

Calculation of the basins of attraction

PEF5737 - Non-linear dynamics and stability

Prof. Dr. Carlos E. N. Mazzilli
Prof. Dr. **Guilherme Rosa Franzini**

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- 3 Mapped region and attractors
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- 5 Example of application: Helmholtz-Duffing oscillator
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- To present a numerical method for calculating the basins of attraction (BoA) of a non-linear equation;
- To discuss how to obtain two measurements of dynamic integrity;
- To present an example of basins of attraction obtained for a 1-dof system;
- To introduce the study of higher-order models.

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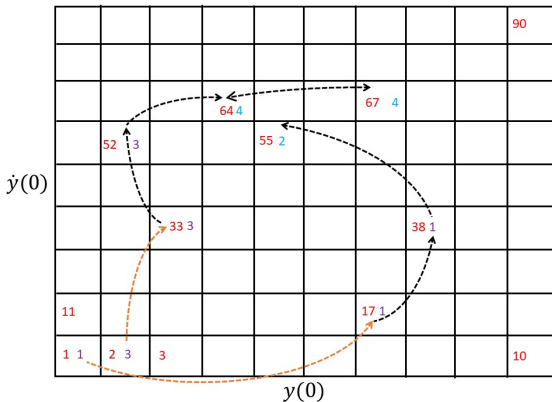
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- **Basins of Attraction:** Set of initial conditions associated with a particular attractor;
- **Focus of this class:** The grid-of-start and the modified cell-mapping methods applied to a (non-autonomous) harmonically forced system.

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- The attractors are identified as the center of the cells to which the trajectory initiated at a particular initial condition converges when $t \rightarrow \infty$;

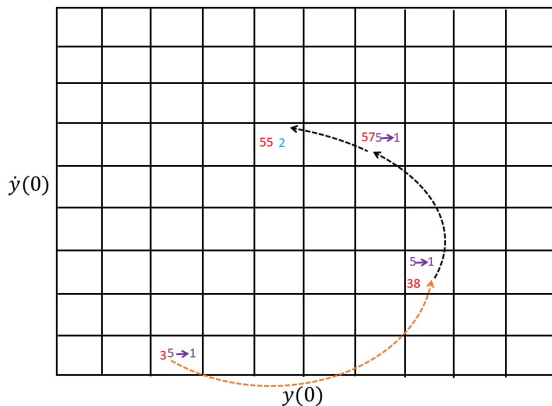
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- The attractors are identified as the center of the cells to which the trajectory initiated at a particular initial condition converges when $t \rightarrow \infty$;
- Aiming at knowing your system, it is desirable to simulate the response for some initial conditions. For a non-autonomous system, you can simulate for a number of forcing period (say, for example, 500) and neglecting the first 100 (for example) periods as the transitory regime. The attractor can be identified by considering the stroboscopic Poincaré's sections for the last 400 periods.



- Once you have selected the region of interest and the discretization to be employed, each cell must be numbered. So, you must be able to associate the center of the cell with its number and *vice-versa*. In above figure, the number of each cell is indicated in red. Notice that the dimension y was discretized in 10 elements and the direction \dot{y} in 9 elements;

- We firstly start the simulations considering as initial condition the center of the cell number 1. Imagine that, at the end of the transitory you choose (for example, after 100 periods of the forcing), the trajectory falls inside cell number 17. In this class, the trajectories are highlighted in purple. So the first two cells visited by trajectory 1 are $1 \rightarrow 17$;
- We simulate the system for one forcing period using as initial condition the point obtained at the end of the transitory. It is important to remark that, if the classical cell mapping is used, you need to adopt as initial condition the center of the cell to which the trajectory lies at the end of the transitory response;
- Imagine that, after one period, the trajectory stops in a point pertaining to cell 38. Hence, the updated trajectory number 1 is $1 \rightarrow 17 \rightarrow 38$. Repeating the procedure and simulating for another period, the trajectory falls in a certain point of cell 55. The updated trajectory 1 is, then, $1 \rightarrow 17 \rightarrow 38 \rightarrow 55$;
- Consider that, repeating the procedure, the simulation which starts at the end of the previous simulation leads, after one period, to another point inside cell 55. In this case, the center of cell 55 is a period-1 attractor, to which we will assign number 2. It is common to use odd number for the trajectory (basins of attraction) and even numbers for the attractors. So, part of the basin of attraction of attractor 2 is $1 \rightarrow 17 \rightarrow 38$.

- We start to compute the new trajectory (trajectory 3) using as initial condition the center of cell 2. We repeat the approach and we find, for example, that trajectory 3 is given by $2 \rightarrow 33 \rightarrow 52 \rightarrow 64 \rightarrow 67$;
- Imagine that the numerical integration for one period using as initial condition the last mapped point (not the center of cell 67) leads to a point pertaining to cell 64. In this case, cells 64 and 67 compose attractor 4 and trajectory 3 is $2 \rightarrow 33 \rightarrow 52$.
- It is natural to start trajectory 5 using as initial condition the center of cell 3. Imagine that, at the end of the transitory, the trajectory falls in a point inside cell 38. So, the updated trajectory 5 is $3 \rightarrow 38$.



- Imagine that, after another forcing period, the trajectory visits cell **55**. Notice that this cell has already been visited and corresponds to attractor **2**. In this case, you must assign **1** to this trajectory, provided it is part of the basins of attraction of attractor **2**;

- What happens when the trajectory escapes the mapped region?
- There are three choices.
 - ① If you are considering the extended region defined by ghost-cells and the trajectory falls in this extended region, you can wait a number of cycles (for example, 20) for the trajectory returning to the mapped region. In this case, you can proceed with the calculations;
 - ② If the trajectory does not return to the mapped region after a certain number of cycles, you consider it as part of the basins of attraction of the divergent attractor, assigning to it -1;
 - ③ If the trajectory visits a point outside the extended region, you assign to it the divergent attractor.
- The consideration of the ghost cells is important, provided it avoids to assign the divergent attractor to a trajectory that escapes the mapped region just for some periods.

- Lets assume the the trajectory does not visit the same cell twice. In this case, we can assign the chaotic attractor to a trajectory of length larger than a certain number (typically 1000). Notice that a 1001-periodic attractor is, then, assumed to be a chaotic one;
- It is important to emphasize that the proposed algorithm has two approximations. The first one is associated with numerical approximations inherently of the integration of the mathematical model. The second one arises from the discretization of the phase space into cells.

- At the end of the algorithm, you will have a vector of length equal to product of the adopted discretizations in the two directions of the phase space;
- This vector contains even numbers (attractors), odd numbers (basins) or -1 (divergent attractor). Now, you choose colors for the basins (for example, basin of attraction 1 and attractor 2 are colored in blue and you include a marker for identifying the attractor);
- The plot can be made by converting this vector into a matrix and using standard functions such as, for example, `patch` (in MATLAB[®]).

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- Local Integrity Measurement (LIM): Largest normalized radius of the circle, centered at the attractor, fully inside the basins of attraction;
- Integrity Factor (IF): Largest normalized radius of the circle fully inside the basins of attraction;
- Both LIM and IF can be obtained using functions for binary image processing.
 - ① Convert your basins of attraction into a binary matrix, in which 1 is the basins you would like to analyze;
 - ② Functions `bwdist` (MATLAB[®]) and `DistanceTransform` (Mathematica[®]) calculate, for a binary image, the distance (in pixels) between the number assigned as 1 and the closest zero for each position of the matrix;
 - ③ If your basin has equal discretization in the $y(0)$ and $\dot{y}(0)$ direction, you just need to re-scale. For a more general case, you need to program your own tool;
 - ④ With the distance calculated for each point of the basin, the distance associated with the attractor is readily obtained (LIM). The IF is the maximum distance and the center of this circle is also easily found from the corresponding position in the matrix.

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- Mathematical model

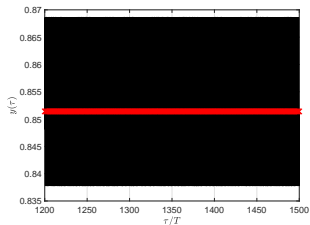
$$\ddot{y} + a\dot{y} + by + cy^2 + dy^3 = e \sin \Omega t \quad (1)$$

- Investigated scenarios

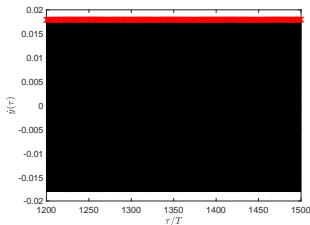
Scenario	a	b	c	d	e	Ω
1	0.10	1.20	-0.30	2	0.020	1.17
2					0.077	

Scenario 1 - two attractors are identified

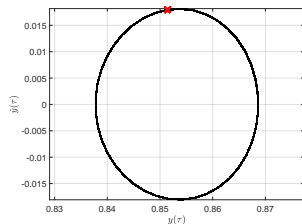
Attractor 1: Red crosses indicate the Poincaré's sections.



(a) $y(\tau)$.



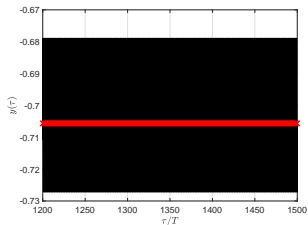
(b) $\dot{y}(\tau)$.



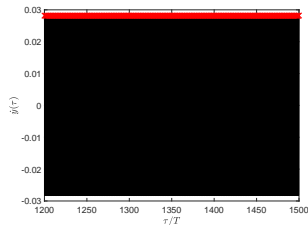
(c) $y(\tau) \times \dot{y}(\tau)$.

Scenario 1 - two attractors are identified

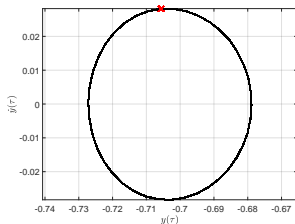
Attractor 2: Red crosses indicate the Poincaré's sections.



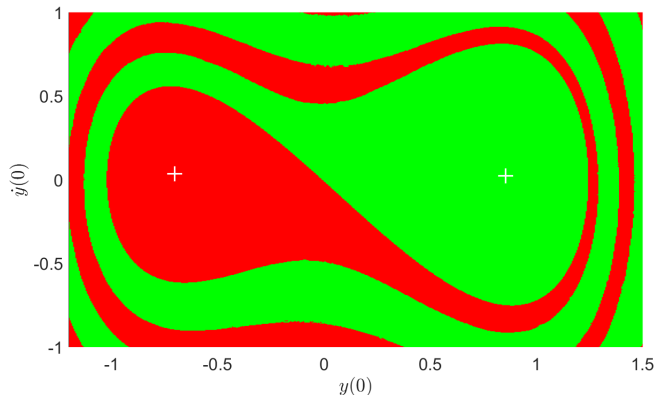
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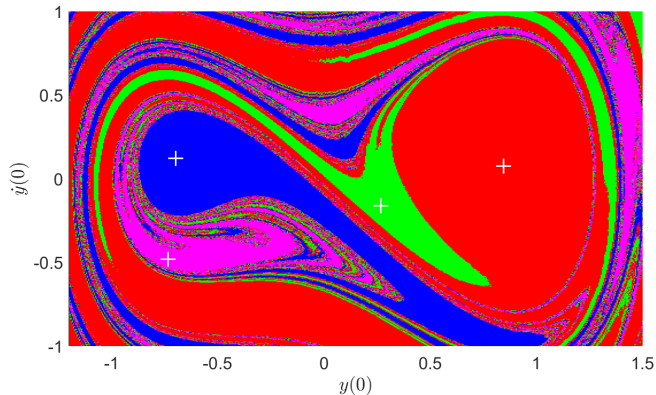
(b) $\dot{y}(\tau)$.



(c) $y(\tau) \times \dot{y}(\tau)$.

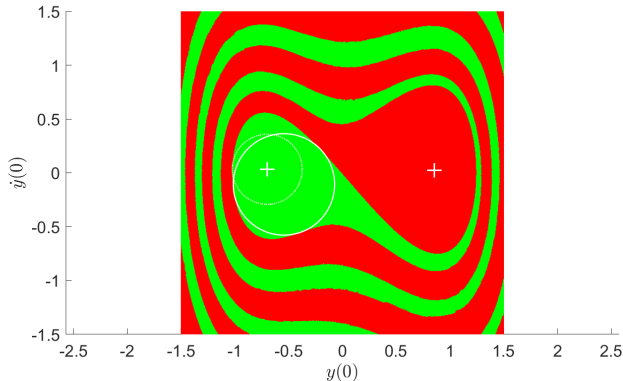


- Two attractors (white crosses) identified as the center of the cells;
- Compact basins of attractions;
- 500×500 grid: 1.8 hour for running the basins



- Four attractors (white crosses) identified as the center of the cells;
- Not compact basins of attractions;
- 500×500 grid: 1.8 hour for running the basins

- In this case, the adopted discretization is the same in the two directions. The dashed circle represents LIM and the continuous one is associated with IF.
- For plotting the integrity profiles, you need to compute the basins of attraction and the dynamic integrity measurements for each varied parameter (in the case, the amplitude of the external excitation).



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- If the system has more than one degree of freedom, some alternatives appear. The first one is to extend the mentioned idea.
- For a 2-dof system and using a discretization of 100 cells in each direction, you need to map 10^8 cells. This has a huge computational cost and tools such as MATLAB[®] are not useful (the group has the idea of creating an in-house code name PoliBoA[®], based on C++);
- You can proceed with the bidimensional grid-of-start method, in which you discretize the complete phase space but just use as initial conditions those constrained to a bidimensional cross-section;
- You can also use the above idea, but combined with the bidimensional cell-mapping. In this case, if the trajectory crosses the bi-dimensional cross-section (a hyper-volume, specifically), you compute this point in the trajectory, saving some time if compared to the grid-of-start method.

- Consider the 2-dof system governed by

$$\ddot{y} - \beta_1 \sin \theta \ddot{\theta} - \beta_1 \cos \theta \dot{\theta}^2 + \beta_2 \dot{y} + (1 + \delta \sin n\tau)y = 0$$

$$\ddot{\theta} - \frac{1}{\hat{r}} \sin \theta \ddot{y} + \beta_3 \dot{\theta} = 0$$

An example of bidimensional basins of attractor is presented below. More cross-sections are needed for studying the problem. **Ongoing work!**

$$\theta(0) = 0, \dot{\theta}(0) = -0.5$$

