Using the Jacobi identity and the commutation relations (2.116) we have that if α and β are roots then

$$\begin{aligned} [H_i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \\ &= (\alpha_i + \beta_i) [E_\alpha, E_\beta] \end{aligned} \tag{2.122}$$

Since the algebra is closed under the commutator we have that $[E_\alpha, E_\beta]$ must be an element of the algebra. We have then three possibilities

1. $\alpha + \beta$ is a root of the algebra and then $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$
2. $\alpha + \beta$ is not a root and then $[E_\alpha, E_\beta] = 0$
3. $\alpha + \beta = 0$ and consequently $[E_\alpha, E_\beta]$ must be an element of the Cartan subalgebra since it commutes with all H_i .

Since in a semisimple Lie algebra the roots are not degenerated (see (2.121)), we conclude from (2.122) that 2α is never a root.

We then see that the knowlegde of the roots of the algebra provides all the information about the commutation relations and consequently about the structure of the algebra. From what we have learned so far, we can write the commutation relations of a semisimple Lie algebra $\mathcal{G}$ as

$$[H_i, H_j] = 0 \tag{2.123}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.124}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ H_\alpha & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.125}$$

where $H_\alpha \equiv 2\alpha.H/\alpha^2$, $i, j = 1, 2, ...$ rank $\mathcal{G}$ (see discussion leading to (2.129) and (2.130)). The structure constants $N_{\alpha\beta}$ will be determined later. The basis $\{H_i, E_\alpha\}$ is called the *Weyl-Cartan basis* of a semisimple Lie algebra.

Using the cyclic property of the trace (2.47) (or equivalently, the invariance property (2.46)) we get that, in a given representation

$$Tr([H_i, E_\alpha]E_\beta) = Tr(E_\alpha[E_\beta, H_i]) \tag{2.126}$$

and so

$$(\alpha_i + \beta_i)Tr(E_\alpha E_\beta) = 0 \tag{2.127}$$

The step operators are orthogonal unless they have equal and opposite roots. In particular E_α is orthogonal to itself. If it was orthogonal to all others, the Killing form would have vanishing determinant and the algebra would not be semisimple. Therefore for semisimple algebras if α is a root then $-\alpha$ must also be a root, and $Tr(E_\alpha E_{-\alpha}) \neq 0$. The value of $Tr(E_\alpha E_{-\alpha})$ is connected to the structure constant of the second relation in (2.125). We know that $[E_\alpha, E_{-\alpha}]$ must be an element of the Cartan subalgebra. Therefore we write

$$[E_\alpha, E_{-\alpha}] = x_i H_i \tag{2.128}$$

Using (2.108) and the cyclic property of the trace we get

$$[\,E_\alpha\,,\,E_{-\alpha}\,] = \frac{2}{\alpha^2}\,\alpha_i\,H_i$$

$$\begin{aligned} Tr(x_i H_i H_j) &= x_j \\ &= Tr([E_\alpha, E_{-\alpha}]H_j) \\ &= Tr([H_j, E_\alpha]E_{-\alpha}) \\ &= \alpha_j Tr(E_\alpha E_{-\alpha}) \end{aligned} \tag{2.129}$$

Consequently $[E_\alpha, E_{-\alpha}]$ must be proportional to $\alpha.H$. Normalizing the step operators such that

$$Tr(E_\alpha E_{-\alpha}) = \frac{2}{\alpha^2} \tag{2.130}$$

we obtain the second relation in (2.125).

Again using the invariance property (2.46) we have that

$$Tr([H_i, E_\alpha]H_j) = Tr([H_j, H_i]E_\alpha) \tag{2.131}$$

and so

$$\alpha_i Tr(H_j E_\alpha) = 0 \tag{2.132}$$

Since by assumption α is a root and therefore different from zero we get

$$Tr(H_i E_\alpha) = 0 \tag{2.133}$$

From the above results and (2.108) we see that we can normalize the Cartan subalgebra generators H_i and the step operator E_α such that the Killing form becomes

$$\begin{aligned} Tr(H_i H_j) &= \delta_{ij} \; ; \quad i, j = 1, 2, ...rank\ \mathcal{G} \\ Tr(H_i E_\alpha) &= 0 \\ Tr(E_\alpha E_\beta) &= \frac{2}{\alpha^2}\delta_{\alpha+\beta,0} \end{aligned} \tag{2.134}$$

This is the usual normalization of the *Weyl-Cartan basis*.

Notice that linear combinations $(E_\alpha \pm E_{-\alpha})$ diagonalizes the Killing form (2.134). However, by taking real linear combinations of H_i, $(E_\alpha + E_{-\alpha})$ and $i(E_\alpha - E_{-\alpha})$ one obtains a compact algebra since the eigenvalues of the Killing form are all of the same sign. On the hand, if one takes real linear combinations of H_i, $(E_\alpha + E_{-\alpha})$ and $(E_\alpha - E_{-\alpha})$ one obtains a non compact algebra.

Example 2.6 *In section 2.5 we have discussed the algebra of the group $SU(2)$. In that case the Cartan subalgebra is generated by T_3 only. The step operators are T_+ and T_- corresponding to the roots $+1$ and -1 respectively . So the rank of $SU(2)$ is one. We can represent these roots by the diagram 2.1*

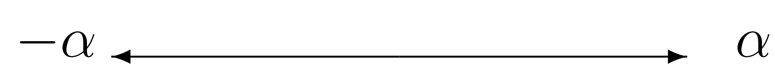


Figure 2.1: The root diagram of A_1 ($su(2)$,so(3) or $sl(2)$)

The algebra $SU(3)$

The group $SU(3)$: 3×3 complex unitary matrices with unity determinant

$g = e^{i\,T} \longrightarrow$ T is hemitian

det g =1 $\longrightarrow$ Tr(T) =0

$2.3^2 - 3^2 - 1 = 8$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ;$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; ;$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; ;$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \; ; \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$Tr(\lambda_i \lambda_j) = 2\delta_{ij}$$

simple and compact

Commutation relations

$$[\lambda_i, \lambda_j] = i f_{ijk} \lambda_k$$

i	j	k	f_{ijk}
1	2	3	2
1	4	7	1
1	5	6	-1
2	4	6	1
2	5	7	1
3	4	5	1
3	6	7	-1
4	5	8	$\sqrt{3}$
6	7	8	$\sqrt{3}$

Cartan subalgebra: λ_3 and λ_8

$$H_1 = \frac{1}{\sqrt{2}}\lambda_3 \; ; \qquad H_2 = \frac{1}{\sqrt{2}}\lambda_8 \; ; \qquad Tr(H_i H_j) = \delta_{ij}$$

$$E_{\pm\alpha_1} = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \; ; \qquad E_{\pm\alpha_2} = \frac{1}{2}(\lambda_6 \pm i\lambda_7) \qquad Tr(E_{\alpha_m} E_{-\alpha_n}) = \delta_{mn}$$

$$E_{\pm\alpha_3} = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$

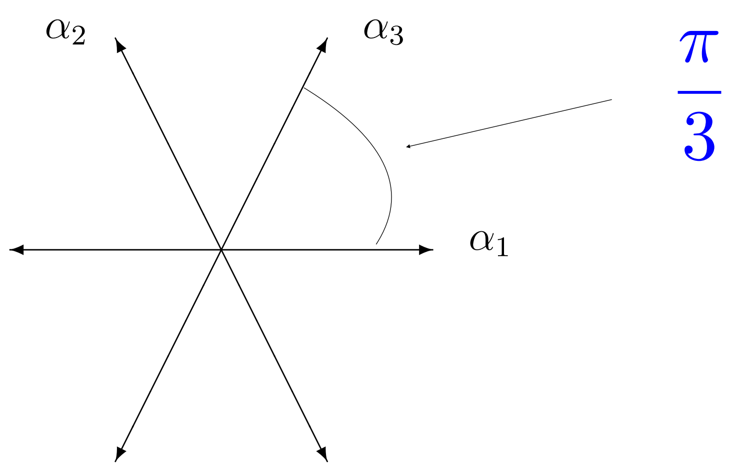
Roots

$$[H_1, E_{\pm\alpha_1}] = \pm\sqrt{2}E_{\pm\alpha_1} \ ; \qquad [H_2, E_{\pm\alpha_1}] = 0 \ ;$$

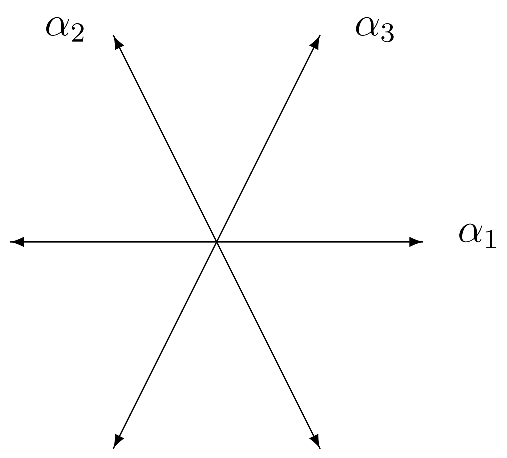
$$[H_1, E_{\pm\alpha_2}] = \mp\frac{\sqrt{2}}{2}E_{\pm\alpha_2} \ ; \qquad [H_2, E_{\pm\alpha_2}] = \pm\sqrt{\frac{3}{2}}E_{\pm\alpha_2} \ ;$$

$$[H_1, E_{\pm\alpha_3}] = \pm\frac{\sqrt{2}}{2}E_{\pm\alpha_3} \ ; \qquad [H_2, E_{\pm\alpha_3}] = \pm\sqrt{\frac{3}{2}}E_{\pm\alpha_3}$$

root diagram



$$\alpha_1 = (\sqrt{2}, 0) \ ; \quad \alpha_2 = (-\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}) \ ; \quad \alpha_3 = (\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}) \qquad \alpha^2 = 2$$



From the root diagram one can read:

$$[E_{\alpha_1}, E_{\alpha_3}] = [E_{\alpha_3}, E_{\alpha_2}] = [E_{\alpha_2}, E_{-\alpha_1}] = 0$$
$$[E_{-\alpha_1}, E_{-\alpha_3}] = [E_{-\alpha_3}, E_{-\alpha_2}] = [E_{-\alpha_2}, E_{\alpha_1}] = 0$$

$$\begin{aligned}
[E_{\alpha_1}, E_{-\alpha_1}] &= \sqrt{2}H_1 \\
[E_{\alpha_2}, E_{-\alpha_2}] &= -\frac{\sqrt{2}}{2}H_1 + \sqrt{\frac{3}{2}}H_2 \\
[E_{\alpha_3}, E_{-\alpha_3}] &= \frac{\sqrt{2}}{2}H_1 + \sqrt{\frac{3}{2}}H_2
\end{aligned}$$

$$\begin{aligned}
[E_{\alpha_1}, E_{\alpha_2}] = E_{\alpha_3} \ ; &\qquad [E_{-\alpha_1}, E_{-\alpha_2}] = -E_{-\alpha_3} \\
[E_{\alpha_1}, E_{-\alpha_3}] = -E_{-\alpha_2} \ ; &\qquad [E_{-\alpha_1}, E_{\alpha_3}] = E_{\alpha_2} \ ; \\
[E_{\alpha_3}, E_{-\alpha_2}] = E_{\alpha_1} \ ; &\qquad [E_{-\alpha_3}, E_{\alpha_2}] = -E_{-\alpha_1}
\end{aligned}$$

Algebra of $SU(3)$: real linear combinations of

$$H_i \;; \qquad (E_{\alpha_m} + E_{-\alpha_m}) \;; \qquad i\,(E_{\alpha_m} - E_{-\alpha_m}) \qquad \begin{array}{l} i = 1,2 \\ m = 1,2,3 \end{array}$$

Algebra of $SL(3)$: real linear combinations of

$$H_i \;; \qquad (E_{\alpha_m} + E_{-\alpha_m}) \;; \qquad (E_{\alpha_m} - E_{-\alpha_m})$$

Some other non-compact form: real linear combinations of

$$H_i \;; \qquad (E_{\alpha_a} + E_{-\alpha_a}) \;; \qquad i\,(E_{\alpha_a} - E_{-\alpha_a}) \qquad i = 1,2$$

$$(E_{\alpha_3} + E_{-\alpha_3}) \qquad (E_{\alpha_3} - E_{-\alpha_3}) \qquad a = 1,2$$

2.8 The Properties of roots

We have seen that for a semisimple Lie algebra $\mathcal{G}$, if α is a root then, $-\alpha$ is also a root. This means that for each step operator E_α there exists a corresponding step operator E_α. Together with $H_\alpha = 2\alpha.H/\alpha^2$ they constitute a $sl(2)$ subalgebra of $\mathcal{G}$, since from (2.124) and (2.125) one gets

$$\begin{aligned} [H_\alpha, E_{\pm\alpha}] &= \pm 2E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha \end{aligned} \tag{2.145}$$

This subalgebra is isomorphic to $sl(2)$ since H_α plays the role of $2T_3$, E_α and $E_{-\alpha}$ play the role of T_+ and T_- respectively (see section 2.5). Therefore to each pair of roots α and $-\alpha$ we can construct a $sl(2)$ subalgebra. These subalgebras, however, do not have to commute among themselves.

We have learned in section 2.5 that T_3 , the third component of the angular momentum, has half integer eigenvalues, and consenquently H_α ($\equiv 2T_3$) must have integer eigenvalues. From (2.124) we have

$$[H_\alpha, E_\beta] = \frac{2\alpha.\beta}{\alpha^2} E_\beta \tag{2.146}$$

Therefore if $\mid m\rangle$ is an eigenstate of H_α with an integer eigenvalue m them the state $E_\beta \mid m\rangle$ has eigenvalue $m + \frac{2\alpha.\beta}{\alpha^2}$ since

$$\begin{aligned} H_\alpha E_\beta \mid m\rangle &= (E_\beta H_\alpha + [H_\alpha, E_\beta]) \mid m\rangle \\ &= \left(m + \frac{2\alpha.\beta}{\alpha^2}\right) E_\beta \mid m\rangle \end{aligned} \tag{2.147}$$

This implies that

$$\frac{2\alpha.\beta}{\alpha^2} = integer \tag{2.148}$$

for any roots α and β. This result is crucial in the study of the structure of semisimple Lie algebras. In order to satisfy this condition the roots must have some very special properties. From Schwartz inequality we get (The roots live in a Euclidean space since they inherit the scalar product from the Killing form of $\mathcal{G}$ restricted to the Cartan subalgebra by $\alpha.\beta \equiv Tr(\alpha.H\beta.H) = \sum_{i=1}^{rank\mathcal{G}} \alpha_i\beta_i$)

$$\alpha.\beta = \mid \alpha \mid\mid \beta \mid \cos\theta \leq \mid \alpha \mid\mid \beta \mid \tag{2.149}$$

where θ is the angle between α and β. Consenquently

$$\frac{2\alpha.\beta}{\alpha^2}\frac{2\alpha.\beta}{\beta^2} = mn = 4(\cos\theta)^2 \leq 4 \tag{2.150}$$

where m and n are integers according to (2.148), and so

$$0 \leq mn \leq 4 \tag{2.151}$$

$\frac{2\alpha.\beta}{\alpha^2}$	$\frac{2\alpha.\beta}{\beta^2}$	θ	$\frac{\alpha^2}{\beta^2}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

$$m\,n = 4$$

1. $\frac{2\alpha.\beta}{\alpha^2} = \pm 2$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 2$ $\longrightarrow$ $\alpha = \pm\beta$
2. $\frac{2\alpha.\beta}{\alpha^2} = \pm 1$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 4$
3. $\frac{2\alpha.\beta}{\alpha^2} = \pm 4$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 1$

$\alpha = \pm 2\,\beta$

$\frac{\alpha^2}{\beta^2} = 2$ and $\frac{\alpha^2}{\gamma^2} = 3$ then it follows that $\frac{\gamma^2}{\beta^2} = \frac{2}{3}$

only two lengths allowed

2.9 The Weyl group

In the section 2.8 we have shown that to each pair of roots α and $-\alpha$ of a semisimple Lie algebra we can construct a $sl(2)$ (or $su(2)$) subalgebra generated by the operators H_α , E_α and $E_{-\alpha}$ (see eq. (2.145)). We now define the hermitian operators:

$$\begin{aligned} T_1(\alpha) &= \frac{1}{2}(E_\alpha + E_{-\alpha}) \\ T_2(\alpha) &= \frac{1}{2i}(E_\alpha - E_{-\alpha}) \end{aligned} \tag{2.152}$$

which satisfy the commutation relations

$$\begin{aligned} [H_i, T_1(\alpha)] &= i\alpha_i T_2(\alpha) \\ [H_i, T_2(\alpha)] &= -i\alpha_i T_1(\alpha) \\ [T_1(\alpha), T_2(\alpha)] &= \frac{i}{2} H_\alpha \end{aligned} \tag{2.153}$$

The operator $T_2(\alpha)$ is the generator of rotations about the 2-axis, and a rotation by π is generated by the element

$$S_\alpha = \exp(i\pi T_2(\alpha)) \tag{2.154}$$

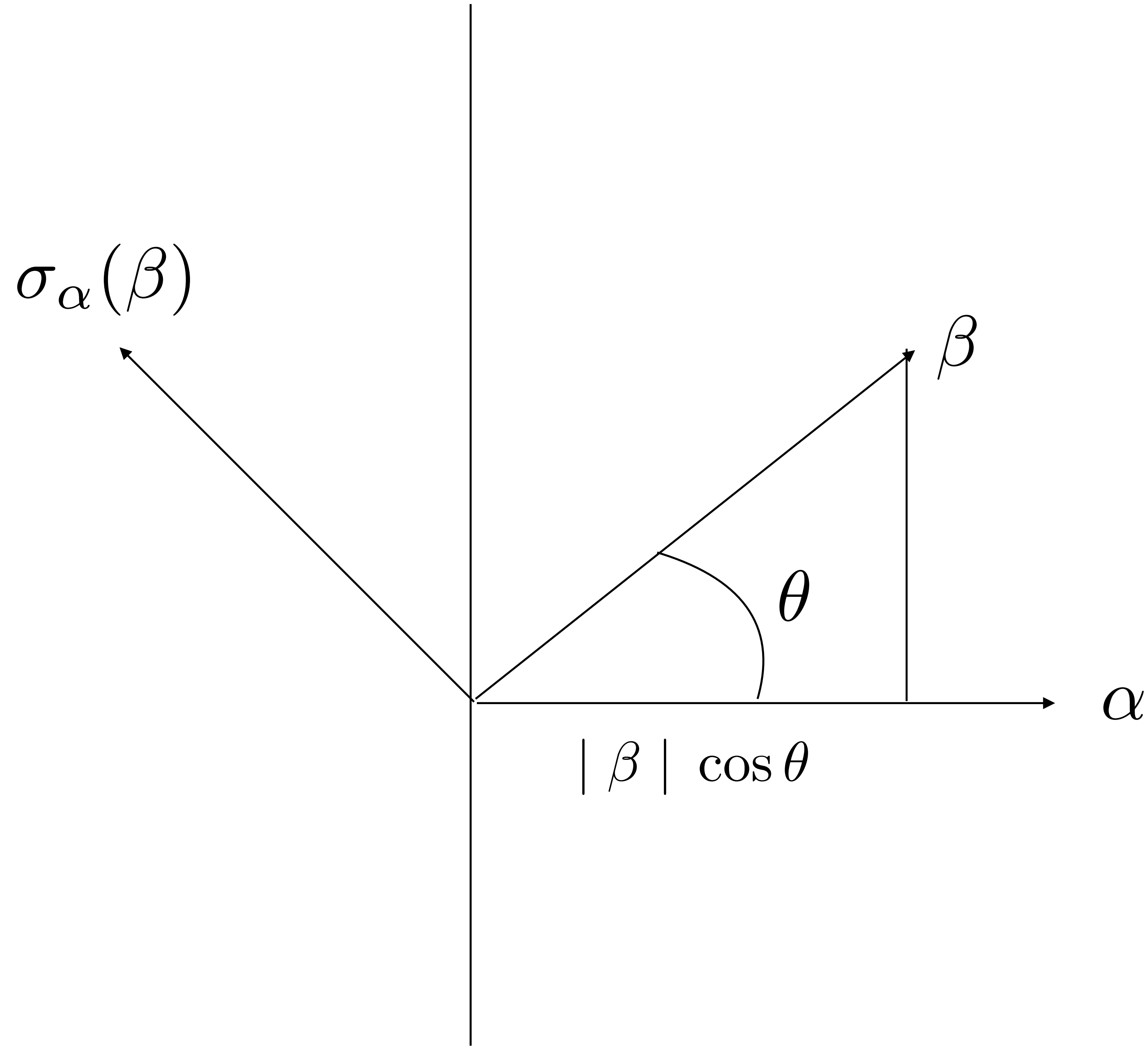
Using (2.27) and (2.153) one can check that

$$\begin{aligned} S_\alpha (x.H) S_\alpha^{-1} &= x.H + x.\alpha T_1(\alpha) \sin\pi + \frac{x.\alpha}{\alpha^2} \alpha.H (\cos\pi - 1) \\ &= \left(x_i - 2\frac{x.\alpha}{\alpha^2} \alpha_i \right) H_i \\ &= \sigma_\alpha(x).H \end{aligned} \tag{2.155}$$

where we have defined the operator σ_α, acting on the root space, by

$$\sigma_\alpha(x) \equiv x - 2\frac{x.\alpha}{\alpha^2}\alpha \tag{2.156}$$

This operator corresponds to a reflection w.r.t the plane perpendicular to α. Indeed, if θ is the angle between x and α then $\frac{x.\alpha}{\alpha^2}\alpha = \mid x \mid \cos\theta \frac{\alpha}{|\alpha|}$. Therefore $\sigma_\alpha(x)$ is obtained from x by subtracting a vector parallel (or anti-parallel) to α and with lenght twice the projection of x in the direction of α. These reflections are called *Weyl reflections* on the root space.



$$\sigma_\alpha(\beta) = \beta - 2\frac{\alpha\cdot\beta}{\alpha^2}\,\alpha = \beta - 2\ |\beta|\ \cos\theta\,\frac{\alpha}{|\alpha|}$$

We now want to show that if α and β are roots of a given Lie algebra $\mathcal{G}$, then $\sigma_\alpha(\beta)$ is also a root. Let us introduce the operator

$$\tilde{E}_\beta \equiv S_\alpha E_\beta S_\alpha^{-1} \tag{2.157}$$

where E_β is a step operator of the algebra and S_α is defined in (2.154). From the fact that (see (2.124))

$$[x.H, E_\beta] = x.\beta E_\beta \tag{2.158}$$

we get, using (2.155) that

$$\begin{aligned}
S_\alpha[x.H, E_\beta]S_\alpha^{-1} &= [S_\alpha x.H S_\alpha^{-1}, S_\alpha E_\beta S_\alpha^{-1}] \\
&= [\sigma_\alpha(x).H, \tilde{E}_\beta] \qquad (2.159)\\
&= x.\beta S_\alpha E_\beta S_\alpha^{-1} \qquad (2.160)\\
&= x.\beta \tilde{E}_\beta \qquad (2.161)
\end{aligned}$$

and so

$$[\sigma_\alpha(x).H, \tilde{E}_\beta] = x.\beta \tilde{E}_\beta \tag{2.162}$$

However, if we perform a reflection twice we get back to where we started, i.e., $\sigma^2 = 1$. Therefore denoting $\sigma_\alpha(x)$ by y we get that $\sigma_\alpha(y) = x$, and then from (2.162)

$$[y.H, \tilde{E}_\beta] = \sigma_\alpha(y).\beta \tilde{E}_\beta \tag{2.163}$$

and so

$$[H_i, \tilde{E}_\beta] = \sigma_\alpha(\beta)_i \tilde{E}_\beta \tag{2.164}$$

Therefore $\tilde{E}_\beta$, defined in (2.157), is a step operator corresponding to the root $\sigma_\alpha(\beta)$. Consequently if α and β are roots, $\sigma_\alpha(\beta)$ is necessarily a root (similarly $\sigma_\beta(\alpha)$).

Example 2.7 *In section 2.7 we have discussed the algebra of the group $SU(3)$. The root diagram with the planes perpendicular to the roots is given in figure 2.3. One can sees that the root diagram is invariant under Weyl reflections. We have*

$$\begin{array}{llll} \sigma_1: & \alpha_1 \leftrightarrow -\alpha_1 & \alpha_2 \leftrightarrow \alpha_3 & -\alpha_2 \leftrightarrow -\alpha_3 \\ \sigma_2: & \alpha_1 \leftrightarrow \alpha_3 & \alpha_2 \leftrightarrow -\alpha_2 & -\alpha_1 \leftrightarrow -\alpha_3 \\ \sigma_3: & \alpha_1 \leftrightarrow -\alpha_2 & \alpha_2 \leftrightarrow -\alpha_1 & \alpha_3 \leftrightarrow -\alpha_3 \end{array}$$

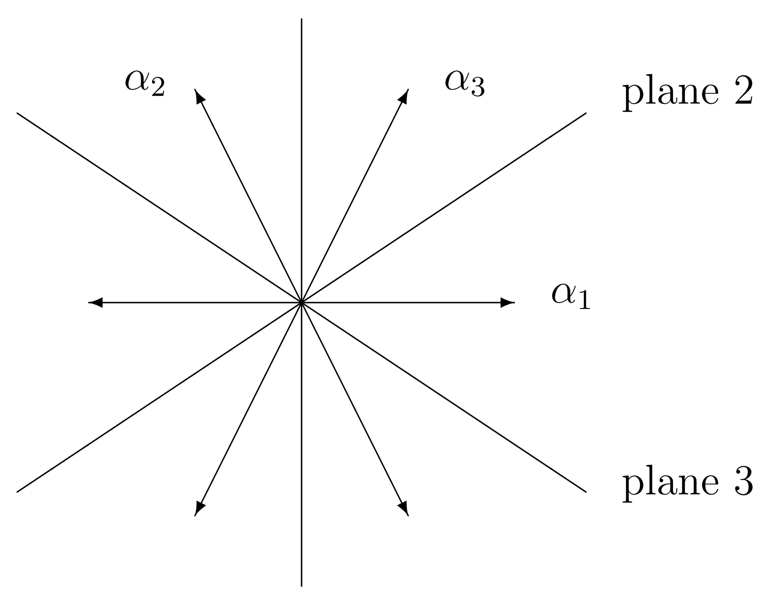


Figure 2.3: The planes orthogonal to the roots of A_2 ($SU(3)$ or $SL(3)$)

$$\begin{array}{ll} \sigma_1\sigma_2: & \left\{ \begin{array}{lll} \alpha_1 \to \alpha_2 & \alpha_2 \to -\alpha_3 & \alpha_3 \to -\alpha_1 \\ -\alpha_1 \to -\alpha_2 & -\alpha_2 \to \alpha_3 & -\alpha_3 \to \alpha_1 \end{array} \right. \\ \sigma_2\sigma_1: & \left\{ \begin{array}{lll} \alpha_1 \to -\alpha_3 & \alpha_2 \to \alpha_1 & \alpha_3 \to -\alpha_2 \\ -\alpha_1 \to \alpha_3 & -\alpha_2 \to -\alpha_1 & -\alpha_3 \to \alpha_2 \end{array} \right. \end{array} \tag{2.165}$$

Notice that the composition of Weyl reflections is not necessarily a reflection and that reflections do not commute. In this particular case the operation $\sigma_2\sigma_1$ is a rotation by an angle of $\frac{2\pi}{3}$ and $\sigma_1\sigma_2$ is its inverse. The set of a Weyl reflexions and the composition of two or more of them form a group called the Weyl group*. It leaves the root diagram of $su(3)$ invariant. This group is isomorphic to S_3 , and in fact the Weyl group of $su(N)$ is S_N , the group of permutations of N elements.*

Definition 2.15 *The* Weyl group *of a Lie algebra, or of its root system, is the finite discrete group generated by the Weyl reflections.*

$$\sigma_\alpha \ , \ \sigma_\beta \ , \ \sigma_\gamma \ \ldots$$

$$\sigma_\alpha \, \sigma_\beta \ , \ \sigma_\alpha \, \sigma_\gamma \ , \ \sigma_\gamma \, \sigma_\beta \ \ldots$$

$$\sigma_\alpha \, \sigma_\beta \, \sigma_\gamma \ \ldots$$

It is generated by reflection, but not all elements are reflections

It leaves the root system invariant.

But it does not necessarily contain all the symmetries of the root system.

inversion $\alpha \leftrightarrow -\alpha$ rotations of $\frac{\pi}{3}$ are symmetries of the $SU(3)$ root system.

But do not belong to its Weyl group.

Automorphisms

Conjugation by $S_\alpha = \exp(i\pi T_2(\alpha))$ map

$x \cdot H \to \sigma_\alpha(x) \cdot H$ $E_\beta \to E_{\sigma_\alpha(\beta)}$

Those are inner automorphisms

Symmetries of the root system which are not in the Weyl group give rise to outer automorphisms

Example:

$H_i \to -H_i,\ E_\alpha \to -E_{-\alpha}$ and $E_{-\alpha} \to -E_\alpha$

Due to the invariance of root system under $\alpha \leftrightarrow -\alpha$

Root Systems or Root Diagrams

Definition 2.16 *A set Φ of vectors in a Euclidean space is the* root system *or* root diagram *of a semisimple Lie algebra $\mathcal{G}$ if*

1. *Φ does not contain zero, spans an Euclidean space of the same dimension as the rank of the Lie algebra $\mathcal{G}$ and the number of elements of Φ is equal to dim $\mathcal{G}$ - rank $\mathcal{G}$.*

2. *If $\alpha \in \Phi$ then the only multiples of α in Φ are $\pm\alpha$*

3. *If $\alpha, \beta \in \Phi$, then $\frac{2\alpha.\beta}{\alpha^2}$ is an integer*

4. *If $\alpha, \beta \in \Phi$, then $\sigma_\alpha(\beta) \in \Phi$, i.e., the Weyl group leaves Φ invariant.*

Root diagrams of simple Lie algebras
can not decompose into orthogonal sub-diagrams

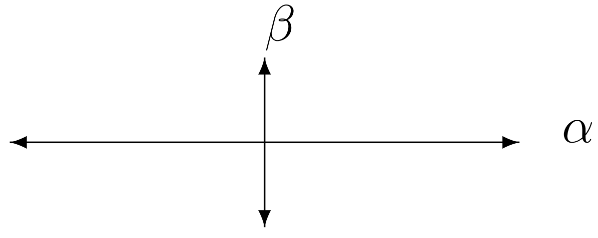


Figure 2.4: The root diagram of $su(2) \oplus su(2)$

Weyl Chambers

The hyperplanes perpendicular to the roots, defined in section 2.9 partition the root space into finitely many regions. These connected regions (without the hyperplanes) are called *Weyl Chambers* . Due to the regularity of the root systems all the Weyl chambers have the same form and are equivalent.

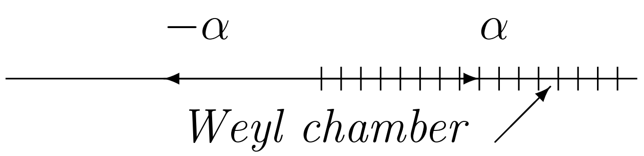


Figure 2.5: The Weyl chambers of A_1 ($su(2)$,so(3) or $sl(2)$)

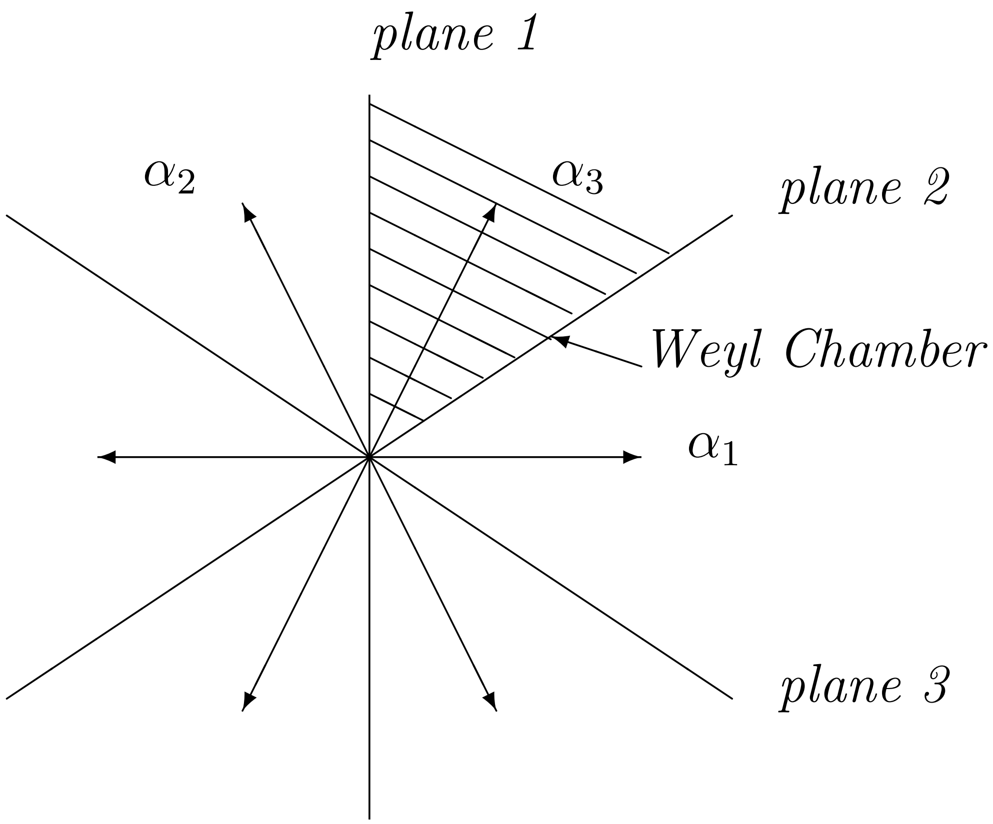
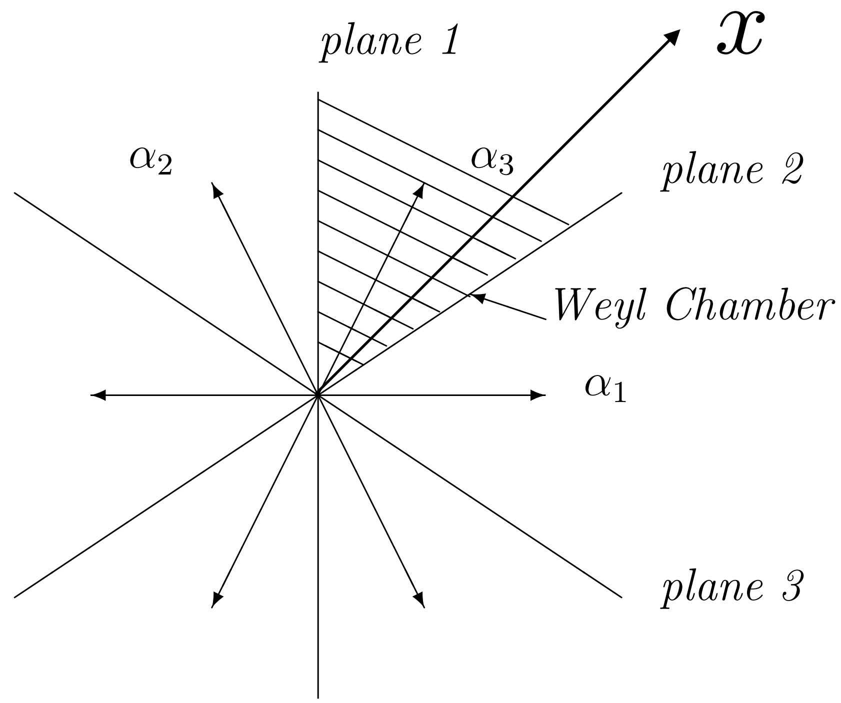


Figure 2.6: The Weyl chambers of A_2 ($SU(3)$ or $SL(3)$)

Fundamental Weyl Chamber (FWC): just a choice

Positive roots



$$x \cdot \alpha > 0 \to \alpha > 0$$

Definition 2.17 *Let x be any vector inside the Fundamental Weyl chamber. We say α is a* positive root *if $\alpha.x > 0$ and a* negative root *if $\alpha.x < 0$.*

Simple Roots

Definition 2.18 *We say a positive root is a* simple root *if it can not be written as the sum of two positive roots.*

Theorem 2.5 *Let α and β be non proportional roots. Then*

1. *if $\alpha.\beta > 0$, $\alpha - \beta$ is a root*

2. *if $\alpha.\beta < 0$, $\alpha + \beta$ is a root*

Proof If $\alpha.\beta > 0$ we see from table 2.2 that either $\frac{2\alpha.\beta}{\alpha^2}$ or $\frac{2\alpha.\beta}{\beta^2}$ is equal to 1. Without loss of generality we can take $\frac{2\alpha.\beta}{\alpha^2} = 1$. Therefore

$$\sigma_\alpha(\beta) = \beta - \frac{2\alpha.\beta}{\alpha^2}\alpha = \beta - \alpha \tag{2.166}$$

So, from the invariance of the root system under the Weyl group, $\beta - \alpha$ is also a root, as well as $\alpha - \beta$. The proof for the case $\alpha.\beta < 0$ is similar. $\Box$

Theorem 2.6 *Let α and β be distinct simple roots. Then $\alpha - \beta$ is not a root and $\alpha.\beta \leq 0$.*

Proof Suppose $\alpha - \beta \equiv \gamma$ is a root. If γ is positive we write $\alpha = \gamma + \beta$, and if it is negative we write $\beta = \alpha + (-\gamma)$. In both cases we get a contradiction to the fact α and β are simple. Therefore $\alpha - \beta$ can not be a root. From theorem 2.5 we conclude $\alpha.\beta$can not be positive. $\Box$

Theorem 2.7 *Let $\alpha_1, \alpha_2, \ldots \alpha_r$ be the set of all simple roots of a semisimple Lie algebra $\mathcal{G}$. Then $r = \text{rank}\, \mathcal{G}$ and each root α of $\mathcal{G}$ can be written as*

$$\alpha = \sum_{a=1}^{r} n_a \alpha_a \tag{2.167}$$

where n_a are integers, and they are positive or zero if α is a positive root and negative or zero if α is a negative root.

Proof Suppose the simple roots are linear dependent. Denote by x_a and $-y_a$ the positive and negative coefficients, respectively, of a vanishing linear combination of the simple roots. Then write

$$\sum_{a=1}^{s} x_a \alpha_a = \sum_{b=s+1}^{r} y_b \alpha_b \equiv v \tag{2.168}$$

with each α_a being different from each α_b. Therefore

$$v^2 = \sum_{ab} x_a y_b \alpha_a . \alpha_b \leq 0 \tag{2.169}$$

Since v is a vector on an Euclidean space it follows that that the only possibility is $v^2 = 0$, and so $v = 0$. But this implies $x_a = y_b = 0$ and consequently the simple roots must be linear independent. Now let α be a positve root. If it is not simple then $\alpha = \beta + \gamma$ with β and γ both positive. If β and/or γ are not simple we can write them as the sum of two positive roots. Notice that α can not appear in the expansion of β and/or γ in terms of two positive roots, since if x is a vector of the Fundamental Weyl Chamber we have $x.\alpha = x.\beta + x.\gamma$. Since they are all positive roots we have $x.\alpha > x.\beta$ and $x.\alpha > a.\gamma$. Therefore β or γ can not be written as $\alpha + \delta$ with δ a positive root. For the same reason β and γ will not appear in the expansion of any further root appearing in this process. Thus, we can continue such process until α is written as a sum of simple roots, i.e. $\alpha = \sum_{a=1}^{r} n_a \alpha_a$ with each n_a being zero or a positive integer. Since, for semisimple Lie algebras, the roots come in pairs (α and $-\alpha$) it follows that the negative roots are written in terms of the simple roots in the same way, with n_a being zero or negative integers. We then see that the set of simple roots span the root space. Since they are linear independent, they form a basis and consequently $r = \text{rank}\, \mathcal{G}$. $\square$

Cartan Matrix

$$K_{ab} \equiv \frac{2\alpha_a.\alpha_b}{\alpha_b^2} \qquad a\,,\,b = 1, 2, 3\ldots, r \equiv \text{rank}$$

Wraps away redundancy in choice of FWC

$SU(2):$ $$K = 2$$

$SO(4) = SU(2) \oplus SU(2):$ $$K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$SU(3):$ $$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

1. It provides the angle between any two simple roots since

$$K_{ab}K_{ba} = 4\frac{\alpha_a.\alpha_b}{\alpha_b^2}\frac{\alpha_a.\alpha_b}{\alpha_a^2} \tag{2.171}$$

with no summation on a or b, and so

$$\cos\theta = -\frac{1}{2}\sqrt{K_{ab}K_{ba}} \tag{2.172}$$

where θ is the angle between α_a and α_b. We take the minus sign because, according to theorem 2.6, the simple roots always form obtuse angles.

2. The Cartan matrix gives the ratio of the lenghts of any two simple roots since

$$\frac{K_{ab}}{K_{ba}} = \frac{\alpha_a^2}{\alpha_b^2} \tag{2.173}$$

3. $K_{aa} = 2$. The diagonal elements do not give any information.

4. From the properties of the roots discussed in section 2.8 we see that

$$K_{ab}K_{ba} = 4\left(\cos\theta\right)^2 = 0, 1, 2, 3 \tag{2.174}$$

we do not get 4 because we are taking $a \neq b$. But from theorem 2.6 we have $\alpha_a.\alpha_b \leq 0$ and so the off diagonal elements of the Cartan matrix can take the values

$$K_{ab} = 0, -1, -2, -3 \tag{2.175}$$

with $a \neq b$. From the table 2.2 we see that if $K_{ab} = -2$ or -3 then we necessarily have $K_{ba} = -1$.

5. If α_a and α_b are orthogonal, obviously $K_{ab} = K_{ba} = 0$. At the end of section 2.9 we have shown that if the root diagram decomponses into two or more mutually orthogonal subdiagrams then the corresponding algebra is not simple. As a consequence of that if follows that the Cartan matrix of a Lie algebra, which is not simple, necessarily has a block-diagonal form.

6. The Cartan matrix is symmetric only when all roots have the same lenght.

Dynkin Diagrams

1. Draw r points, each corresponding to one of the r simple roots of the algebra (r is the rank of the algebra).

2. Join the point a to the point b by $K_{ab}K_{ba}$ lines. Remember that the number of lines can be 0, 1, 2 or 3.

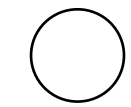
3. If the number of lines joining the points a and b exceeds 1 put an arrow on the lines directed towards the one whose corresponding simple root has a shorter lenght than the other.

When $K_{ab}K_{ba} = 2$ or 3 the corresponding simple roots, α_a and α_b , have different lenghts. In order to see this, remember that K_{ab} or K_{ba} is equal to -1. Taking $K_{ab} = -1$, we have $K_{ba} = -K_{ab}K_{ba} = -2$ or -3. But

$$\frac{\alpha_a^2}{\alpha_b^2} = \frac{K_{ab}}{K_{ba}} = \frac{1}{K_{ab}K_{ba}} \tag{2.180}$$

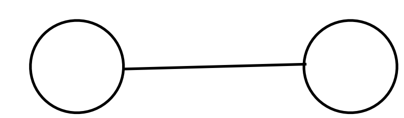
and consenquently $\alpha_b^2 \geq \alpha_a^2$. So the number of lines joining two points in a Dynkin diagram gives the ratio of the squared lenghts of the corresponding simple roots.

$SU(2):$ $\quad K = 2$



$SO(4) = SU(2) \oplus SU(2):$ $\quad K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$SU(3):$ $\quad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$



$B_2 \equiv SO(5)$

Example 2.14 *The algebra of* $SO(5)$ *has dimension* 10 *and rank* 2*. So it has* 8 *roots. It root diagram is shown in figure 2.7. The Fundamental Weyl Chamber is the shaded region. Notice that all roots lie on the hyperplanes perpendicular to the roots. The positive roots are* α_1, α_2, α_3 *and* α_4 *as shown on the diagram. All the others are negative. The simple roots are* α_1 *and* α_2*, and the ratio of their squared lenghts is* 2*. The angle between them is* $\frac{3\pi}{4}$*. The Cartan matrix of* $so(5)$ *is*

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{2.178}$$

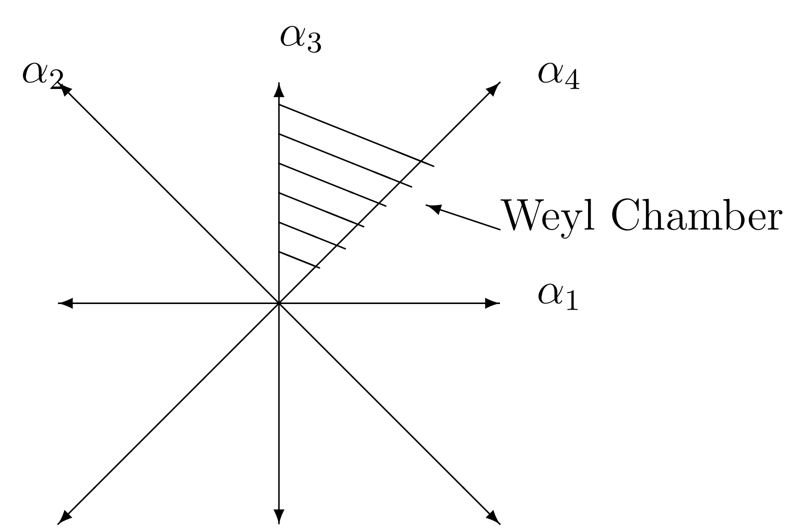
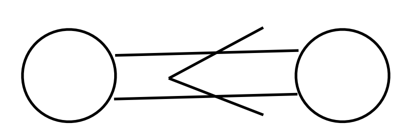


Figure 2.7: The root diagram and Fundamental Weyl chamber of $so(5)$ (or $sp(2)$)



G_2

Example 2.15 *The last simple Lie algebra of rank* 2 *is the exceptional algebra* G_2 *. Its root diagram is shown in figure 2.8. It has* 12 *roots and therefore dimension* 14*. The Fundamental Weyl Chamber is the shaded region. The positive roots are the ones labelled from* 1 *to* 6 *on the diagram. The simple roots are* α_1 *and* α_2*. The Cartan matrix is given by*

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{2.179}$$

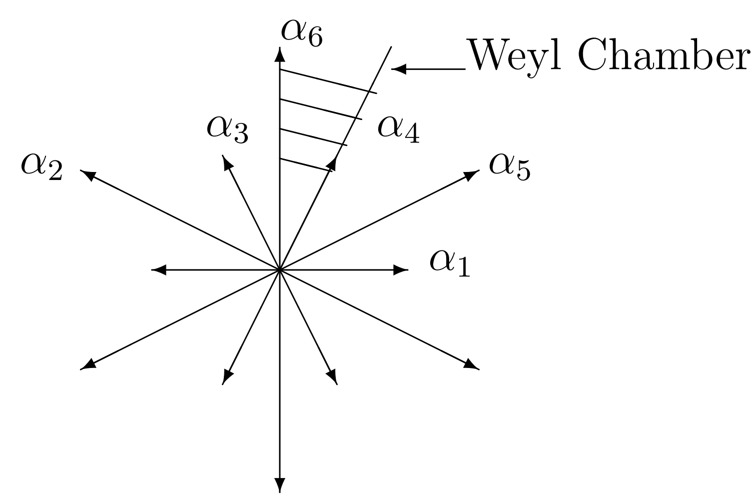


Figure 2.8: The root diagram and Fundamental Weyl Chamber of G_2