a uniform gravitational field, since such a field in no way affects the equations of motion for the relative co-ordinates $\boldsymbol{r}_i^*$. Note that a laboratory is subjected to *other* external forces besides the Earth's gravitational field, since it is supported by the ground. Indeed if the Earth were removed, but the supporting forces somehow retained, there would be no observable difference inside the laboratory, which would of course be accelerated upwards with acceleration g! This was an important consideration in the argument which led to Einstein's general theory of relativity.

8.6 Summary

The centre of mass of any system moves like a particle of mass M acted on by a force equal to the total force on the system. The contribution of this motion to the angular momentum or kinetic energy may be completely separated from the contributions of the relative motion, and $\boldsymbol{J}$ or T may be written as a sum of two corresponding terms. (Of course, the *only* contribution to $\boldsymbol{P}$ comes from the centre-of-mass motion.)

When the internal forces are central, the rate of change of angular momentum is equal to the sum of the moments of the external forces. When they are conservative, the rate of change of the kinetic energy plus the internal potential energy is equal to the rate of working of the external forces. In both cases, the same thing is true for the motion relative to the centre of mass.

If the external forces are also central, or conservative, then the total angular momentum, or total energy (including external potential energy), respectively, are conserved. In particular, for an isolated system, $\boldsymbol{P}$, $\boldsymbol{J}$ and $T + V_{\text{int}}$ are all constants.

Problems

1. A rocket is launched from the surface of the Earth, to reach a height of 50 km. Find the required velocity impulse, neglecting the variation of g with height (and the Earth's rotation). Given that the mass of the payload and rocket without fuel is 100 kg, and the ejection velocity is $2\,\text{km}\,\text{s}^{-1}$, find the required initial mass.
2. A satellite is orbiting the Earth in a circular orbit 230 km above the equator. Calculate the total velocity impulse needed to place it in a

synchronous orbit (see Chapter 4, Problem 1), using an intermediate semi-elliptical transfer orbit which just touches both circles. (*Hint*: From the orbit parameters a and ae, find l, and hence the velocities.) Given that the final mass to be placed in orbit is 30 kg, and the ejection velocity of the rocket is $2.5\,\mathrm{km\,s^{-1}}$, find the necessary initial mass.

3. Assume that the residual mass of a rocket, without payload or fuel, is a given fraction λ of the initial mass including fuel (but still without payload). Show that the total take-off mass required to accelerate a payload m to velocity v is

$$M_0 = m\,\frac{1-\lambda}{\mathrm{e}^{-v/u}-\lambda}.$$

If $\lambda = 0.15$, what is the upper limit to the velocity attainable with an ejection velocity of $2.5\,\mathrm{km\,s^{-1}}$?

4. Find a formula analogous to that of Problem 3 for a two-stage rocket, in which each stage produces the same velocity impulse. (The first-stage rocket is discarded when its fuel is burnt out.) With the figures of Problem 3, what is the minimum number of stages required to reach escape velocity ($11.2\,\mathrm{km\,s^{-1}}$)? With that number, what take-off mass is required, if the payload mass is 100 kg?

5. Find the velocity impulse needed to launch the spacecraft on its trip to Jupiter, described in Chapter 4, Problem 16. (*Hint*: Use energy conservation to find the velocity at the surface of the Earth needed to give the appropriate relative velocity for the spacecraft once it has escaped.) If a three-stage rocket is used, and the parameter λ of Problem 3 is 0.1, what is the minimum required ejection velocity? Given that $u = 2.5\,\mathrm{km\,s^{-1}}$, and that the mass of the payload is 500 kg, find the total take-off mass.

6. Find the gain in kinetic energy when a rocket emits a small amount of matter. Hence calculate the total energy which must be supplied from chemical or other sources to accelerate the rocket to a given velocity. Show that this is equal to the energy required if an equal amount of matter is ejected while the rocket is held fixed on a test-bed.

7. *A rocket with take-off mass M_0 is launched vertically upward, as in Problem 1. Consider the effect of a finite burn-up time. Show that, if the rocket ejects matter at a constant rate a, then its height at time t is

$$z = ut - \frac{uM}{a}\ln\frac{M_0}{M} - \tfrac{1}{2}gt^2, \qquad \text{with} \qquad M = M_0 - at.$$

Hence, show that if it burns out after a time t_1, leaving a final mass M_1, then [provided that $t_1 < (u/g)\ln(M_0/M_1)$], the maximum height reached is

$$z = \frac{u^2}{2g}\left(\ln\frac{M_0}{M_1}\right)^2 - ut_1\left(\frac{M_0}{M_0 - M_1}\ln\frac{M_0}{M_1} - 1\right).$$

Using the same values of u, M_0 and M_1 as in Problem 1, find the maximum height reached if the burn-up time is (a) 10 s, and (b) 30 s.

8. *A satellite is to be launched into a synchronous orbit directly from the surface of the Earth, using a rocket launched vertically from a point on the equator. Find the required launch speed to achieve the desired apogee, and the required velocity increment. With the same figures as in Problem 2, find the required take-off mass.
9. Two billiard balls are resting on a smooth table, and just touching. A third identical ball moving along the table with velocity v perpendicular to their line of centres strikes both balls simultaneously. Find the velocities of the three balls immediately after impact, assuming that the collision is elastic.
10. A spherical satellite of radius r is moving with velocity v through a uniform tenuous atmosphere of density ρ. Find the retarding force on the satellite if each particle which strikes it (a) adheres to the surface, and (b) bounces off it elastically. Can you explain why the two answers are equal, in terms of the scattering cross-section of a hard sphere?
11. *If the orbit of the satellite of Problem 10 is highly elliptical, the retarding force is concentrated almost entirely in the lowest part of the orbit. Replace it by an impulsive force of impulse I delivered once every orbit, at perigee. By considering changes in energy and angular momentum, find the changes in the parameters a and l. Show that $\delta l = \delta a(1-e)^2$, and hence that the effect is to decrease the period and apogee distance, while leaving the perigee distance unaffected. (The orbit therefore becomes more and more circular with time.) Show that the velocity at apogee increases, while that at perigee decreases.
12. *Suppose that the satellite of Problems 10 and 11 has achieved a circular orbit of radius a. Find the rates of change of energy and angular momentum, and hence show that the rates of change of a and l are equal, so that the orbit remains approximately circular. Show also that the velocity of the satellite must *increase*.
13. If the satellite orbit of Problem 12 is 500 km above the Earth's surface, the mass and radius of the satellite are 30 kg and 0.7 m, and

$\rho = 10^{-13}\,\text{kg}\,\text{m}^{-3}$, find the changes in orbital period and height in a year. If the height is 200 km and $\rho = 10^{-10}\,\text{kg}\,\text{m}^{-3}$, find the changes in a single orbit.

14. Find the lengths of the 'day' and the 'month' (a) when the Moon was 10 Earth radii away, and (b) when the solar and lunar tides become equal in magnitude. (See Chapter 6, Problem 16.)
15. Show that in a conservative N-body system, a state of minimal total energy for a given total z-component of angular momentum is necessarily one in which the system is rotating as a rigid body about the z-axis. [Use the method of Lagrange multipliers (see Appendix A, Problem 11), and treat the components of the positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$ as independent variables.]
16. *A planet of mass M is surrounded by a cloud of small particles in orbits around it. Their mutual gravitational attraction is negligible. Due to collisions between the particles, the energy will gradually decrease from its initial value, but the angular momentum will remain fixed, $\boldsymbol{J} = \boldsymbol{J}_0$, say. The system will thus evolve towards a state of minimum energy, subject to this constraint. Show that the particles will tend to form a ring around the planet. [As in Problem 15, the constraint may be imposed by the method of Lagrange multipliers. In this case, because there are three components of the constraint equation, we need three Lagrange multipliers, say $\omega_x, \omega_y, \omega_z$. We have to minimize the function $E - \boldsymbol{\omega} \cdot (\boldsymbol{J} - \boldsymbol{J}_0)$ with respect to variations of the positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$, and with respect to $\boldsymbol{\omega}$. Show by minimizing with respect to $\dot{\boldsymbol{r}}_i$ that once equilibrium has been reached the cloud rotates as a rigid body, and by minimizing with respect to $\boldsymbol{r}_i$ that all particles occupy the same orbit.] What happens to the energy lost? Why does the argument not necessarily apply to a cloud of particles around a hot star?
17. *An N-body system is interacting only through the gravitational forces between the bodies. Show that the potential energy function V satisfies the equation

$$\sum_i \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_i V = -V,$$

where $\boldsymbol{\nabla}_i = (\partial/\partial x_i, \partial/\partial y_i, \partial/\partial z_i)$. (*Hint*: Show that each two-body term V_{ij} satisfies this equation. This condition expresses the fact that V is a homogeneous function of the co-ordinates of degree -1.)

18. *Under the conditions of Problem 17, show that the total kinetic and potential energies T and V satisfy the *virial equation*,

$$2T + V = \frac{\mathrm{d}^2 K}{\mathrm{d}t^2}, \qquad \text{where} \qquad K = \tfrac{1}{2}\sum_i m_i \boldsymbol{r}_i^2.$$

(Note that K relates to the overall scale of the system. We may define a root-mean-square radius r by $K = \frac{1}{2}Mr^2$.) Deduce that, if the scale of the system as measured by K is, on average, neither growing nor shrinking, then the time-averaged value of the total energy is equal to minus the time-averaged value of the kinetic energy — the *virial theorem*. (Compare Chapter 4, Problem 19.)

In general, it is convenient to describe the rotational motion in terms of a set of principal axes. Normally these rotate with the body, though in the special case of a symmetric rigid body (with $I_1 = I_2$) they may be chosen as in §9.9. In that case, the orientation of the body is conveniently described by Euler's angles. We shall see in the following chapter that the Lagrangian method is very useful for obtaining equations of motion in terms of Euler's angles.

Problems

1. A uniform solid cube of edge length $2a$ is suspended from a horizontal axis along one edge. Find the length of the equivalent simple pendulum. Given that the cube is released from rest with its centre of mass level with the axis, find its angular velocity when it reaches the lowest point.
2. An insect of mass 100 mg is resting on the edge of a flat uniform disc of mass 3 g and radius 50 mm, which is rotating at 60 r.p.m. about a smooth pivot. The insect crawls in towards the centre of the disc. Find the angular velocity when it reaches it, and the gain in kinetic energy. Where does this kinetic energy come from, and what happens to it when the insect crawls back out to the edge?
3. A uniform solid cube of edge $2a$ is sliding with velocity v on a smooth horizontal table when its leading edge is suddenly brought to rest by a small ridge on the table. Which dynamical variables are conserved (a) before impact, (b) during impact, and (c) after impact? Find the angular velocity immediately after impact, and the fractional loss of kinetic energy. Determine the minimum value of v for which the cube topples over rather than falling back.
4. A pendulum consists of a light rigid rod of length 250 mm, with two identical uniform solid spheres of radius 50 mm attached one on either side of its lower end. Find the period of small oscillations (a) perpendicular to the line of centres, and (b) along it.
5. A uniform rod of mass M and length $2a$ hangs from a smooth hinge at one end. Find the length of the equivalent simple pendulum. It is struck sharply, with impulse X, at a point a distance b below the hinge. Use the angular momentum equation to find the initial value of the angular velocity. Find also the initial momentum. Determine the point at which the rod may be struck without producing any impulsive reaction at the hinge. Show that, if the rod is struck elsewhere, the

direction of the impulsive reaction depends on whether the point of impact is above or below this point.

6. *(a) A simple pendulum supported by a light rigid rod of length l is released from rest with the rod horizontal. Find the reaction at the pivot as a function of the angle of inclination.
   (b) For the cube of Problem 1, find the horizontal and vertical components of the reaction on the axis as a function of its angular position. Compare your answer with the corresponding expressions for the equivalent simple pendulum.
7. *Find the principal moments of inertia of a flat rectangular plate of mass 30 g and dimensions 80 mm×60 mm. Given that the plate is rotating about a diagonal with angular velocity $15\,\mathrm{rad\,s^{-1}}$, find the components of the angular momentum parallel to the edges. Given that the axis is of total length 120 mm, and is held vertical by bearings at its ends, find the horizontal component of the force on each bearing.
8. Find the principal moments of inertia of a uniform solid cube of mass m and edge length $2a$ (a) with respect to the mid-point of an edge, and (b) with respect to a vertex.
9. Find the moment of inertia about an axis through its centre of a uniform hollow sphere of mass M and outer and inner radii a and b. (*Hint*: Think of it as a sphere of density ρ and radius a, with a sphere of density ρ and radius b removed.)
10. A spaceship of mass 3 t has the form of a hollow sphere, with inner radius 2.5 m and outer radius 3 m. Its orientation in space is controlled by a uniform circular flywheel of mass 10 kg and radius 0.1 m. Given that the flywheel is set spinning at 2000 r.p.m., find how long it takes the spaceship to rotate through 1°. Find also the energy dissipated in this manoeuvre.
11. A long, thin hollow cylinder of radius a is balanced on a horizontal knife edge, with its axis parallel to it. It is given a small displacement. Calculate the angular displacement at the moment when the cylinder ceases to touch the knife edge. (*Hint*: This is the moment when the radial component of the reaction falls to zero.)
12. *Calculate the principal moments of inertia of a uniform, solid cone of vertical height h, and base radius a about its vertex. For what value of the ratio h/a is every axis through the vertex a principal axis? For this case, find the position of the centre of mass and the principal moments of inertia about it.

13. A top consists of a uniform, solid cone of height 50 mm and base radius 20 mm. It is spinning with its vertex fixed at 7200 r.p.m. Find the precessional period of the axis about the vertical.
14. A gyroscope consisting of a uniform solid sphere of radius 0.1 m is spinning at 3000 r.p.m. about a horizontal axis. Due to faulty construction, the fixed point is not precisely at the centre, but 20 μm away from it along the axis. Find the time taken for the axis to move through 1°.
15. A gyroscope consisting of a uniform circular disc of mass 100 g and radius 40 mm is pivoted so that its centre of mass is fixed, and is spinning about its axis at 2400 r.p.m. A 5 g mass is attached to the axis at a distance of 100 mm from the centre. Find the angular velocity of precession of the axis.
16. *A uniformly charged sphere is spinning freely with angular velocity $\boldsymbol{\omega}$ in a uniform magnetic field $\boldsymbol{B}$. Taking the z axis in the direction of $\boldsymbol{\omega}$, and $\boldsymbol{B}$ in the xz-plane, write down the moment about the centre of the magnetic force on a particle at $\boldsymbol{r}$. Evaluate the total moment of the magnetic force on the sphere, and show that it is equal to $(q/2M)\boldsymbol{J}\wedge\boldsymbol{B}$, where q and M are the total charge and mass, respectively. Hence show that the axis will precess around the direction of the magnetic field with precessional angular velocity equal to the Larmor frequency of §5.5. What difference would it make if the charge distribution were spherically symmetric, but non-uniform?
17. *A wheel of radius a, with its mass concentrated on the rim, is rolling with velocity v round a circle of radius R ($\gg a$), maintaining a constant inclination α to the vertical. Show that $v = a\omega = R\Omega$, where ω is the angular velocity of the wheel about its axis, and Ω ($\ll \omega$) is the precessional angular velocity of the axis. Use the momentum equation to find the horizontal and vertical components of the force at the point of contact. Then show from the angular momentum equation about the centre of mass that $R = 2v^2/g\tan\alpha$. Evaluate R for $v = 5\,\mathrm{m\,s^{-1}}$ and $\alpha = 30^\circ$.
18. A solid rectangular box, of dimensions 100 mm$\times$60 mm$\times$20 mm, is spinning freely with angular velocity 240 r.p.m. Determine the frequency of small oscillations of the axis, if the axis of rotation is (a) the longest, and (b) the shortest, axis.
19. *A rigid body of spheroidal shape, spinning rapidly about its axis of symmetry, is placed on a smooth flat table. Show by considering the moment of the force at the point of contact that its axis will precess in one direction if it is oblate ($c < a = b$, *e.g.* a discus) and in the

opposite direction if it is prolate ($c > a = b$, *e.g.* a rugby ball). Show also that if there is a small frictional force, the axis will become more nearly vertical, so that if the body is oblate its centre of mass will fall, but if it is prolate it will *rise*.

20. *The average moment exerted by the Sun on the Earth is, except for sign, identical with the expression found in Chapter 6, Problem 26, provided we interpret m as the mass of the Sun, and r as the distance to the Sun. Show that $Q = -2(I_3 - I_1)$ and hence that the precessional angular velocity produced by this moment is

$$\boldsymbol{\Omega} = -\frac{3}{2}\frac{I_3 - I_1}{I_3}\frac{\varpi^2}{\omega}\cos\alpha\,\boldsymbol{n},$$

where ϖ is the Earth's orbital angular velocity, and $\alpha = 23.45°$ is the tilt between the Earth's axis and the normal to the orbital plane (the ecliptic). Show also that $(I_3 - I_1)/I_3 \approx \epsilon$, the oblateness of the Earth, and hence evaluate $\boldsymbol{\Omega}$. Why is this effect less sensitive to the distribution of density within the Earth than the complementary one discussed in §6.5?

21. *The axis of a gyroscope is free to rotate within a smooth horizontal circle in colatitude λ. Due to the Coriolis force, there is a couple on the gyroscope. To find the effect of this couple, use the equation for the rate of change of angular momentum in a frame rotating with the Earth (*e.g.*, that of Fig. 5.7), $\dot{\boldsymbol{J}} + \boldsymbol{\Omega}\wedge\boldsymbol{J} = \boldsymbol{G}$, where $\boldsymbol{G}$ is the couple restraining the axis from leaving the horizontal plane, and $\boldsymbol{\Omega}$ is the Earth's angular velocity. (Neglect terms of order Ω^2, in particular the contribution of $\boldsymbol{\Omega}$ to $\boldsymbol{J}$.) From the component along the axis, show that the angular velocity ω about the axis is constant; from the vertical component show that the angle φ between the axis and east obeys the equation

$$I_1\ddot{\varphi} - I_3\omega\Omega\sin\lambda\cos\varphi = 0.$$

Show that the stable position is with the axis pointing north. Determine the period of small oscillations about this direction if the gyroscope is a flat circular disc spinning at 6000 r.p.m. in latitude 30° N. Explain why this system is sensitive to the horizontal component of $\boldsymbol{\Omega}$, and describe the effect qualitatively from the point of view of an inertial observer.