Escalonamento e existência de soluções

MAP 2110 - Diurno

IME USP

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Matriz na forma escalonada

Vamos estudar o que acontece quando aplicamos o algoritmo da Eliminação de Gauss em sistemas lineares de qualquer dimensão, isto é, com n incógnitas e m equações.

Sistema Linear

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Exemplo

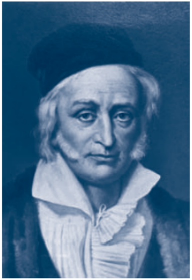
Vamos ver o seguinte exemplo com 4 equações e 5 incógnitas.

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 + x_5 &= a \\ 2x_1 + x_3 + x_4 + 3x_5 &= b \\ 3x_1 + x_2 + 3x_4 + 4x_5 &= c \\ x_1 - x_2 + x_4 &= d \end{aligned}$$

Matriz aumentada do sistema

Matriz aumentada = matriz dos coeficientes junto com os elemntos do lado direito da equação

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left[\begin{array}{ccccc|c}
1 & 1 & -1 & 2 & 1 & a \\
2 & 0 & 1 & 1 & 3 & b \\
3 & 1 & 0 & 3 & 4 & c \\
1 & -1 & 0 & 1 & 0 & d
\end{array}\right]
\end{array}
$$



Carl Friedrich Gauss. Photo

Gaussian[4] Algorithm[5]

Step 1. *If the matrix consists entirely of zeros, stop—it is already in row-echelon form*

Step 2. *Otherwise, find the first column from the left containing a nonzero entry (ca it a), and move the row containing that entry to the top position.*

Step 3. *Now multiply the new top row by $1/a$ to create a leading 1.*

Step 4. *By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.*

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. *Repeat steps 1–4 on the matrix consisting of the remaining rows.*

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Aplicando o método de Eliminação

$$\left[\begin{array}{ccccc|c} \color{red}{1} & 1 & -1 & 2 & 1 & a \\ 2 & 0 & 1 & 1 & 3 & b \\ 3 & 1 & 0 & 3 & 4 & c \\ 1 & -1 & 0 & 1 & 0 & d \end{array}\right]$$

Faremos as operações elementares:

$$L_2 = L_2 - 2L_1$$
$$L_3 = L_3 - 3L_1$$
$$L_4 = L_4 - L_1$$

Aplicando o método de Eliminação- 2

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & {\color{red}-2} & 3 & -3 & 1 & b-2a \\ 0 & -2 & 3 & -3 & 1 & c-3a \\ 0 & -2 & 1 & -1 & -1 & d-a \end{array}\right]$$

$$L_3 = L_3 - L_2$$
$$L_4 = L_4 - L_2$$

Aplicando o método de Eliminação- 3

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & -2 & 3 & -3 & 1 & b-2a \\ 0 & 0 & 0 & 0 & 0 & c-b-a \\ 0 & 0 & -2 & 2 & -2 & d-b+a \end{array}\right]$$

$$L_3 = L_4$$
$$L_4 = L_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & -2 & 3 & -3 & 1 & b-2a \\ 0 & 0 & -2 & 2 & -2 & d-b+a \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

Na forma escalonada teríamos

Forma escalonada

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & 1 & -3/2 & 3/2 & -1/2 & (2a-b)/2 \\ 0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

podemos fazer o escalonamento usando operações elementares diferentes

Vamos partir do segundo passo do escalonamento acima:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & -2 & 3 & -3 & 1 & b-2a \\ 0 & -2 & 3 & -3 & 1 & c-3a \\ 0 & -2 & 1 & -1 & -1 & d-a \end{array}\right]$$

E fazer agora primeiro a troca da linha 4 e 2

$$L_2 = L_4$$
$$L_4 = L_2$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & -2 & 1 & -1 & -1 & d-a \\ 0 & -2 & 3 & -3 & 1 & c-3a \\ 0 & -2 & 3 & -3 & 1 & b-2a \end{array}\right]$$

$$L_3 = L_3 - L_2$$
$$L_4 = L_4 - L_2$$

$$
\left[\begin{array}{ccccc|c}
1 & 1 & -1 & 2 & 1 & a \\
0 & -2 & 1 & -1 & -1 & d-a \\
0 & 0 & 2 & -2 & 2 & c-d-2a \\
0 & 0 & 2 & -2 & 2 & b-d-a
\end{array}\right]
$$

$$L_4 = L4 - L_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & -2 & 1 & -1 & -1 & d-a \\ 0 & 0 & 2 & -2 & 2 & c-d-2a \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & 1 & -1/2 & 1/2 & 1/2 & (a-d)/2 \\ 0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$

$$
\left[\begin{array}{ccccc|c}
1 & 1 & -1 & 2 & 1 & a \\
0 & 1 & -1/2 & 1/2 & 1/2 & (a-d)/2 \\
0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\
0 & 0 & 0 & 0 & 0 & b+a-c
\end{array}\right]
$$

$$
\neq \left[\begin{array}{ccccc|c}
1 & 1 & -1 & 2 & 1 & a \\
0 & 1 & -3/2 & 3/2 & -1/2 & (2a-b)/2 \\
0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\
0 & 0 & 0 & 0 & 0 & c-b-a
\end{array}\right]
$$

Forma escalonada reduzida

Este método também é conhecido como algoritmo de Gauss Jordan. Vamos começar com a segunda versão do processo de escalonamento:

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & 1 & -1/2 & 1/2 & 1/2 & (a-d)/2 \\ 0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$

Continuando com as operações elementares

$$L_2 = L_2 + 1/2 * L_3$$
$$L_1 = L_1 + L_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 2 & (c-d)/2 \\ 0 & 1 & 0 & 0 & 1 & (c-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$

$$L_1 = L_1 - L_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & (d+c)/4 \\ 0 & 1 & 0 & 0 & 1 & (c-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$

Esta é a forma escalonada reduzida da matriz

O processo de Jordan com a segunda Matriz escalonada

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & a \\ 0 & 1 & -3/2 & 3/2 & -1/2 & (2a-b)/2 \\ 0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

$$L_2 = L_2 + 3/2 * L_3$$
$$L_1 = L_1 + L_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 2 & (b+a-d)/2 \\ 0 & 1 & 0 & 0 & 1 & (a+b-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

$$L_1 = L_1 - L_2$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & (b+a+d)/4 \\ 0 & 1 & 0 & 0 & 1 & (a+b-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

Comparando as formas reduzidas

Obtidas com diferentes operações elementares.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & (b+a+d)/4 \\ 0 & 1 & 0 & 0 & 1 & (a+b-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (b-a-d)/2 \\ 0 & 0 & 0 & 0 & 0 & c-b-a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & (d+c)/4 \\ 0 & 1 & 0 & 0 & 1 & (c-3d)/4 \\ 0 & 0 & 1 & -1 & 1 & (c-d-2a)/2 \\ 0 & 0 & 0 & 0 & 0 & b+a-c \end{array}\right]$$