MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

NÃO ESQUEÇA DE INICIAR A GRAVAÇÃO

MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

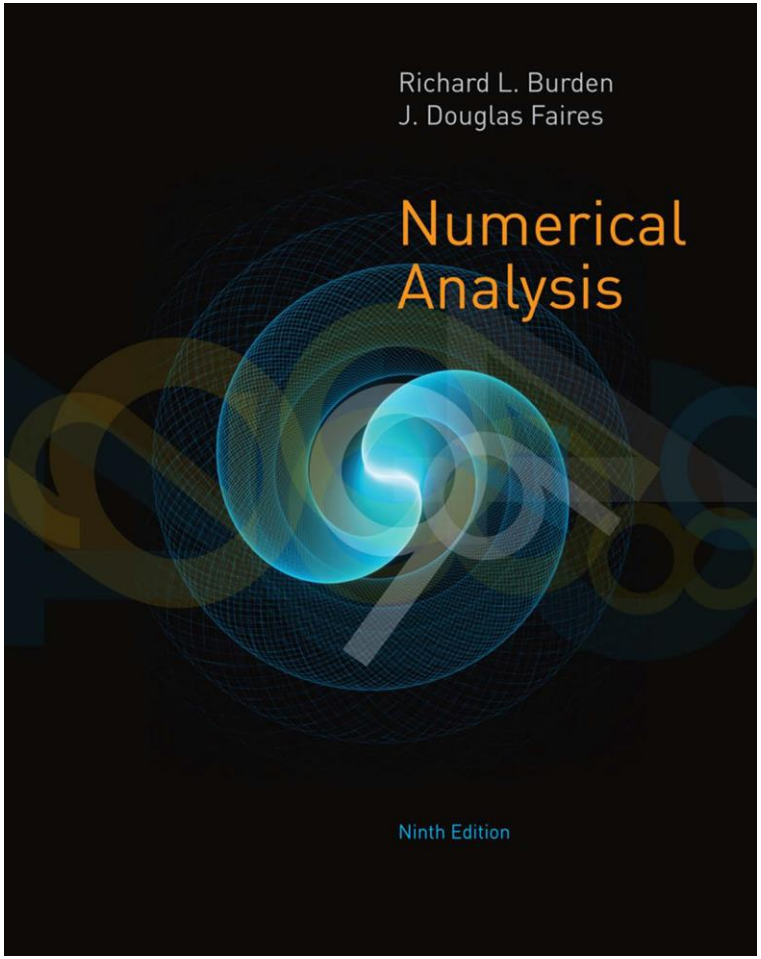
Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

MAP2210



Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

MAP2210

6 Direct Methods for Solving Linear Systems 357

6.1 Linear Systems of Equations 358
6.2 Pivoting Strategies 372
6.3 Linear Algebra and Matrix Inversion 381
6.4 The Determinant of a Matrix 396
6.5 Matrix Factorization 400
6.6 Special Types of Matrices 411
6.7 Survey of Methods and Software 428

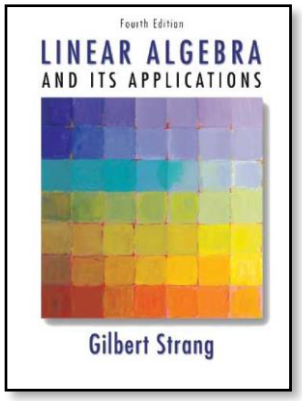
7 Iterative Techniques in Matrix Algebra 431

7.1 Norms of Vectors and Matrices 432
7.2 Eigenvalues and Eigenvectors 443
7.3 The Jacobi and Gauss-Siedel Iterative Techniques 450
7.4 Relaxation Techniques for Solving Linear Systems 462
7.5 Error Bounds and Iterative Refinement 469
7.6 The Conjugate Gradient Method 479
7.7 Survey of Methods and Software 495

9 Approximating Eigenvalues 561

9.1 Linear Algebra and Eigenvalues 562
9.2 Orthogonal Matrices and Similarity Transformations 570
9.3 The Power Method 576
9.4 Householder's Method 593
9.5 The QR Algorithm 601
9.6 Singular Value Decomposition 614
9.7 Survey of Methods and Software 626

+

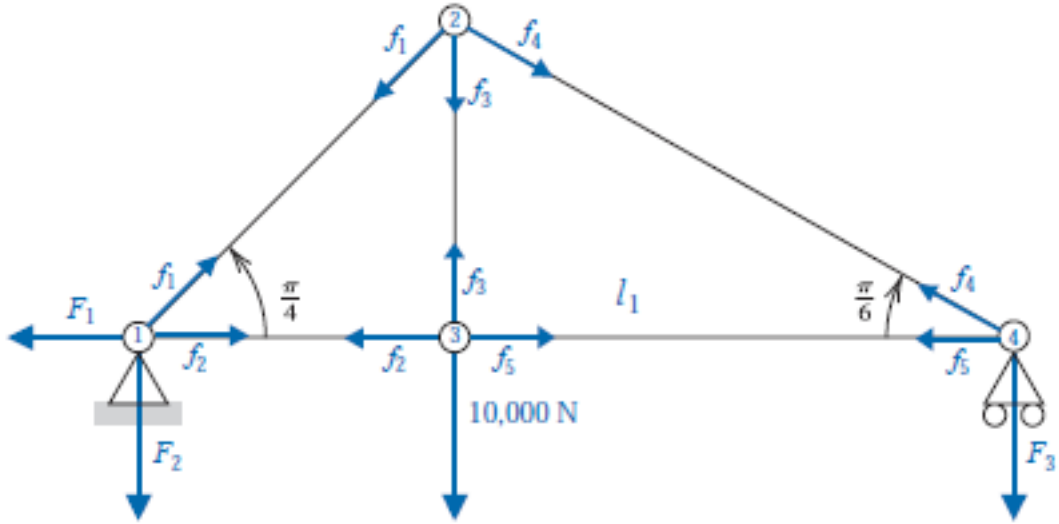


+

CHAPTER 7

Iterative Techniques in Matrix Algebra

Introduction



Joint	Horizontal Component	Vertical Component
①	$-F_1 + \frac{\sqrt{2}}{2} f_1 + f_2 = 0$	$\frac{\sqrt{2}}{2} f_1 - F_2 = 0$
②	$-\frac{\sqrt{2}}{2} f_1 + \frac{\sqrt{3}}{2} f_4 = 0$	$-\frac{\sqrt{2}}{2} f_1 - f_3 - \frac{1}{2} f_4 = 0$
③	$-f_2 + f_5 = 0$	$f_3 - 10{,}000 = 0$
④	$-\frac{\sqrt{3}}{2} f_4 - f_5 = 0$	$\frac{1}{2} f_4 - F_3 = 0$

This linear system can be placed in the matrix form

$$\begin{bmatrix} -1 & 0 & 0 & \frac{\sqrt{2}}{2} & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10{,}000 \\ 0 \\ 0 \end{bmatrix}.$$

8 x 8 = 64 elementos
47 zeros
17 não zeros
Esparsidade 73.4%
Densidade 26.5%

Diversos problemas aplicados dão origem a sistemas de matrizes esparsas

O princípio dos métodos iterativos é substituir o problema linear original:

$$A\,x = b$$

Por outro:

$$\mathbf{x}^{(i+1)} = T\mathbf{x}^{(i)} + \mathbf{c}.$$

onde na convergência as soluções são próximas dentro de uma tolerância especificada.

O material desse aula provê uma revisão rápida e preparação para esse tópico, e inicia a discussão sobre o problema de autovalores.

7.1 Norms of Vectors and Matrices

Vector Norms

Let $\mathbb{R}^n$ denote the set of all n-dimensional column vectors with real-number components. To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

Definition 7.1 A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. ■

Definition 7.2 The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$ ■

Distance between Vectors in $\mathbb{R}^n$

The norm of a vector gives a measure for the distance between an arbitrary vector and the zero vector, just as the absolute value of a real number describes its distance from 0. Similarly, the **distance between two vectors** is defined as the norm of the difference of the vectors just as distance between two real numbers is the absolute value of their difference.

Definition 7.4 If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|. \qquad \blacksquare$$

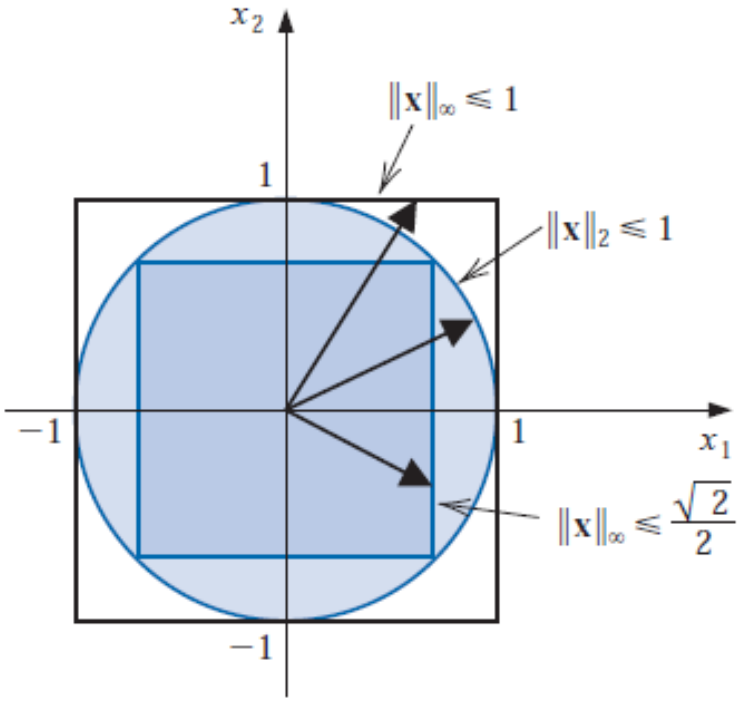
Definition 7.5 A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \ge N(\varepsilon). \qquad \blacksquare$$

Theorem 7.6 The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \ldots, n$. $\blacksquare$

Theorem 7.7 For each $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{n}\|\mathbf{x}\|_\infty. \quad \blacksquare$$



It can be shown that all norms on $\mathbb{R}^n$ are equivalent with respect to convergence; that is, if $\|\cdot\|$ and $\|\cdot\|'$ are any two norms on $\mathbb{R}^n$ and $\{\mathbf{x}^{(k)}\}_{k=1}^\infty$ has the limit $\mathbf{x}$ with respect to $\|\cdot\|$, then $\{\mathbf{x}^{(k)}\}_{k=1}^\infty$ also has the limit $\mathbf{x}$ with respect to $\|\cdot\|'$. The proof of this fact for the general case can be found in [Or2], p. 8. The case for the l_2 and l_∞ norms follows from Theorem 7.7.

Matrix Norms and Distances

Definition 7.8 A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\|\|B\|$. ■

The **distance between $n \times n$ matrices** A and B with respect to this matrix norm is $\|A - B\|$.

Although matrix norms can be obtained in various ways, the norms considered most frequently are those that are natural consequences of the vector norms l_2 and l_∞.

Theorem 7.9 If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \tag{7.2}$$

is a matrix norm. ■

Matrix norms defined by vector norms are called the **natural**, or *induced*, **matrix norm** associated with the vector norm. In this text, all matrix norms will be assumed to be natural matrix norms unless specified otherwise.

For any $\mathbf{z} \neq \mathbf{0}$, the vector $\mathbf{x} = \mathbf{z}/\|\mathbf{z}\|$ is a unit vector. Hence

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\mathbf{z}\neq\mathbf{0}} \left\| A\left(\frac{\mathbf{z}}{\|\mathbf{z}\|}\right)\right\| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|},$$

and we can alternatively write

$$\|A\| = \max_{\mathbf{z}\neq\mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}. \tag{7.3}$$

The following corollary to Theorem 7.9 follows from this representation of $\|A\|$.

Corollary 7.10 For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|.$$

■

An illustration of these norms when $n = 2$ is shown in Figures 7.4 and 7.5 for the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

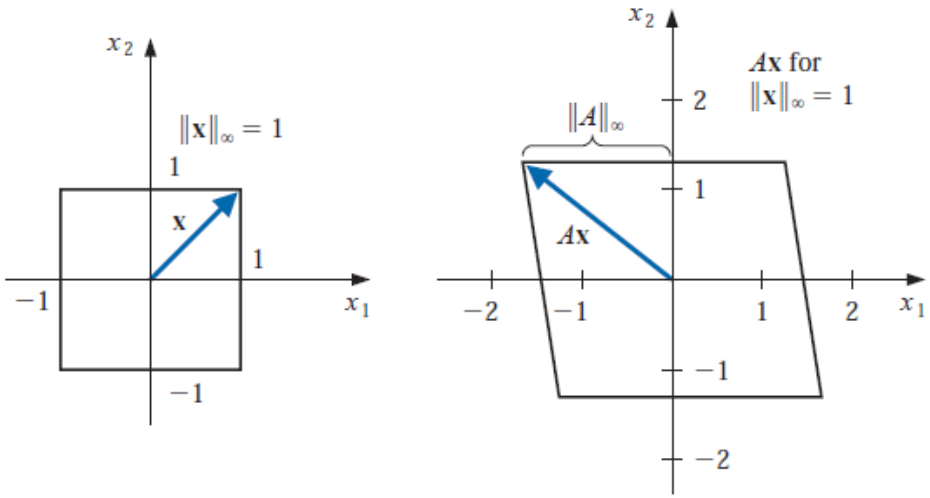


Figure 7.4

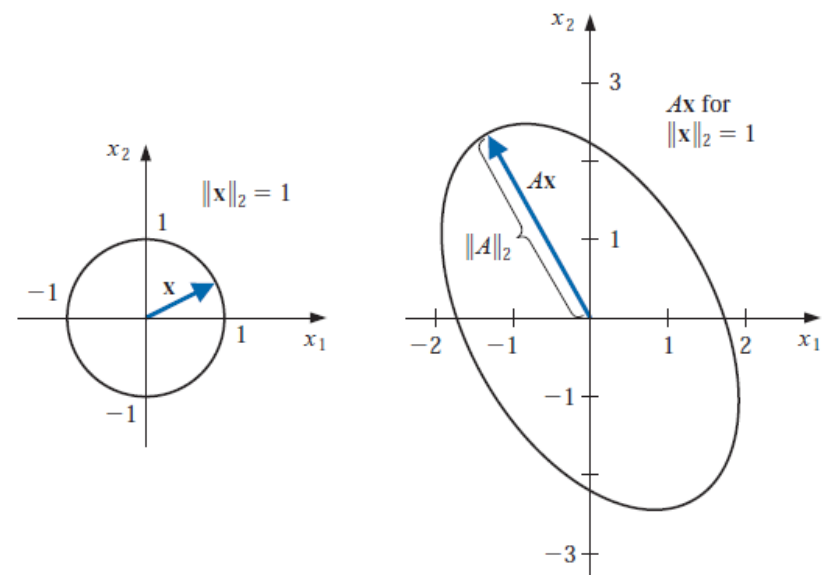


Figure 7.5

The l_∞ norm of a matrix can be easily computed from the entries of the matrix.

Theorem 7.11 If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|. \qquad \blacksquare$$

Proof First we show that $\|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$.

Let $\mathbf{x}$ be an n-dimensional vector with $1 = \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$. Since $A\mathbf{x}$ is also an n-dimensional vector,

$$\|A\mathbf{x}\|_\infty = \max_{1 \le i \le n} |(A\mathbf{x})_i| = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \max_{1 \le j \le n} |x_j|.$$

But $\max_{1 \le j \le n} |x_j| = \|\mathbf{x}\|_\infty = 1$, so

$$\|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|,$$

and consequently,

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|. \tag{7.4}$$

Now we will show the opposite inequality. Let p be an integer with

$$\sum_{j=1}^{n} |a_{pj}| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|,$$

and $\mathbf{x}$ be the vector with components

$$x_j = \begin{cases} 1, & \text{if } a_{pj} \ge 0, \\ -1, & \text{if } a_{pj} < 0. \end{cases}$$

Then $\|\mathbf{x}\|_\infty = 1$ and $a_{pj}x_j = |a_{pj}|$, for all $j = 1, 2, \ldots, n$, so

$$\|A\mathbf{x}\|_\infty = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij}x_j \right| \ge \left| \sum_{j=1}^{n} a_{pj}x_j \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

This result implies that

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty \ge \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Putting this together with Inequality (7.4) gives $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$. ■ ■ ■

7.2 Eigenvalues and Eigenvectors

An $n \times m$ matrix can be considered as a function that uses matrix multiplication to take m-dimensional column vectors into n-dimensional column vectors. So an $n \times m$ matrix is actually a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$. A square matrix A takes the set of n-dimensional vectors into itself, which gives a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$. In this case, certain nonzero vectors $\mathbf{x}$ might be parallel to $A\mathbf{x}$, which means that a constant λ exists with $A\mathbf{x} = \lambda\mathbf{x}$. For these vectors, we have $(A - \lambda I)\mathbf{x} = \mathbf{0}$. There is a close connection between these numbers λ and the likelihood that an iterative method will converge. We will consider this connection in this section.

Definition 7.12 If A is a square matrix, the **characteristic polynomial** of A is defined by

$$p(\lambda) = \det(A - \lambda I).$$

■

Definition 7.13 If p is the characteristic polynomial of the matrix A, the zeros of p are **eigenvalues**, or characteristic values, of the matrix A. If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is an **eigenvector**, or characteristic vector, of A corresponding to the eigenvalue λ. ■

To determine the eigenvalues of a matrix, we can use the fact that

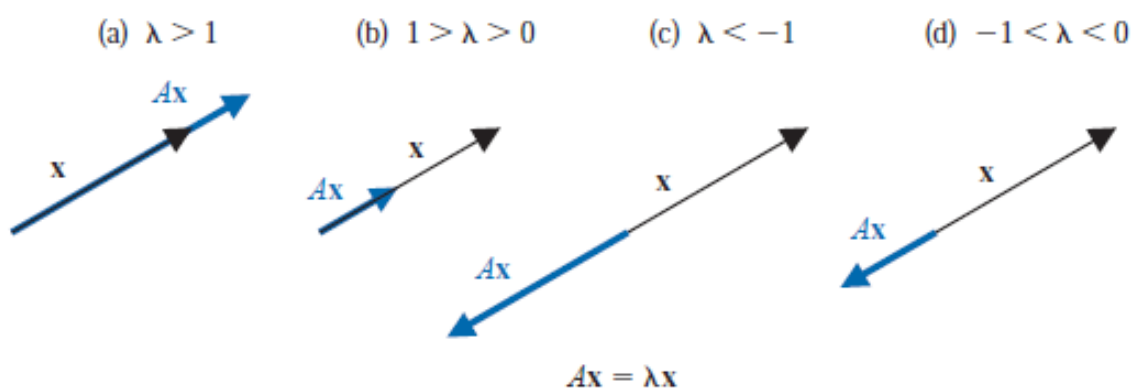
- λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Once an eigenvalue λ has been found a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ is determined by solving the system

- $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}$ is an eigenvector associated with the real eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$, so the matrix A takes the vector $\mathbf{x}$ into a scalar multiple of itself.

- If λ is real and $\lambda > 1$, then A has the effect of stretching $\mathbf{x}$ by a factor of λ, as illustrated in Figure 7.6(a).
- If $0 < \lambda < 1$, then A shrinks $\mathbf{x}$ by a factor of λ (see Figure 7.6(b)).
- If $\lambda < 0$, the effects are similar (see Figure 7.6(c) and (d)), although the direction of $A\mathbf{x}$ is reversed.



Notice also that if $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ and α is any nonzero constant, then $\alpha\mathbf{x}$ is also an eigenvector since

$$A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha(\lambda\mathbf{x}) = \lambda(\alpha\mathbf{x}).$$

An important consequence of this is that for any vector norm $||\cdot||$ we could choose the constant $\alpha = \pm||\mathbf{x}||^{-1}$, which would result in $\alpha\mathbf{x}$ being an eigenvector with norm 1. So

- For every eigenvalue and any vector norm there are eigenvectors with norm 1.

Example 2 Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

Solution The characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$

$$= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) = -(\lambda - 3)(\lambda - 2)^2,$$

so there are two eigenvalues of A: $\lambda_1 = 3$ and $\lambda_2 = 2$.

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$. For example, when $x_3 = 1$ we have the eigenvector $\mathbf{x}_1 = (0, 1, 1)^t$, and any eigenvector of A corresponding to $\lambda = 3$ is a nonzero multiple of $\mathbf{x}_1$.

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

In this case the eigenvector has only to satisfy the equation

$$x_1 - x_2 + 2x_3 = 0,$$

which can be done in various ways. For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$. We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$. Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$ that is not a multiple of $\mathbf{x}_2$. The eigenvectors of A corresponding to the eigenvalue $\lambda_2 = 2$ generate an entire plane. This plane is described by all vectors of the form

$$\alpha\mathbf{x}_2 + \beta\mathbf{x}_3 = (-2\beta, 2\alpha, \alpha + \beta)^t,$$

for arbitrary constants α and β, provided that at least one of the constants is nonzero. ■

Spectral Radius

Definition 7.14 The **spectral radius** $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \qquad \text{where } \lambda \text{ is an eigenvalue of } A.$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.) ■

Theorem 7.15 If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$,

(ii) $\rho(A) \leq \|A\|$, for any natural norm $\|\cdot\|$. ■

Proof The proof of part (i) requires more information concerning eigenvalues than we presently have available. For the details involved in the proof, see [Or2], p. 21.

To prove part (ii), suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$. Then $A\mathbf{x} = \lambda\mathbf{x}$ and

$$|\lambda| = |\lambda| \cdot \|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\| = \|A\|.$$

Thus

$$\rho(A) = \max |\lambda| \leq \|A\|.$$

■ ■ ■

Part (i) of Theorem 7.15 implies that if A is symmetric, then $\|A\|_2 = \rho(A)$ (see Exercise 14).

An interesting and useful result, which is similar to part (ii) of Theorem 7.15, is that for any matrix A and any $\varepsilon > 0$, there exists a natural norm $\|\cdot\|$ with the property that $\rho(A) < \|A\| < \rho(A) + \varepsilon$. Consequently, $\rho(A)$ is the greatest lower bound for the natural norms on A. The proof of this result can be found in [Or2], p. 23.

Example 3 Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Solution To apply Theorem 7.15 we need to calculate $\rho(A^tA)$, so we first need the eigenvalues of A^tA.

$$A^tA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}.$$

If

$$\begin{aligned} 0 = \det(A^tA - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda = -\lambda(\lambda^2 - 14\lambda + 42), \end{aligned}$$

then $\lambda = 0$ or $\lambda = 7 \pm \sqrt{7}$. By Theorem 7.15 we have

$$||A||_2 = \sqrt{\rho(A^tA)} = \sqrt{\max\{0, 7-\sqrt{7}, 7+\sqrt{7}\}} = \sqrt{7+\sqrt{7}} \approx 3.106. \qquad \blacksquare$$

Convergent Matrices

In studying iterative matrix techniques, it is of particular importance to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called *convergent*.

Definition 7.16 We call an $n \times n$ matrix A **convergent** if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n. \qquad \blacksquare$$

Example 4 Show that

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is a convergent matrix.

Solution Computing powers of A, we obtain:

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}, \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix},$$

and, in general,

$$A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}.$$

So A is a convergent matrix because

$$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{k}{2^{k+1}} = 0. \qquad \blacksquare$$

Notice that the convergent matrix A in Example 4 has $\rho(A) = \frac{1}{2}$, because $\frac{1}{2}$ is the only eigenvalue of A. This illustrates an important connection that exists between the spectral radius of a matrix and the convergence of the matrix, as detailed in the following result.

Theorem 7.17 The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$. ■

The proof of this theorem can be found in [IK], p. 14.

O princípio dos métodos iterativos é substituir o problema linear original:

$$A\,x = b$$

Por outro:

$$\mathbf{x}^{(i+1)} = T\mathbf{x}^{(i)} + \mathbf{c}.$$

onde na convergência as soluções são próximas dentro de uma tolerância especificada

Consequentemente para um método iterativo convergir a matriz T deve ser convergente.

Na próxima aula veremos formas de construir a matriz T a partir da matriz A e analisaremos a convergência

EXERCISE SET 7.2

1. Compute the eigenvalues and associated eigenvectors of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **e.** $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}$ **f.** $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

Fim...

AULA 08