	**7.58***	7.83	13.11	**12.70**	13.82	14.37
0.50	0.50	23.86	**23.61**	24.44	21.05	20.78	**20.67***
0.50	0.75	**12.89**	12.92	14.08	**12.19***	12.23	12.21
0.50	1.00	**6.68***	6.85	8.46	**7.59**	7.85	7.94
0.75	0.75	**8.71**	8.76	8.78	7.49	**7.42***	7.46
0.75	1.00	**5.40**	5.51	5.67	**4.93***	4.95	5.00
1.00	1.00	**4.00**	4.10	3.95	3.44	**3.42***	3.48
	UCL	3	4	4	6	7	8
	UDL	1.380	0.989	1.602	1.143	0.978	0.736

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table II. *ARL* values for $n=6$ and $\rho=0.5$

k_x	k_y	np_{xy} chart			np_w chart		
0.0	0.00	370.21	370.05	370.40	370.40	370.40	370.40
0.00	0.25	150.16	**148.34***	156.94	**151.31**	151.60	153.47
0.00	0.50	**55.57***	56.89	68.42	**66.94**	68.76	72.03
0.00	0.75	**20.92***	22.54	**31.14**	32.24	34.53	38.07
0.00	1.00	**8.72***	9.81	**15.03**	16.96	19.17	22.50
0.25	0.25	81.44	**76.34**	74.48	67.51	**66.49***	66.77
0.25	0.50	38.38	**36.21**	36.25	32.53	**32.27***	32.85
0.25	0.75	17.11	**16.89***	18.37	**17.01**	17.29	18.14
0.25	1.00	**7.90***	8.24	9.83	**9.68**	10.20	11.14
0.50	0.50	23.05	**21.06**	19.61	17.02	**16.73***	16.97
0.50	0.75	12.57	**11.69**	11.00	9.64	**9.56***	9.82
0.50	1.00	6.68	**6.52**	6.48	**5.91***	5.99	6.29
0.75	0.75	8.35	**7.64**	6.79	5.88	**5.82***	5.95
0.75	1.00	5.23	**4.90**	4.38	3.88	**3.87***	4.00
1.00	1.00	3.83	**3.57**	3.08	2.74	**2.72***	2.81
	UCL	3	4	4	6	7	8
	UDL	1.469	1.094	1.494	1.040	0.836	0.633

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table III. *ARL* values for $n=6$ and $\rho=0.3$

k_x	k_y	np_{xy} chart			np_w chart		
0.0	0.00	370.21	370.29	370.40	370.40	370.40	370.40
0.00	0.25	147.69	**146.16**	148.00	**143.13***	143.58	145.78
0.00	0.50	**56.08***	56.15	60.96	**60.26**	62.10	65.44
0.00	0.75	**21.97***	22.50	**26.55**	27.95	30.08	33.40
0.00	1.00	**9.45***	9.92	**12.50**	14.37	16.32	19.26
0.25	0.25	75.03	**72.82**	67.96	61.66	**60.81***	61.26
0.25	0.50	35.07	**34.20**	31.92	28.76	**28.60***	29.26
0.25	0.75	16.13	**16.07**	15.71	**14.68***	14.97	15.80
0.25	1.00	**7.79***	7.97	8.27	**8.24**	8.71	9.57
0.50	0.50	20.03	**19.43**	16.96	14.79	**14.58***	14.85
0.50	0.75	10.94	**10.75**	9.37	8.27	**8.24***	8.50
0.50	1.00	**6.04**	6.08	5.49	**5.06***	5.14	5.43
0.75	0.75	7.04	**6.94**	5.77	5.04	**5.00***	5.15
0.75	1.00	**4.46**	4.47	3.73	**3.34***	3.34	3.47
1.00	1.00	**3.22**	3.24	2.65	2.38	**2.38***	2.46
	UCL	3	4	4	6	7	8
	UDL	1.502	1.138	1.429	0.978	0.773	.569

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table IV. *ARL* values for $n = 6$ and $\rho = 0$

k_x	k_y	np_{xy} chart			np_w chart		
0.0	0.00	370.16	370.38	370.40	370.40	370.40	370.40
0.00	0.25	143.99	**141.31**	136.29	**131.18***	131.52	133.67
0.00	0.50	54.11	**53.46**	52.01	**51.42***	53.07	56.08
0.00	0.75	**21.20***	21.39	**21.42**	22.68	24.50	27.28
0.00	1.00	**9.17***	9.50	**9.80**	11.35	12.93	15.26
0.25	0.25	70.59	**67.31**	59.63	53.18	**52.28***	52.71
0.25	0.50	32.39	**30.88**	26.64	23.62	**23.43***	24.02
0.25	0.75	14.86	**14.45**	12.63	**11.67***	11.90	12.60
0.25	1.00	**7.24**	7.24	6.53	**6.47***	6.84	7.53
0.50	0.50	20.03	**19.43**	13.79	11.80	**11.60***	11.86
0.50	0.75	10.94	**10.75**	7.47	6.50	**6.47***	6.70
0.50	1.00	**6.04**	6.08	4.35	**3.98***	4.04	4.28
0.75	0.75	6.15	**5.89**	4.58	3.97	**3.94***	4.07
0.75	1.00	3.90	**3.81**	2.99	**2.66***	2.67	2.78
1.00	1.00	2.80	**2.76**	2.15	**1.94***	1.95	2.02
	UCL	3	4	4	6	7	8
	UDL	1.531	1.182	1.342	0.885	0.678	0.472

np_{xy} and np_w charts proposed by Ho & Costa (2015)

Table XI. np_{xy}, np_w, and T^2 control charts

		ρ									
		0.5					0				
		T^2	np_{xy}	np_{xy}	np_w	np_w	T^2	np_{xy}	np_{xy}	np_w	np_w
		n					n				
k_x	k_y	3	3	6	3	6	3	3	6	3	6
0	0	370.37	370.11	370.21	370.40	370.40	370.37	370.04	370.16	370.40	370.40
0	0.25	**161.87**	184.93	**150.16**	194.65	151.31	230.61	179.14	143.99	**172.30**	**131.18**
0	0.5	**42.26**	86.82	**55.57**	107.22	66.94	90.90	**83.46**	54.11	84.30	**51.42**
0	0.75	**12.82**	40.58	**20.92**	61.56	32.24	**35.18**	39.30	**21.20**	44.00	22.68
0	1	**4.98**	19.78	**8.72**	36.89	16.96	**14.99**	19.43	**9.17**	24.80	11.35
0.25	0.25	214.84	111.38	81.44	**105.45**	**67.51**	159.77	102.35	70.59	**86.60**	**53.18**
0.25	0.5	84.34	61.37	38.38	**60.16**	**32.53**	72.26	55.01	32.39	**45.50**	**23.62**
0.25	0.75	**25.26**	32.42	17.11	36.01	**17.01**	30.40	29.04	14.86	**25.30**	**11.67**
0.25	1	**8.71**	17.21	**7.90**	22.54	9.68	**13.61**	15.65	7.24	15.00	**6.47**
0.5	0.5	77.10	39.57	23.05	**35.43**	**17.02**	41.18	33.88	20.03	**25.60**	**11.80**
0.5	0.75	35.08	23.93	12.57	**21.96**	**9.64**	20.69	20.14	10.94	**15.10**	**6.50**
0.5	1	**13.29**	14.10	6.68	14.29	**5.91**	10.46	11.96	6.04	**9.50**	**3.98**
0.75	0.75	28.02	16.49	8.35	**14.06**	**5.88**	12.41	13.39	6.15	**9.50**	**3.97**
0.75	1	14.88	10.87	5.23	**9.46**	**3.88**	7.24	8.77	3.90	**6.30**	**2.66**
1	1	11.55	7.99	3.83	**6.56**	**2.73**	4.82	6.29	2.80	**4.30**	**1.94**
UCL		11.829	2	3	3	6	11.829	2	3	3	6
UDL			1.394	0.690	1.394	1.040		1.459	1.531	1.166	0.885

Max D proposed by Melo et al. (2017*b*)

- Each item is classified as approved or disapproved in respect to each quality characteristic by a gauge
- An item is classified as disapproved in i-th quality characteristic if its value is out of discriminating limits: $w_L; w_U$
- Let D_i = number of disapproved items in i-th quality characteristics in a sample of n units
- The monitor statistic is Max D$=\max(D_1, D_2, \dots, D_p)$
- A signal is triggered whenever Max $D > L$, L, the control limit set to satisfy a performance metric

Max D: comparing to T^2

Table 1 Values of ARL_1 of T^2 and $Max\,D$ control charts: $n = 1$

		ρ											
		0.0			0.3			0.5			0.8		
δ_x	δ_y	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2
0.00	0.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00
	0.25	0.06	*230.76*	310.80	0.07	*231.14*	305.80	0.08	*231.47*	294.47	0.17	*230.52*	237.56
	0.50	0.25	*131.91*	202.04	0.27	*132.46*	192.34	0.33	*132.40*	172.07	0.69	129.21	97.76
	1.00	1.00	*41.47*	67.27	1.10	*41.53*	60.43	1.33	*41.32*	47.85	2.78	38.98	16.91
	2.00	4.00	*6.25*	9.40	4.40	*6.25*	8.05	5.33	6.20	5.84	11.11	5.87	1.93
0.25	0.25	0.13	*167.75*	265.73	0.10	*168.60*	285.10	0.08	*169.84*	294.47	0.07	*173.06*	305.19
	0.50	0.31	*108.66*	178.87	0.26	*109.42*	197.63	0.25	*110.48*	202.04	0.31	*112.16*	178.87
	1.00	1.06	*38.90*	62.82	1.00	*39.16*	67.06	1.08	*39.30*	61.43	1.84	38.12	31.18
	2.00	4.06	*6.20*	9.16	4.13	*6.21*	8.90	4.75	*6.18*	7.08	9.06	5.87	2.54
0.50	0.50	0.50	*80.63*	129.68	0.38	*81.45*	156.96	0.33	*82.73*	172.07	0.28	*86.04*	191.19
	1.00	1.25	*34.65*	51.83	1.04	*35.14*	64.09	1.00	*35.64*	67.27	1.25	*35.94*	51.83
	2.00	4.25	*6.10*	8.51	4.01	*6.14*	9.36	4.33	*6.13*	8.24	7.36	5.87	3.47
1.00	1.00	2.00	*22.20*	27.71	1.54	*22.85*	39.80	1.33	*23.59*	47.85	1.11	*25.23*	59.66
	2.00	5.00	*5.62*	6.50	4.18	*5.76*	8.76	4.00	*5.86*	9.40	5.00	*5.80*	6.50
2.00	2.00	8.00	3.42	3.06	6.15	*3.66*	4.62	5.33	*3.87*	5.84	4.44	*4.27*	7.90
UCL			0	11.827		0	11.827		0	11.827		0	11.827
w_U			2.999			2.997			2.990			2.999	

Max D: comparing to T^2

Table 2 Values of ARL_1 of T^2 and $Max\,D$ control charts: $n = 3$

		ρ											
		0.0			0.3			0.5			0.8		
δ_x	δ_y	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2
0.00	0.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00
	0.25	0.19	*170.92*	230.39	0.21	*171.16*	221.35	0.25	*171.17*	202.04	0.52	170.30	125.55
	0.50	0.75	*70.11*	90.82	0.82	*70.23*	92.68	1.00	70.09	67.27	2.08	68.44	26.12
	1.00	3.00	*14.09*	14.98	3.30	14.09	12.89	4.00	14.03	9.40	8.33	13.53	2.87
	2.00	12.00	*2.09*	2.76	13.19	2.09	1.58	16.00	2.09	1.32	33.33	2.05	1.01
0.25	0.25	0.38	*111.22*	159.63	0.29	*111.74*	187.26	0.25	*112.70*	202.04	0.21	*115.57*	220.27
	0.50	0.94	*57.43*	72.21	0.78	*57.99*	87.06	0.75	*58.49*	90.92	0.94	*59.76*	72.21
	1.00	3.19	*13.53*	13.60	3.01	*13.60*	14.92	3.25	13.63	13.19	5.52	13.38	5.52
	2.00	12.19	2.09	1.73	12.40	2.09	1.69	14.25	2.09	1.46	27.19	2.05	1.03
0.50	0.50	1.50	*38.92*	41.15	1.15	*39.49*	57.07	1.00	*40.15*	67.24	0.83	*42.17*	81.75
	1.00	3.75	12.23	10.45	3.13	*12.42*	13.99	3.00	*12.59*	14.98	3.75	12.80	10.45
	2.00	12.75	2.07	1.64	12.03	2.08	1.73	13.00	2.08	1.61	22.08	2.05	1.09
1.00	1.00	6.00	7.43	4.82	4.62	7.71	7.42	4.00	*7.99*	9.40	3.33	*8.70*	12.66
	2.00	15.00	1.95	1.40	12.53	1.99	1.67	12.00	2.02	1.76	15.00	2.04	1.40
2.00	2.00	24.00	1.38	1.06	18.46	1.44	1.19	16.00	1.50	1.32	13.33	1.63	1.57
UCL			2	11.827		2	11.827		2	11.827		2	11.827
w_U			1.223			1.222			1.220			1.203	

Max D: comparing to T^2

Table 3 Values of ARL_1 of T^2 and $Max\,D$ control charts: $n = 6$

		ρ											
		0.0			0.3			0.5			0.8		
δ_x	δ_y	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2	d^2	$Max\,D$	T^2
0.00	0.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00	0.00	370.00	370.00
	0.25	0.38	*126.95*	159.63	0.42	*127.80*	149.71	0.50	*127.17*	149.71	1.04	128.61	64.25
	0.50	1.50	*38.81*	41.15	1.64	39.06	36.29	2.00	38.80	36.30	4.16	37.98	8.79
	1.00	6.00	5.85*	4.82	6.60	5.87*	4.13	8.00	5.85*	4.13	26.66	5.71*	1.28
	2.00	24.00	1.17*	1.06	26.38	1.17*	1.04	32.00	1.17*	1.04	66.66	1.16*	1.00
0.25	0.25	0.76	*76.71*	90.82	0.58	*77.85*	115.37	70.50	*78.02*	115.37	0.42	*80.30*	148.55
	0.50	1.88	32.44	30.37	1.56	*32.93*	38.88	1.50	*33.05*	38.88	1.88	33.70	30.37
	1.00	6.38	5.71*	4.36	6.02	5.76*	4.80	6.50	5.76*	4.80	11.04	5.68*	1.95
	2.00	24.38	1.17*	1.06	24.80	1.17*	1.05	28.5	1.17*	1.05	54.38	1.16*	1.00
0.50	0.50	3.00	20.70	14.98	2.30	*21.21*	22.46	2.00	*21.53*	22.46	1.66	*22.74*	35.76
	1.00	7.50	5.27*	3.37	6.26	5.38*	4.49	6.00	5.45*	4.49	7.50	5.52*	3.37
	2.00	25.50	1.16*	1.04	24.06	1.17*	1.06	26.00	1.17*	1.06	44.16	1.16*	1.00
1.00	1.00	12.00	3.22*	1.76	9.24	3.38*	2.48	8.00	3.52*	2.48	6.66	*3.82**	4.06
	2.00	30.00	1.14*	1.02	25.06	1.15*	1.05	24.00	1.16*	1.05	30.00	1.16*	1.02
2.00	2.00	48.00	1.02*	1.00	37.92	1.03*	1.00	32.00	1.04*	1.00	26.66	1.07*	1.03
UCL			4	11.827		4	11.827		4	11.827		4	11.827
			3*			3*			3*			3*	
w_U			0.866			0.867			0.864			0.856	
			1.271*			1.272*			1.270*			1.261*	

Max D

Table 4 Minimum sample size (MSS) needed for $Max\,D$ control chart to outperform T^2 control chart with $n = 3$

		ρ											
		0.0			0.3			0.5			0.8		
		T^2	$Max\,D$		T^2	$Max\,D$		T^2	$Max\,D$		T^2	$Max\,D$	
δ_x	δ_y	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS	ARL_1	ARL_1	MSS
0.00	0.25	230.39	193.33	2	221.35	193.66	2	202.04	194.02	2	125.55	116.43	7
0.00	0.50	90.82	90.37	2	82.68	70.23	3	67.27	57.38	4	26.12	24.27	9
0.00	1.00	14.98	14.09	3	12.89	9.95	4	9.40	7.35	5	2.87	2.54	11
0.00	2.00	1.76	1.52	4	1.58	1.52	4	1.32	1.31	5	1.01	1.01	11
0.25	0.25	159.63	130.97	2	187.26	131.65	2	202.04	132.82	2	220.27	135.78	2
0.25	0.50	72.21	57.43	3	87.06	74.55	2	90.82	75.29	2	72.21	59.76	3
0.25	1.00	13.60	13.53	3	14.92	13.60	3	13.19	9.70	4	5.52	4.57	7
0.25	2.00	1.73	1.52	4	1.69	1.52	4	1.46	1.31	5	1.03	1.03	9
0.50	0.50	41.15	38.92	3	57.07	52.34	2	67.27	53.28	2	81.75	55.64	2
0.50	1.00	10.45	8.79	4	13.99	12.42	3	14.98	12.59	3	10.45	9.17	4
0.50	2.00	1.64	1.51	4	1.75	1.52	4	1.61	1.52	4	1.09	1.05	8
1.00	1.00	4.82	3.99	5	7.42	5.52	4	9.40	7.99	3	12.66	8.70	3
1.00	2.00	1.40	1.26	5	1.67	1.48	4	1.76	1.49	4	1.40	1.30	5
2.00	2.00	1.06	1.06	5	1.19	1.17	4	1.32	1.21	4	1.57	1.27	4

Monitoring bivariate means by attribute+variable charts

- Max D-T^2 chart proposed by Melo et al. (2017*a*)
- The sample of n units is split into 2 sub-samples: n_1 and $n_2 = n - n_1$
- Evaluate n_1 attributively by a gauge and get the statistic Max D
- If Max D $>$ C, then measure n_2 units and calculate T^2. If $T^2 > L$, then the process is stopped for adjustment

Max D-T^2

Table I. Some designs of *Max D* – T^2 control chart

							Max D			T^2	
ρ	δ_1	δ_2	n_1	n_2	*ASS*	ARL_1	*C*	*w*	α_D	*L*	α_{T^2}
0.0	0	0.5	2	4	2.721	57.098	1	0.503	0.180	8.399	0.015
			3	6	4.622	34.972	1	0.705	0.270	9.210	0.010
			6	6	6.404	27.066	3	0.694	0.068	6.438	0.040
			7	5	7.118	27.801	4	0.690	0.024	4.326	0.115
	0.5	0.5	2	4	2.721	21.052	1	0.503	0.180	8.399	0.015
			3	6	4.622	11.617	1	0.705	0.270	9.210	0.010
			6	6	6.649	9.024	3	0.583	0.108	7.378	0.025
			7	5	7.300	9.947	4	0.513	0.060	6.202	0.045
0.5	0	0.5	2	4	3.081	43.741	1	0.237	0.270	9.210	0.010
			3	6	6.243	25.310	1	0.226	0.541	10.597	0.005
			6	6	6.811	20.829	3	0.506	0.135	7.824	0.020
			7	5	7.270	22.868	4	0.524	0.054	5.991	0.050
	0.5	0.5	2	4	2.360	33.236	1	0.758	0.090	7.013	0.030
			3	6	3.811	19.726	1	0.956	0.135	7.824	0.020
			6	6	6.295	14.102	3	0.753	0.049	5.801	0.055
			7	5	7.113	14.418	4	0.691	0.023	4.241	0.120
0.8	0	0.5	2	4	4.162	16.837	1	0.698	0.541	10.597	0.005
			3	6	6.243	8.811	1	0.140	0.541	10.597	0.005
			6	6	9.243	8.073	3	0.316	0.541	10.597	0.005
			7	5	8.351	10.047	3	0.418	0.270	9.210	0.010
	0.5	0.5	2	4	2.270	40.099	1	0.817	0.068	6.438	0.040
			3	6	3.649	24.665	1	0.993	0.108	7.378	0.025
			6	6	6.203	17.075	3	0.813	0.034	5.051	0.080
			7	5	7.073	16.864	4	0.752	0.015	3.375	0.185

ASS: average sample size.

Max D-T^2

Table II. Values of ARL_1 of $Max\ D - T^2$: $\rho = 0.0, \delta_1 = 0.0$ and $\delta_2 = 0.5$

n_2	Sub-sample size n_1: Attribute chart						
	1	2	3	4	5	6	7
1	132.207$^{\#}$	90.416	70.173	57.402	46.138	38.879	32.972
	(1.003)	(2.003)	(3.003)	(4.003)	(5.003)	(6.003)	(7.003)
2	107.162	86.619	70.013	57.342	46.090	38.838	32.938
	(1.216)	(2.031)	(3.007)	(4.005)	(5.005)	(6.005)	(7.005)
3	80.928$^{\bigstar}$	71.260	62.591	52.704	44.708	38.536	32.834
	(1.811)	(2.270)	(3.090)	(4.054)	(5.024)	(6.014)	(7.011)
4	62.426	57.098$^{\bullet}$	51.488	45.175	39.924	35.262	30.841
	(3.162)	(2.721)	(3.541)	(4.216)	(5.103)	(6.075)	(7.044)
5	49.061	46.103	42.221$^{\spadesuit}$	38.164	34.645	31.028	27.801
	(3.703)	(3.351)	(3.901)	(4.541)	(5.270)	(6.193)	(7.118)
6	39.642	37.913	34.972	32.291$^{\blacktriangle}$	29.861$^{\blacksquare}$	27.066	24.693
	(4.243)	(5.243)	(4.622)	(4.811)	(5.541)	(6.405)	(7.249)
7	32.741	31.313	29.438	27.501	25.645	23.609$^{\clubsuit}$	21.838
	(4.784)	(5.784)	(4.892)	(5.261)	(6.261)	(6.757)	(7.473)
T^2 chart	**202.043$^{\#}$**	**129.684$^{\bigstar}$**	**90.824$^{\bullet}$**	**67.268$^{\spadesuit}$**	**51.833$^{\blacktriangle}$**	**41.150$^{\blacksquare}$**	**33.445$^{\clubsuit}$**
n_{T^2}	(1)	(2)	(3)	(4)	(5)	(6)	(7)

Max D-T^2

Table III. Values of ARL_1 of $Max\ D - T^2$: $\rho = 0.0$, $\delta_1 = 0.5$ and $\delta_2 = 0.5$

	Sub-sample size n_1: Attribute chart						
n_2	1	2	3	4	5	6	7
1	73.705#	51.297	39.057	31.319	24.792	20.722	17.470
	(1.025)	(2.005)	(3.003)	(4.003)	(5.003)	(6.003)	(7.003)
2	48.794	39.118	32.939	27.085	22.881	19.739	16.907
	(1.270)	(2.098)	(3.049)	(4.047)	(5.023)	(6.023)	(7.013)
3	33.189	28.315	24.699	21.065	18.498	16.306	14.402
	(1.811)	(2.324)	(3.270)	(4.147)	(5.095)	(6.081)	(7.056)
4	23.904 ★	21.052•	18.676	16.439	14.803	13.228	11.942
	(2.081)	(2.721)	(3.541)	(4.360)	(5.240)	(6.197)	v7.144)
5	17.823	16.149	14.544	13.087	11.994	10.844	9.947
	(3.703)	(3.351)	(3.901)	(4.676)	(5.450)	(6.386)	(7.300)
6	13.741	12.804 ♠	11.617	10.636	9.841 ■	9.024	8.376
	(4.243)	(3.622)	(4.622)	(5.081)	(5.811)	(6.649)	(7.463)
7	10.962	10.395	9.504 ▲	8.835	8.213	7.620 ♣	7.144
	(4.784)	(5.784)	(4.892)	(5.261)	(6.261)	(6.946)	(7.757)
T^2 chart	**129.684#**	**67.268★**	**41.150•**	**27.708♠**	**19.900▲**	**14.982■**	**11.697♣**
n_{T^2}	(1)	(2)	(3)	(4)	(5)	(6)	(7)

Max D-T^2

Table IV. Values of ARL_1 of $Max\ D-T^2$: $\rho = 0.5, \delta_1 = 0.0$ and $\delta_2 = 0.5$

	Sub-sample size n_1: Attribute chart						
n_2	1	2	3	4	5	6	7
1	127.731# (1.020)	90.533 (2.003)	70.132 (3.003)	57.416 (4.003)	46.067 (5.003)	38.851 (6.003)	32.924 (7.003)
2	90.053 (1.541)	77.231 ★ (2.108)	66.086 (3.032)	55.024 (4.021)	45.726 (5.009)	38.796 (6.006)	32.881 (7.005)
3	63.158 (2.622)	57.861 (2.541)	52.017 (3.324)	45.468 (4.147)	40.070 (5.071)	35.342 (6.054)	30.868 (7.035)
4	46.105 (3.162)	43.741 • (3.081)	40.134 (3.721)	36.413 (4.432)	33.209 (5.240)	29.802 (6.180)	26.722 (7.120)
5	35.230♠ (3.703)	33.708 (4.703)	31.497 (4.351)	29.283 ▲ (4.901)	27.205 (5.676)	24.877 (6.450)	22.868 (7.270)
6	27.852 (4.243)	26.649 (5.243)	25.310 (6.243)	23.838 (5.622)	22.402 (6.081)	20.829 (6.811)	19.451 (7.541)
7	22.613 (4.784)	21.636 (5.784)	20.549 (6.784)	19.773 ■ (5.892)	18.642 ♣ (6.892)	17.569 (7.261)	16.603 (7.946)
T^2 chart n_{T^2}	172.071# (1)	101.527★ (2)	67.268• (3)	47.854♠ (4)	35.755▲ (5)	27.708■ (6)	22.090♣ (7)

Max D-T^2

Table V. Values of ARL_1 of $Max\ D - T^2$: $\rho = 0.5$, $\delta_1 = 0.5$ and $\delta_2 = 0.5$

n_2	Sub-sample size n_1: Attribute chart						
	1	2	3	4	5	6	7
1	82.726	53.304	40.191	32.396	25.726	21.562	18.233
	(1.003)	(2.003)	(3.003)	(4.003)	(5.003)	(6.003)	(7.003)
2	67.868#	50.176	39.656	32.108	25.693	21.534	18.209
	(1.108)	(2.026)	(3.011)	(4.009)	(5.005)	(6.005)	(7.005)
3	51.378	41.259	34.622	28.570	24.059	20.804	17.789
	(1.405)	(2.147)	(3.068)	(4.051)	(5.029)	(6.020)	(7.018)
4	39.423★	33.236	28.832	24.299	21.097	18.536	16.210
	(2.081)	(2.360)	(3.270)	(4.144)	(5.090)	(6.075)	(7.050)
5	30.933	26.973	23.705	20.539	18.224	16.182	14.418
	(2.351)	(2.676)	(3.541)	(4.300)	(5.193)	(6.159)	(7.113)
6	24.906	22.177•	19.726	17.442	15.732	14.102	12.746
	(4.243)	(3.081)	(3.811)	(4.541)	(5.360)	(6.295)	(7.203)
7	20.221	18.442♠	16.601	14.940▲	13.648■	12.334♣	11.276
	(4.784)	(3.892)	(4.261)	(4.757)	(5.631)	(6.473)	(7.344)
T^2 chart	172.071#	101.527★	67.268•	47.854♠	35.755▲	27.708■	22.090♣
n_{T^2}	(1)	(2)	(3)	(4)	(5)	(6)	(7)

Max D-T^2

Table VI. Values of ARL_1 of *Max D* $- T^2$: $\rho = 0.8, \delta_1 = 0.0$ and $\delta_2 = 0.5$

	Sub-sample size n_1: Attribute chart						
n_2	1	2	3	4	5	6	7
1	86.298# (1.270)	74.112 (2.054)	63.712 (3.018)	53.079 (4.012)	44.777 (5.003)	37.950 (6.003)	32.238 (7.003)
2	43.709★ (2.081)	41.571 (2.541)	38.007 (3.360)	34.529 (4.216)	31.556 (5.135)	28.311 (6.098)	25.566 (7.064)
3	26.228 (2.622)	25.095 (3.622)	23.850 (4.622)	22.472 (4.811)	21.082 (5.541)	19.619 (6.405)	18.352 (7.270)
4	17.597• (3.162)	16.837 (4.162)	16.002 (5.162)	15.503 (6.162)	14.792 (6.081)	14.058 (7.081)	13.348 (7.721)
5	12.713♠ (3.703)	12.164 (4.703)	11.561 (5.703)	11.201 (6.703)	11.077 (6.351)	10.528 (7.351)	10.047 (8.351)
6	9.690 (4.243)	9.271 (5.243)	8.811 (6.243)	8.537 (7.243)	8.268 (8.243)	8.073 (9.243)	7.890 (10.243)
7	7.693▲ (4.784)	7.361■ (5.784)	6.996 (6.784)	6.778 (7.784)	6.565 (8.784)	6.409 (9.784)	6.264 (10.784)
T^2 chart n_{T^2}	97.755# (1)	45.449★ (2)	26.118• (3)	16.912♠ (4)	11.850▲ (5)	8.790■ (6)	6.811♣ (7)

Max D-T^2

Table VII. Values of ARL_1 of $Max\ D - T^2$: $\rho = 0.8, \delta_1 = 0.5$ and $\delta_2 = 0.5$

	Sub-sample size n_1: Attribute chart						
n_2	1	2	3	4	5	6	7
1	84.643 (1.003)	55.684 (2.003)	42.222 (3.003)	33.989 (4.003)	27.150 (5.003)	22.788 (6.003)	19.310 (7.003)
2	76.715# (1.064)	54.876 (2.013)	42.174 (3.005)	33.951 (4.005)	27.119 (5.005)	22.762 (6.005)	19.288 (7.005)
3	60.950 (1.324)	47.924 ★ (2.085)	39.435 (3.037)	32.218 (4.028)	26.638 (5.015)	22.676 (6.011)	19.259 (7.009)
4	48.204 (1.721)	40.099 (2.270)	34.389 (3.127)	28.541 (4.094)	24.409 (5.055)	21.293 (6.048)	18.371 (7.031)
5	38.652 (2.351)	33.405 (2.541)	29.082 (3.386)	24.827 (4.208)	21.742 (5.129)	19.156 (6.104)	16.864 (7.073)
6	31.660 (2.622)	28.009 • (3.081)	24.665 (3.649)	21.543 (4.405)	19.208 (5.249)	17.075 (6.203)	15.265 (7.141)
7	26.206 (4.784)	23.728 (3.261)	21.099 ♠ (3.946)	18.747 ▲ (4.631)	16.960 ■ (5.420)	15.198 ♣ (6.344)	13.755 (7.236)
T^2 chart n_{T^2}	**191.190#** (1)	**119.089★** (2)	**81.747•** (3)	**59.659♠** (4)	**45.449▲** (5)	**35.755■** (6)	**28.848♣** (7)

Principal Component chart

- T^2 control chart is effective if p (the number of quality characteristics) is not very large
- As p increases, the performance metric as ARL_1 to detect a specified shift also increases
- It looks like the shift "diluted" in the p-dimensional space of variables
- Most common alternative - monitor by principal component charts

Principal Component chart

- Original variables: $\mathbf{X}$=$(X_1, \ldots X_p)$ find new variables $\mathbf{Y}$=$(Y_1, \ldots, Y_p)$ as

$$\mathbf{Y} = \mathbf{XC}$$

  c_{ij}, constants to be determined such $\mathbf{Y}$ are no correlated variables
- $\mathbf{C_{p\times p}}$ is determined such that

$$\mathbf{C}'\boldsymbol{\Sigma}\mathbf{C} = \boldsymbol{\lambda}$$

- $\boldsymbol{\lambda}$ - a diagonal matrix, the main diagonal elements $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of the matrix $\boldsymbol{\Sigma}$

Principal component chart

- Properties: $\boldsymbol{\Sigma}$ and $\boldsymbol{\lambda}$:

$$tr(\boldsymbol{\Sigma}) = \sum_{i=1}^{p} \sigma_i^2 = \sum_{i=1}^{p} \lambda_i$$

  σ_i^2 - the variance of the X_i
- $\lambda_1 \geq \lambda_2 \ldots, \geq \lambda_p \geq 0$
- λ_i is the variance of the new variable Y_i
- $\mathbf{C} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p)$, $c_i = (c_{1i}, c_{2i}, \ldots, c_{pi})$ is the eigenvector related to the eigenvalue λ_i

Principal Component chart

- For the j-th observation $\mathbf{x}_j=(x_{1j}, \ldots, x_{pj})$
- Principal component scores can be obtained as

$$y_{1j} = c_{11}x_{1j} + \ldots + c_{1p}x_{pj}$$

$$y_{2j} = c_{21}x_{1j} + \ldots + c_{2p}x_{pj}$$

$$\ldots$$

$$y_{pj} = c_{p1}x_{1j} + \ldots + c_{pp}x_{pj}$$

- In general the first r components are retained for analysis such that

$$\frac{\sum_{i=1}^{r} \lambda_i}{\sum_{i=1}^{p} \lambda_i} > k$$

Principal component chart - General framework

- In general the first two principal components are retained
- A 95% (or another level) confidence contour is drawn, and score values of z_{1i} and z_{2i} are plotted.

Principal component chart

Chemical Process Data

			Original Data			
Observation	x_1	x_2	x_3	x_4	z_1	z_2
1	10	20.7	13.6	15.5	0.291681	–0.6034
2	10.5	19.9	18.1	14.8	0.294281	0.491533
3	9.7	20	16.1	16.5	0.197337	0.640937
4	9.8	20.2	19.1	17.1	0.839022	1.469579
5	11.7	21.5	19.8	18.3	3.204876	0.879172
6	11	20.9	10.3	13.8	0.203271	–2.29514
7	8.7	18.8	16.9	16.8	–0.99211	1.670464
8	9.5	19.3	15.3	12.2	–1.70241	–0.36089
9	10.1	19.4	16.2	15.8	–0.14246	0.560808
10	9.5	19.6	13.6	14.5	–0.99498	–0.31493
11	10.5	20.3	17	16.5	0.944697	0.504711
12	9.2	19	11.5	16.3	–1.2195	–0.09129
13	11.3	21.6	14	18.7	2.608666	–0.42176
14	10	19.8	14	15.9	–0.12378	–0.08767
15	8.5	19.2	17.4	15.8	–1.10423	1.472593
16	9.7	20.1	10	16.6	–0.27825	–0.94763
17	8.3	18.4	12.5	14.2	–2.65608	0.135288
18	11.9	21.8	14.1	16.2	2.36528	–1.30494
19	10.3	20.5	15.6	15.1	0.411311	–0.21893
20	8.9	19	8.5	14.7	–2.14662	–1.17849

Principal Component chart

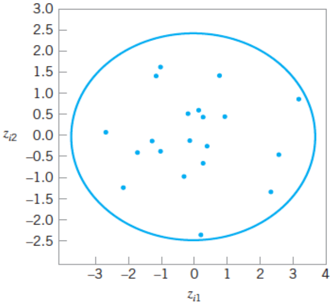


FIGURE 11.16 Scatter plot of the first 20 principal component scores z_{i1} and z_{i2} from Table 11.6, with 95% confidence ellipse.

Principal component chart

New Data						
Observation	x_1	x_2	x_3	x_4	z_1	z_2
21	9.9	20	15.4	15.9	0.074196	0.239359
22	8.7	19	9.9	16.8	−1.51756	−0.21121
23	11.5	21.8	19.3	12.1	1.408476	−0.87591
24	15.9	24.6	14.7	15.3	6.298001	−3.67398
25	12.6	23.9	17.1	14.2	3.802025	−1.99584
26	14.9	25	16.3	16.6	6.490673	−2.73143
27	9.9	23.7	11.9	18.1	2.738829	−1.37617
28	12.8	26.3	13.5	13.7	4.958747	−3.94851
29	13.1	26.1	10.9	16.8	5.678092	−3.85838
30	9.8	25.8	14.8	15	3.369657	−2.10878

Principal Component chart

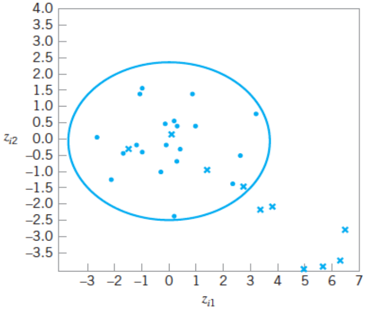


FIGURE 11.17 Principal components trajectory chart, showing the last 10 observations from Table 11.6.

Principal component chart

- If more than 2 components are retained - analysis pairwise scatter plots
- For $r > 4$, may have some difficulties of interpretation of the meaning of the principal components

Monitoring matrix of covariance-variance

- Similar approach of the univariate chart S^2
- The statistic W is calculated

$$W = -pn + pn\ln(n) - n\ln\left(\frac{|\mathbf{A}|}{|\boldsymbol{\Sigma}|}\right) + tr(\boldsymbol{\Sigma}^{-1}\mathbf{A})$$

- $\mathbf{A} = (n-1)\mathbf{S}$, $\mathbf{S}$, the observed matrix of covariance-variance
- W follows asymptotically a Chi-square distribution with $0.5p(p+1)$ degrees of freedom

Monitoring matrix of covariance-variance

- Approach based on the first two moments of $|\mathbf{S}|$
- Central line and control limits build as:

$$CL = E(|\mathbf{S}|) = b_1|\Sigma|, \text{with } b_1 = \frac{1}{(n-1)^p}\prod_{i=1}^{p}(n-i)$$

$$\text{Control limits:} E(|\mathbf{S}|) \pm 3Var(|\mathbf{S}|)$$

$$Var(|\mathbf{S}|) = b_2|\Sigma|^2,$$

$$b_2 = \frac{1}{(n-1)^{2p}}\prod_{i=1}^{p}(n-i)\left[\prod_{j=1}^{p}(n-j-2) - \prod_{j=1}^{p}(n-j)\right]$$

Monitoring matrix of covariance-variance

- Another approach based on asymptotic distribution of $|\mathbf{S}|$
- For $p = 2$,

$$2(n-1)\left(\frac{|\mathbf{S}|}{|\boldsymbol{\Sigma}|}\right)^{0.5}$$

  follows a Chi-square distribution with (2n-4) degrees of freedom
- Let $\mathbf{S}$, a covariance matrix with n degrees of freedom. Then

$$\sqrt{n}\left(\frac{|\mathbf{S}|}{|\boldsymbol{\Sigma}|}-1\right)$$

  is asymptotically normally distributed with mean 0 and variance $2p$

Other approaches: VMax

- VMax chart- proposed by Costa & Machado (2009):
- Let $S_i^2 = \sum_{j=1}^{n} \frac{z_{ij}^2}{n}$, $z_{ij} = \frac{X_{ij} - \mu_i}{\sigma_i}$
- VMax=max$(S_1^2, S_2^2, \ldots, S_p^2)$, a signal is triggered whenever $VMax > L$, L, the control limit satisfying some performance metric

Other approaches: VMax

Table 3 The *ARL* for the VMAX chart and for the |**S**| chart (p=2, ρ=0.5)

		n					
		4			5		
		|S|	VMAX		|S|	VMAX	
			Case I	Case II		Case I	Case
γ^2	UCL	6.134	4.094	4.094	5.375	3.668	3.668
1.0		200.0	200.0	200.0	200.0	200.0	200.0
1.1		146.8	136.6	143.0	141.4	132.5	139.7
1.2		112.5	92.4	107.0	104.6	86.8	102.4
1.3		89.1	63.9	82.9	80.5	58.3	78.0
1.4		73.3	45.7	66.1	64.1	40.7	61.4
1.5		60.4	33.9	54.1	51.9	29.6	49.6
2.0		30.2	11.6	25.4	24.1	9.62	22.3
3.0		13.6	4.09	10.7	10.2	3.38	9.0
5.0		6.37	1.95	4.77	4.58	1.67	3.9

Other approaches: VMax

Table 6 The *ARL* for the VMAX chart and for the |**S**| chart (p=3, ρ_{12}=ρ_{13}=ρ_{23}=0.5)

		n							
		4				5			
		|**S**|	VMAX			|**S**|	VMAX		
			Case I	Case II	Case III		Case I	Case II	Case I
γ^2	*UCL*	4.050	4.313	4.313	4.313	4.620	3.851	3.851	3.851
1.0		200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1		160.4	149.9	155.9	157.8	155.2	146.3	153.0	155.2
1.2		135.3	107.7	123.0	127.7	125.3	101.9	118.6	123.8
1.3		116.9	76.8	98.7	105.8	103.5	70.5	93.5	101.3
1.4		102.9	55.6	80.4	89.5	87.3	49.7	75.1	84.7
1.5		89.4	41.3	66.6	77.1	74.5	36.1	61.4	72.3
2.0		54.6	13.7	31.7	42.6	41.6	11.3	27.8	38.6
3.0		29.8	4.55	12.9	20.9	20.7	3.72	10.9	18.2
5.0		15.8	2.06	5.49	9.86	9.93	1.75	4.55	8.3

Other approaches: VMax

Table 7 The *ARL* for the VMAX chart (p=4, n=5)

ρ value		Value							
ρ_{12}=ρ_{13}		0.5	0.7	0.5	0.7	0.5	0.7	0.5	0.7
ρ_{14}=ρ_{23}		0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
ρ_{24}=ρ_{34}		0.5	0.2	0.5	0.2	0.5	0.2	0.5	0.2
		Case I		Case II		Case III		Case IV	
γ^2	*UCL*	3.980	3.970	3.980	3.970	3.980	3.970	3.980	3.97
1.0		200.0	200.0	200.0	200.0	200.0	200.0	200.0	200
1.1		152.7	158.6	160.0	162.9	162.5	164.9	164.5	166
1.2		112.9	116.9	128.8	130.1	134.7	138.8	135.8	139
1.3		79.4	82.1	105.2	108.1	114.6	114.8	118.0	115
1.4		56.9	57.9	85.4	87.9	97.3	99.9	99.6	102
1.5		41.4	41.9	70.9	73.7	82.9	84.7	88.7	89
2.0		12.6	12.8	32.5	33.7	45.3	48.3	52.9	54
3.0		3.95	3.92	12.4	13.1	20.9	22.8	27.9	28
5.0		1.80	1.80	5.05	5.37	9.49	10.1	13.5	14

Other approaches: VMax

Table 8 The *ARL* for the VMAX chart and for the chart (p=4, n=5, $\rho_{12}=\rho_{13}=\rho_{14}=\rho_{23}=\rho_{24}=\rho_{34}$=0.5)

		|S|	VMAX			
			Case I	Case II	Case III	Case I
γ^2	*UCL*	2.000	3.980	3.980	3.980	3.980
1.0		200.0	200.0	200.0	200.0	200.0
1.1		166.8	152.7	160.0	162.5	164.5
1.2		145.6	112.9	128.8	134.7	135.8
1.3		127.9	79.4	105.2	114.6	118.0
1.4		108.5	56.9	85.4	97.3	99.6
1.5		96.9	41.4	70.9	82.9	88.7
2.0		61.1	12.6	32.5	45.3	52.9
3.0		35.7	3.95	12.4	20.9	27.9
5.0		19.2	1.80	5.05	9.49	13.5

Other approaches: RMax proposed by Costa & Machado (2011)

- For a sample of n units, let R_i=max $(X_{1i}, X_{2i}, \ldots, X_{ni})$ - min $(X_{1i}, X_{2i}, \ldots, X_{ni})$
- RMax=max$(R_1, R_2, \ldots, R_p)$
- A signal is triggered whenever $RMax > L$, L, the control limit

Other approaches: RMax

Table 6. The ARL for the RMAX chart and for the |S| chart ($n = 5$, $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$).

$p = 2$				$p = 3$				
CL		|S|	RMAX	CL			|S|	RMAX
a_1	a_2	5.375	5.145	a_1	a_2	a_3	4.620	5.294
1.0	1.0	200.0	200.0	1.0	1.0	1.0	200.0	200.0
$\sqrt{1.2}$	1.0	104.6	96.9	$\sqrt{1.2}$	1.0	1.0	125.3	96.9
$\sqrt{1.4}$	1.0	64.1	49.6	$\sqrt{1.4}$	1.0	1.0	87.3	49.6
$\sqrt{2}$	1.0	24.1	13.0	$\sqrt{2}$	1.0	1.0	41.6	13.0
$\sqrt{3}$	1.0	10.2	4.53	$\sqrt{3}$	1.0	1.0	20.7	4.53
$\sqrt{5}$	1.0	4.58	2.08	$\sqrt{5}$	1.0	1.0	9.93	2.08
1.0	1.0	200.0	200.0	1.0	1.0	1.0	200.0	200.0
$\sqrt[4]{1.2}$	$\sqrt[4]{1.2}$	104.6	110.6	$\sqrt[4]{1.2}$	$\sqrt[4]{1.2}$	1.0	125.3	127.4
$\sqrt[4]{1.4}$	$\sqrt[4]{1.4}$	64.1	70.0	$\sqrt[4]{1.4}$	$\sqrt[4]{1.4}$	1.0	87.3	85.4
$\sqrt[4]{2}$	$\sqrt[4]{2}$	24.1	27.8	$\sqrt[4]{2}$	$\sqrt[4]{2}$	1.0	41.6	35.0
$\sqrt[4]{3}$	$\sqrt[4]{3}$	10.2	12.0	$\sqrt[4]{3}$	$\sqrt[4]{3}$	1.0	20.7	14.6
$\sqrt[4]{5}$	$\sqrt[4]{5}$	4.58	5.28	$\sqrt[4]{5}$	$\sqrt[4]{5}$	1.0	9.93	6.17
				1.0	1.0	1.0	200.0	200.0
				$\sqrt[6]{1.2}$	$\sqrt[6]{1.2}$	$\sqrt[6]{1.2}$	125.3	132.0
				$\sqrt[6]{1.4}$	$\sqrt[6]{1.4}$	$\sqrt[6]{1.4}$	87.3	94.4
				$\sqrt[6]{2}$	$\sqrt[6]{2}$	$\sqrt[6]{2}$	41.6	46.4
				$\sqrt[6]{3}$	$\sqrt[6]{3}$	$\sqrt[6]{3}$	20.7	23.0
				$\sqrt[6]{5}$	$\sqrt[6]{5}$	$\sqrt[6]{5}$	9.93	11.0

Other approaches: RMax

Table 4. The ARL for the RMAX and VMAX charts ($p = 2, \rho = 0.5$).

			n					
			4			5		
a_1	a_2	CL	VMAX 4.094	RMAX 4.960	% –	VMAX 3.668	RMAX 5.145	P_v (%) –
1.0	1.0		200.0	200.0	0	200.0	200.0	–
1.25	1.0		28.5	35.8	25.6	24.7	31.5	27.5
1.25	1.25		16.2	20.5	26.5	13.9	17.9	28.8
1.5	1.0		8.11	10.9	34.4	6.70	9.09	35.7
1.25	1.5		6.94	9.24	33.1	5.78	7.72	33.6
1.5	1.5		4.69	6.24	33.1	3.91	5.20	33.0

Other approaches: RMax

Table 5. The ARL for the RMAX chart and for the VMAX chart ($p = 3$, $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$).

				n					
				4			5		
a_1	a_2	a_3	CL	VMAX 4.313	RMAX 5.113	% –	VMAX 3.851	RMAX 5.294	P_v (% –
1.0	1.0	1.0		200.0	200.0	0	200.0	200.0	–
1.25	1.0	1.0		34.8	43.9	26.1	30.1	38.8	28.9
1.25	1.25	1.0		19.9	25.6	28.6	17.1	22.3	30.4
1.25	1.25	1.25		14.4	18.4	27.8	12.3	16.0	30.1
1.5	1.5	1.5		4.05	5.38	32.8	3.39	4.45	31.3

Other approaches: VMix proposed by Quinino et al. (2012)- for p=2

- Consider W_1 and W_2 two normal correlated random variables
- Let

$$X_1 = Z_1 \text{ and } X_2 = \frac{Z_2 - \rho X_1}{\sqrt{1-\rho^2}},$$

with $Z_1 = \dfrac{W_1 - \mu_1}{\sigma_1}$, $Z_2 = \dfrac{W_2 - \mu_2}{\sigma_2}$

- $VMix = \dfrac{\sum_{i=1}^{n} X_{1i}^2 + X_{2i}^2}{2n}$, $2n \times VMix$ follows a chi-square distribution with $2n$ degrees of freedom

Other approaches: VMix

TABLE 1 **Performance of VMIX compared to the competitor charts: comparison of ARL_1 values**

$(k_x;\ k_y)$	VMIX	VMAX	|S|	NT	W	v_t
(1.1025; 1)	128.583	130.677	140.291	134.201	193.586	144.88
(1.1025; 1.1025)	88.278	97.108	100.448	95.579	188.343	100.32
(1.21; 1)	84.12	82.983	101.976	87.796	181.837	101.92
(1.21; 1.21)	44.625	52.489	56.024	49.595	161.372	55.72
(1.5625; 1)	27.986	24.653	46.395	31.989	103.051	46.27
(1.5625; 1.5625)	10.391	13.359	15.403	12.485	62.937	15.45
(2.25; 1)	7.98	6.700	18.415	9.493	27.419	18.69
(2.25; 2.25)	2.919	3.669	4.529	3.530	11.979	4.55
(4; 1)	2.399	2.134	6.299	2.849	5.067	6.31
(4; 4)	1.266	1.396	1.692	1.421	2.352	1.69

Other approaches: VMix

TABLE 2 **Performance of the VMIX EWMA compared to the competitor charts: comparison of ARL_1 values**

	VMIX EWMA		VMAX EWMA		v_t EWMA	
$(k_x;\ k_y)$	λ=0.2	λ=0.4	λ=0.2	λ=0.4	λ=0.2	λ=0.4
(1.1025; 1)	73.217	100.908	77.688	104.004	141.383	158.378
(1.1025; 1.1025)	34.206	57.284	40.798	65.261	76.104	107.608
(1.21; 1)	32.274	52.963	33.597	53.039	84.510	98.775
(1.21; 1.21)	9.408	20.718	12.120	25.828	24.669	43.840
(1.5625; 1)	5.015	11.483	4.758	10.845	17.864	31.998
(1.5625; 1.5625)	1.452	3.125	1.701	4.034	3.358	6.888
(2.25; 1)	1.330	2.620	1.263	2.369	4.172	8.694
(2.25; 2.25)	1.002	1.146	1.007	1.253	1.209	1.916
(4; 1)	1.001	1.094	1.000	1.062	1.454	2.558
(4; 4)	1.000	1.000	1.000	1.001	1.004	1.109

MCUSUM

- There are many versions of MCUSUM
- One of them is the proposed by Crosier (1988) which states:

$$\mathbf{S}_t = \begin{cases} (\mathbf{S}_{t-1} + \mathbf{Z}_t - \boldsymbol{\mu})\left(1 - \frac{k}{d_t}\right) \text{ if } d_t > k; \\ \mathbf{0}, \text{ otherwise} \end{cases} \quad (5)$$

with $\boldsymbol{\mu}= E(\mathbf{Z}_t)$, k is the solution for $k^2 = \mathbf{k}'\boldsymbol{\Sigma}^{-1}\mathbf{k}$

$$d_t = \left[(\mathbf{S}_{t-1} + \mathbf{Z}_t - \boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{S}_{t-1} + \mathbf{Z}_t - \boldsymbol{\mu})\right]^{\frac{1}{2}}$$

and $\mathbf{S}_t = (S_{1,t}, S_{2,t}, \ldots, S_{n,t})$ with

$$S_{j,t} = \max\left[0,\ (S_{j,t-1} + Z_{j,t} - \mu_j)\left(1 - \frac{k}{d_t}\right)\right] \quad (6)$$

for $j = 1, \ldots, N$, to include the directional approach presented by Fricker Jr et al. (2008). Starting with $\mathbf{S}_0 = \mathbf{0}$ the control chart signals whenever $C_t = \left(\mathbf{S}_t'\boldsymbol{\Sigma}^{-1}\mathbf{S}_t\right)^{\frac{1}{2}} > h$

MEWMA

- Like MCUSUM, there are several proposal for MEWMA. The one proposed by Lowry et al. (1992) with directional approach of Joner et al. (2008) is shown here

$$\mathbf{Y}_t = \max\left[\mathbf{0},\ \lambda(\mathbf{Z_t} - \boldsymbol{\mu}) + (\mathbf{1} - \lambda)\mathbf{Y_{t-1}}\right] \tag{7}$$

can be obtained, choosing a weight $\lambda \in]0;1[$,
$\mathbf{Y}_t = (Y_{1,t}, Y_{2,t}, \ldots, Y_{n,t})$ with

$$Y_{j,t} = \max[0;\ \lambda(Z_{j,t} - \mu_i) + (1-\lambda)Y_{j,t-1}] \tag{8}$$

for $j = 1, \ldots, N$. Starting at $t = 0$ with $\mathbf{Y}_0 = \mathbf{0}$, MEWMA chart signals whenever $E_t = \mathbf{Y}'_t \boldsymbol{\Sigma}^{-1}_{Y_t} \mathbf{Y}_t > b$ with

$$\boldsymbol{\Sigma}_{Y_t} = \frac{\lambda[1-(1-\lambda)^{2t}]}{2-\lambda}\boldsymbol{\Sigma}_Z \tag{9}$$

$\boldsymbol{\Sigma}_Z = \sigma^2 \mathbf{I}_N$, $\mathbf{I}_N$ is identity matrix $N \times N$.

MCUSUM and MEWMA: Question for Seminar

- Research for other versions of MCUSUM and MEWMA
- Compare them, find common points, advantages and disadvantages, etc

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