Dinamica Non Lineare di Strutture e Sistemi Meccanici

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Lezione 6

Reduced-Order Modelling

Historical notes

- Steindl & Troger (2001): analytical and numerical (finite-difference); linear and non-linear; pre- and post-processing (Karhunen-Loeve)
- Shaw & Pierre & Peschek (2001-2009): analytical projection of dynamics onto non-linear modes defined by invariant manifolds
- Mazzilli & Baracho Neto (2006-2008): analytical projection of dynamics onto non-linear modes defined by invariant manifolds or multiple time scales
- Amabili & Touzé (2007): numerical methods
- Kerschen & Peeters & Golinval & Vakakis (2008): numerical methods

Reduced-Order Modelling

Non-linear Galerkin procedures

Equations of motion of wide class of discrete systems

$$\boxed{M_{rs}\left(\mathbf{p}\right)\ddot{p}_s + D_{rs}\left(\mathbf{p},\dot{\mathbf{p}}\right)\dot{p}_s + K_{rs}\left(\mathbf{p}\right)p_s = R_r}, \; r = 1,2\ldots n, \quad sum\ in\ s = 1\ to\ n,$$

$$M_{rs}\left(\mathbf{p}\right) = M_{rs}^{0} + M_{rsk}^{1}p_k + M_{rsk\ell}^{2}p_k p_\ell$$

$$D_{rs}\left(\mathbf{p},\dot{\mathbf{p}}\right) = D_{rs}^{0} + D_{rsk}^{1}\dot{p}_k + D_{rsk\ell}^{2}\dot{p}_k p_\ell$$

$$K_{rs}\left(\mathbf{p}\right) = K_{rs}^{0} + K_{rsk}^{1}p_k + K_{rsk\ell}^{2}p_k p_\ell$$

$$\boxed{\begin{array}{l} M_{rs}^{0},\ M_{rsk}^{1},\ M_{rsk\ell}^{2} \\ D_{rs}^{0},\ D_{rsk}^{1},\ D_{rsk\ell}^{2} \\ K_{rs}^{0},\ K_{rsk}^{1},\ K_{rsk\ell}^{2} \end{array}}$$

constants

Reduced-Order Modelling

Non-linear Galerkin procedures

Modal relationships

$$p_r\left(\mathbf{U},\dot{\mathbf{U}}\right)=p_{0r}+a_{1r}^{u}U_u+a_{2r}^{u}\dot{U}_u+a_{3r}^{uv}U_uU_v+a_{4r}^{uv}U_u\dot{U}_v+a_{5r}^{uv}\dot{U}_u\dot{U}_v$$
$$+a_{6r}^{uvw}U_uU_vU_w+a_{7r}^{uvw}U_uU_v\dot{U}_w+a_{8r}^{uvw}U_u\dot{U}_v\dot{U}_w+a_{9r}^{uvw}\dot{U}_u\dot{U}_v\dot{U}_w,$$

$$\dot{p}_r\left(\mathbf{U},\dot{\mathbf{U}}\right)=b_{1r}^{u}U_u+b_{2r}^{u}\dot{U}_u+b_{3r}^{uv}U_uU_v+b_{4r}^{uv}U_u\dot{U}_v+b_{5r}^{uv}\dot{U}_u\dot{U}_v$$
$$+b_{6r}^{uvw}U_uU_vU_w+b_{7r}^{uvw}U_uU_v\dot{U}_w+b_{8r}^{uvw}U_u\dot{U}_v\dot{U}_w+b_{9r}^{uvw}\dot{U}_u\dot{U}_v\dot{U}_w,$$

where the coefficients a_1 to a_9 and b_1 to b_9 are supposed known;
they are naturally obtained via the invariant manifold procedure;
they must be 'extracted' from the multiple time scales approach

Reduced-Order Modelling

Non-linear Galerkin procedures

Forced non-linear modal oscillator equation

$$\ddot{U}_u + c_1^u U_u + c_2^u \dot{U}_u + c_3^{uv} U_u U_v + c_4^{uv} U_u \dot{U}_v + c_5^{uv} \dot{U}_u \dot{U}_v$$
$$+ c_6^{uvw} U_u U_v U_w + c_7^{uvw} U_u U_v \dot{U}_w + c_8^{uvw} U_u \dot{U}_v \dot{U}_w + c_9^{uvw} \dot{U}_u \dot{U}_v \dot{U}_w = P_u \ ,$$

where the coefficients c_1 to c_9 are supposed known;

they are naturally obtained via the invariant manifold procedure;

they must be 'extracted' from the multiple time scales approach.

Key-question: how to obtain the modal force P_u ?

Once known, the usual analysis methods can be applied

to the modal oscillator equation...

Reduced-Order Modelling

Non-linear Galerkin procedures

Balance of virtual works

$$R_r \delta p_r = P_u M_u \delta U_u$$

$$\delta p_r\left(\mathbf{U},\dot{\mathbf{U}}\right)=\left[a_{1r}^{u}+\left(a_{3r}^{uv}+a_{3r}^{vu}\right)U_v+a_{4r}^{uv}\dot{U}_v+\left(a_{6r}^{uvw}+a_{6r}^{vuw}+a_{6r}^{wvu}\right)U_vU_w+\left(a_{7r}^{uvw}+a_{7r}^{vuw}\right)U_v\dot{U}_w+a_{8r}^{uvw}\dot{U}_v\dot{U}_w\right]\delta U_u$$

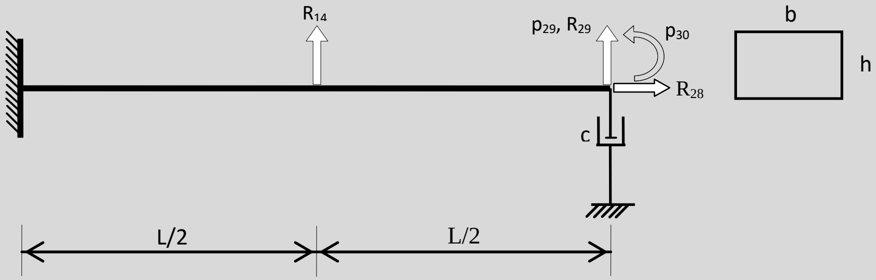
$$\Downarrow$$

$$P_u=\gamma_u^r\left[a_{1r}^{u}+\left(a_{3r}^{uv}+a_{3r}^{vu}\right)U_v+a_{4r}^{uv}\dot{U}_v+\left(a_{6r}^{uvw}+a_{6r}^{vuw}+a_{6r}^{wvu}\right)U_vU_w+\left(a_{7r}^{uvw}+a_{7r}^{vuw}\right)U_v\dot{U}_w+a_{8r}^{uvw}\dot{U}_v\dot{U}_w\right]$$

$$\gamma_u^r=\frac{R_r}{M_u}$$

Reduced-Order Modelling

Example 1 (classical resonance)



$$b = 0.010m$$
$$h = 0.010m$$
$$E = 2.1\times10^{11}\, N/m^2$$
$$\rho = 7,800 kg/m^3$$
$$c = 0.7 Ns/m$$
$$R_{14}(t) = 6.6\cos(82.2\,t)$$
$$R_{29}(t) = 6.6\cos(13.0\,t)$$
$$R_{28}(t) = 0$$

FE model with 30 degrees of freedom.

Chosen modal coordinates: $U_u = p_{29}$, with $u = 1, 2$

Reduced-Order Modelling

Example 1 (classical resonance)

First-mode coefficients

$$c_1^1 = 0.17345E+03 \qquad c_2^1 = 0.17959E+01 \qquad c_6^{111} = -0.40506E+02$$
$$c_7^{111} = -0.28403E+00 \qquad c_8^{111} = 0.21383E+00 \qquad c_9^{111} = -0.59973E\text{-}03$$
$$a_{1,14}^1 = 0.33951E+00 \qquad a_{6,14}^{111} = -0.57529E-02 \qquad a_{7,14}^{111} = -0.91501E-04$$
$$a_{8,14}^{111} = 0.47231-04 \qquad a_{1,29}^1 = 0.10000E+01$$

$$\begin{cases} \omega_{01} = \sqrt{c_1^1} = 13.17\ rad/s \\ \xi_1 = \dfrac{c_2^1}{2\sqrt{c_1^1}} = 0.0681 \end{cases}$$

Second-mode coefficients

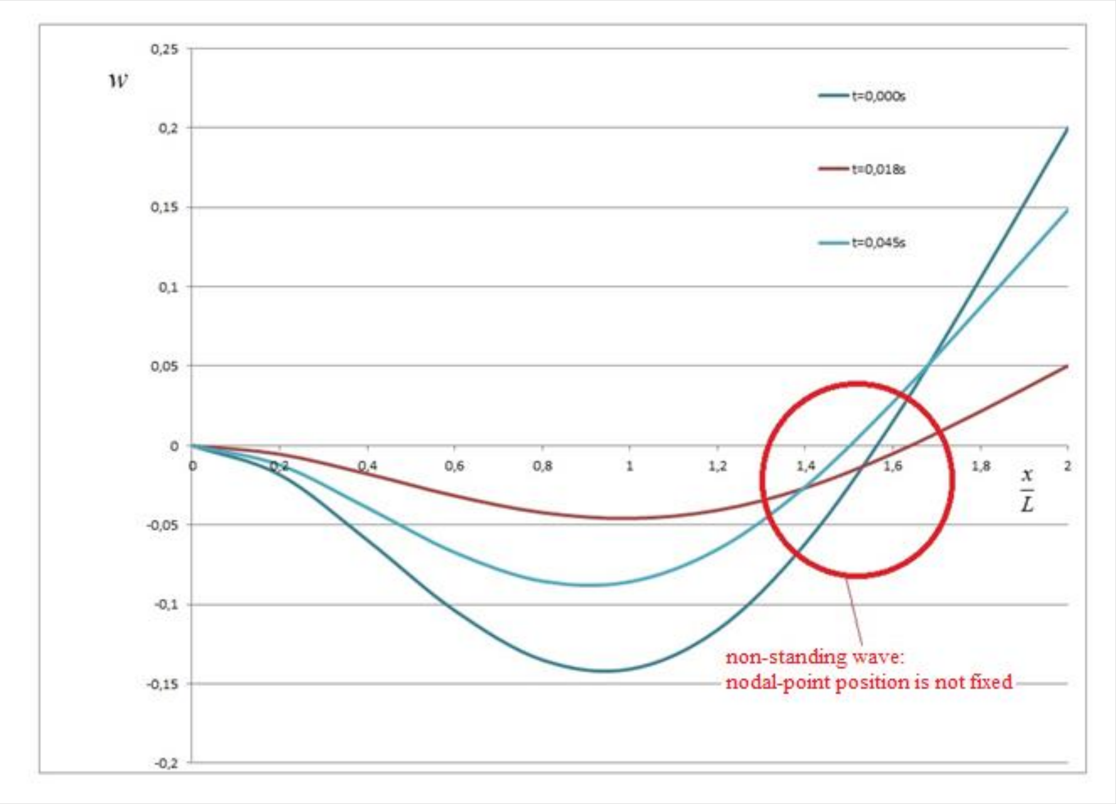
$$c_1^2 = 0.68055E+04 \qquad c_2^2 = 0.17946E+01 \qquad c_6^{222} = -0.10416E+06$$
$$c_7^{222} = 0.18909E+03 \qquad c_8^{222} = 0.27836E+02 \qquad c_9^{222} = -0.93038E\text{-}02$$
$$a_{1,14}^2 = -0.71314E+00 \qquad a_{6,14}^{222} = -0.133399E+01 \qquad a_{7,14}^{222} = 0.75882E-02$$
$$a_{8,14}^{222} = 0.17499E-03 \qquad a_{1,29}^2 = 0.1000E+01$$

$$\begin{cases} \omega_{02} = \sqrt{c_1^2} = 82.49\ rad/s \\ \xi_2 = \dfrac{c_2^2}{2\sqrt{c_1^2}} = 0.0108 \end{cases}$$

Reduced-Order Modelling

Example 1 (classical resonance)

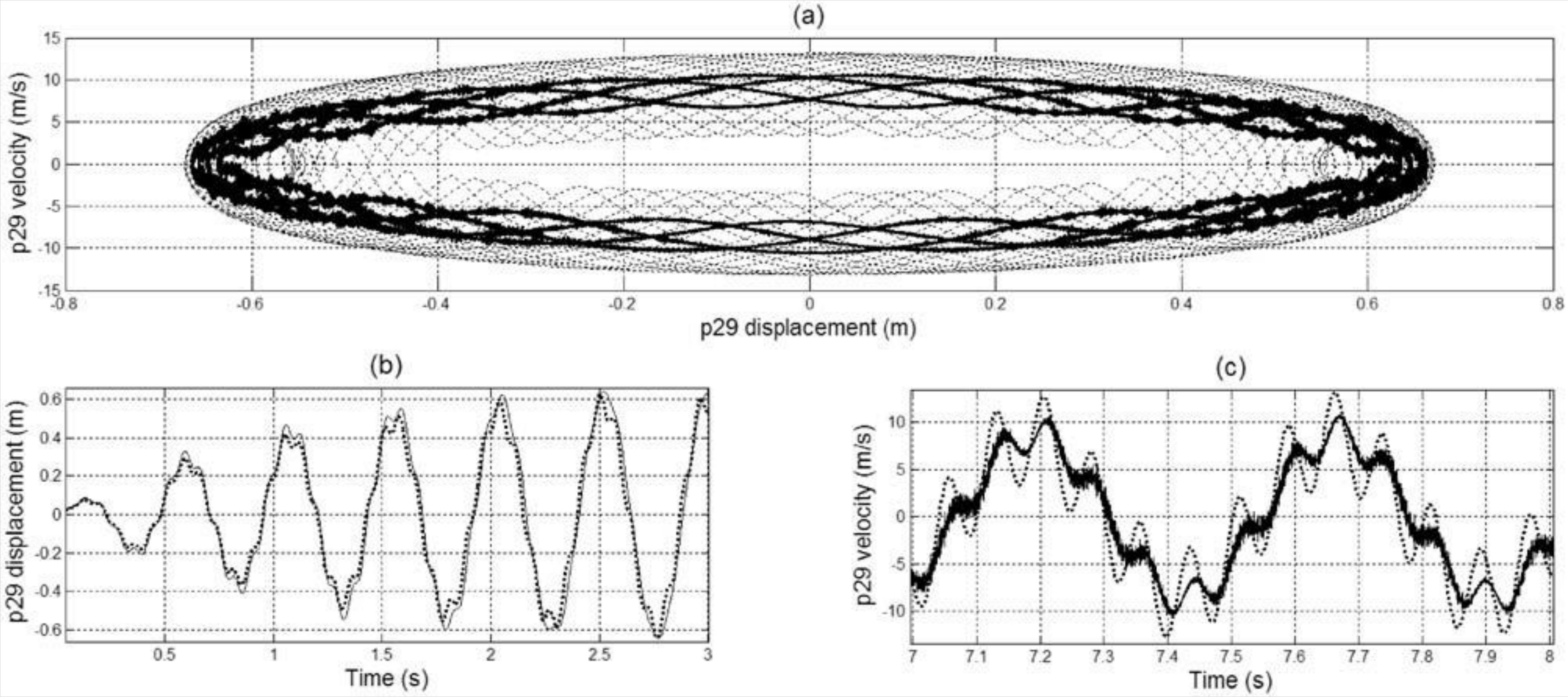
Second mode is not a standing wave!



Reduced-Order Modelling

Example 1 (classical resonance)

First-mode generalized force is not strongly influenced by non-linearities, but second-mode generalized force is influenced by non-linearities



Master (modal) coordinate response (ROM in dotted line; FEM in continuous line)

Reduced-Order Modelling

Example 1 (classical resonance)

Use of modal relationships to recover other generalized displacements and velocities

Example: p_{30} and $\dot{p}_{30}$

$$p_{30} = a_{1,30}^{1} U_1 + a_{1,30}^{2} U_2 + a_{2,30}^{1} \dot{U}_1 + a_{2,30}^{2} \dot{U}_2 + a_{3,30}^{11} \left(U_1\right)^2 + a_{3,30}^{22} \left(U_1\right)^2$$
$$+ a_{4,30}^{11} U_1 \dot{U}_1 + a_{4,30}^{22} U_2 \dot{U}_2 + a_{5,30}^{11} \left(\dot{U}_1\right)^2 + a_{5,30}^{22} \left(\dot{U}_2\right)^2 + a_{6,30}^{111} \left(U_1\right)^3 + a_{6,30}^{222} \left(U_2\right)^3$$
$$+ a_{7,30}^{111} \left(U_1\right)^2 \dot{U}_1 + a_{7,30}^{222} \left(U_2\right)^2 \dot{U}_2 + a_{8,30}^{111} U_1 \left(\dot{U}_1\right)^2 + a_{8,30}^{222} U_2 \left(\dot{U}_2\right)^2 + a_{9,30}^{111} \left(\dot{U}_1\right)^3 + a_{9,30}^{222} \left(\dot{U}_2\right)^3$$

$$\dot{p}_{30} = b_{1,30}^{1} U_1 + b_{1,30}^{2} U_2 + b_{2,30}^{1} \dot{U}_1 + b_{2,30}^{2} \dot{U}_2 + b_{3,30}^{11} \left(U_1\right)^2 + b_{3,30}^{22} \left(U_1\right)^2$$
$$+ b_{4,30}^{11} U_1 \dot{U}_1 + b_{4,30}^{22} U_2 \dot{U}_2 + b_{5,30}^{11} \left(\dot{U}_1\right)^2 + b_{5,30}^{22} \left(\dot{U}_2\right)^2 + b_{6,30}^{111} \left(U_1\right)^3 + b_{6,30}^{222} \left(U_2\right)^3$$
$$+ b_{7,30}^{111} \left(U_1\right)^2 \dot{U}_1 + b_{7,30}^{222} \left(U_2\right)^2 \dot{U}_2 + b_{8,30}^{111} U_1 \left(\dot{U}_1\right)^2 + b_{8,30}^{222} U_2 \left(\dot{U}_2\right)^2 + b_{9,30}^{111} \left(\dot{U}_1\right)^3 + b_{9,30}^{222} \left(\dot{U}_2\right)^3$$

Reduced-Order Modelling

Example 1 (classical resonance)

Use of modal relationships to recover other generalized displacements and velocities

Example: p_{30} and $\dot{p}_{30}$

First-mode coefficients

$$a^{1}_{1,30} = 0.68828E+00 \quad a^{1}_{2,30} = -0.67052E-03 \quad a^{111}_{6,30} = 0.76285E-02$$
$$a^{111}_{7,30} = 0.13231E-03 \quad a^{111}_{8,30} = -0.69597E-04$$
$$b^{1}_{1,30} = 0.11630E+00 \quad b^{1}_{2,30} = 0.68948E+00 \quad b^{111}_{6,30} = -0.50102E-01$$
$$b^{111}_{7,30} = 0.46600E-01 \quad b^{111}_{8,30} = 0.69290E-03 \quad b^{111}_{9,30} = -0.69117E-04$$

Second-mode coefficients

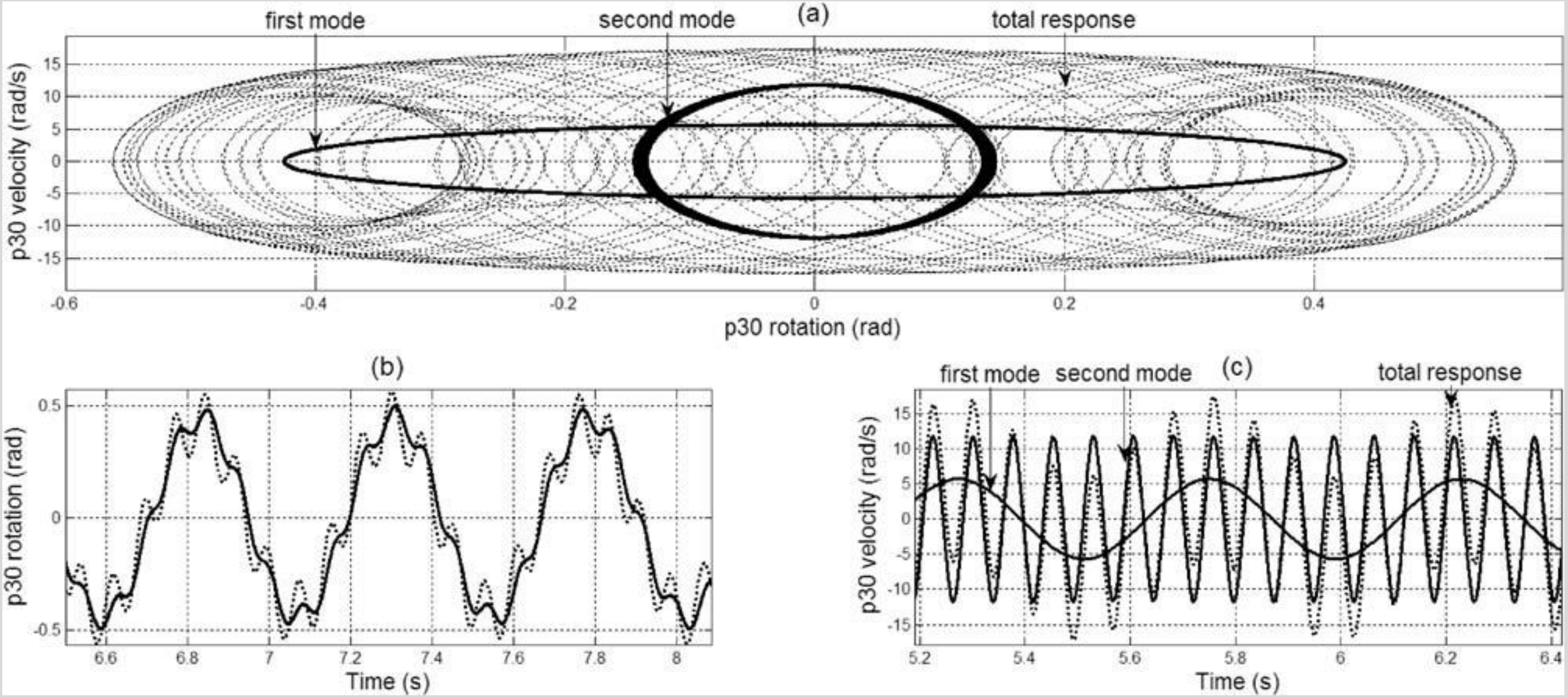
$$a^{2}_{1,30} = 0.23896E+01 \quad a^{2}_{2,30} = -0.59586E-03 \quad a^{222}_{6,30} = 0.20454E+00$$
$$a^{222}_{7,30} = -0.28702E-02 \quad a^{222}_{8,30} = 0.63985E-04$$
$$b^{2}_{1,30} = 0.40551E+01 \quad b^{2}_{2,30} = 0.23907E+01 \quad b^{222}_{6,30} = -0.42533E+02$$
$$b^{222}_{7,30} = -0.79776E+01 \quad b^{222}_{8,30} = 0.17033E-01 \quad b^{222}_{9,30} = 0.63941E-03$$

Reduced-Order Modelling

Example 1 (classical resonance)

Use of modal relationships to recover other generalized displacements and velocities

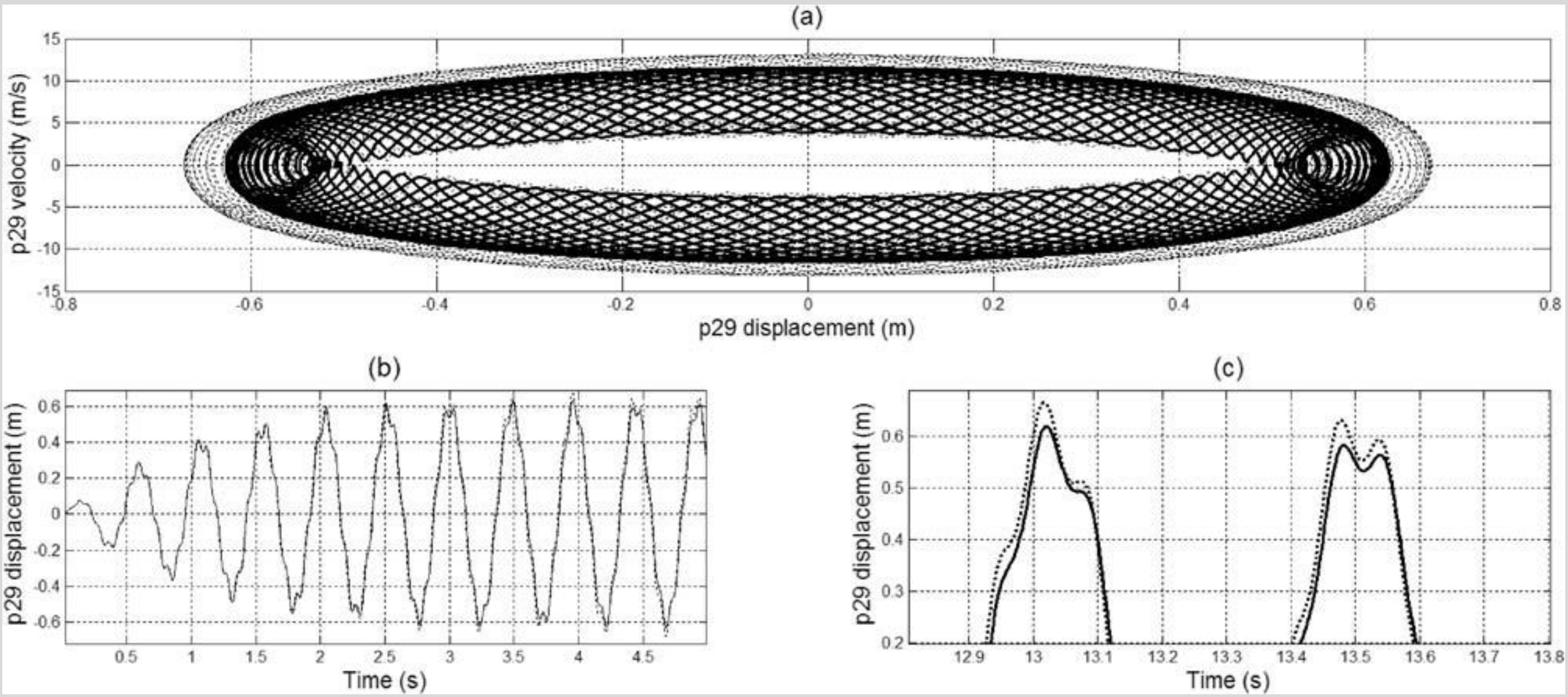
Example: p_{30} and $\dot{p}_{30}$



Slave coordinate response (ROM in dotted line; FEM in continuous line)

Reduced-Order Modelling

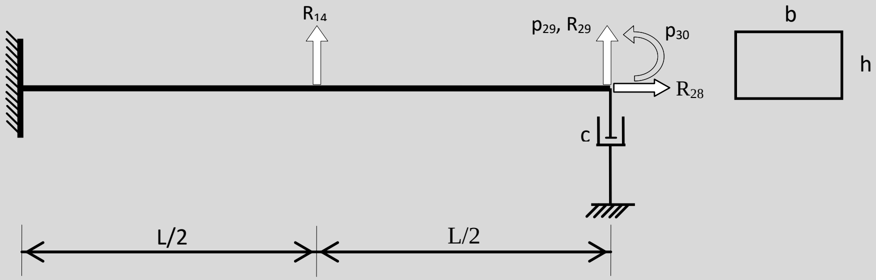
Example 1 (classical resonance)



Master (modal) coordinate response with (dotted line) and without (continuous line) velocity contents in the non-linear mode

Reduced-Order Modelling

Example 2 (parametric resonance)



$$b = 0.040m$$

$$h = 0.003m$$

$$E = 2.1 \times 10^{11} N / m^2$$

$$\rho = 7{,}800 kg / m^3$$

$$c = 0$$

$$R_{14}(t) = R_{29}(t) = 0$$

$$R_{28}(t) = 1.62 \cos(2\omega_{0u} t)$$

FE model with 30 degrees of freedom.

Chosen modal coordinates: $U_u = p_{29}$, with $u = 1, 2$

Reduced-Order Modelling

Example 2 (parametric resonance)

First-mode coefficients

$$c_1^1 = 0.15602E+02 \qquad c_6^{111} = 0.53036E+00 \qquad c_8^{111} = 0.28812E+00$$
$$a_{3,28}^{11} = -0.29049E+00$$

$$\Rightarrow \quad \omega_{01} = \sqrt{c_1^1} = 3.95\, rad/s$$

Second-mode coefficients

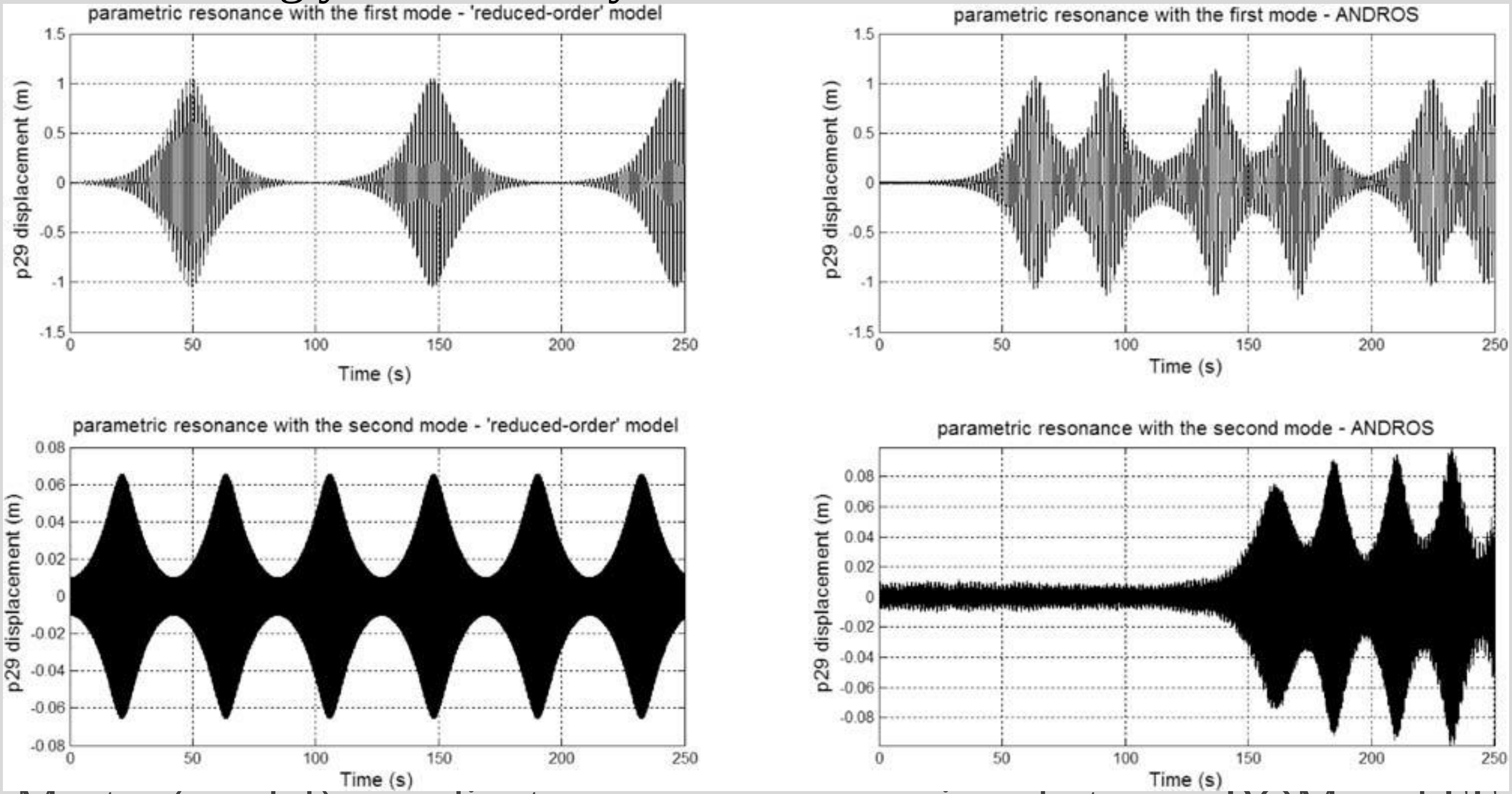
$$c_1^2 = 0.61277E+03 \qquad c_6^{222} = -0.93646E+04 \qquad c_8^{222} = 0.27923E+02$$
$$a_{3,28}^{22} = -0.20261E+01$$

$$\Rightarrow \quad \omega_{02} = \sqrt{c_1^2} = 24.75\, rad/s$$

Reduced-Order Modelling

Example 2 (parametric resonance)

Generalized modal forces and modal oscillators are strongly influenced by non-linearities in both modes



Master (modal) coordinate response: comparison between ROM and FE

Reduced-Order Modelling

Example 2 (parametric resonance)

Use of modal relationships to recover other generalized displacements and velocities

Example: p_{30} and $\dot{p}_{30}$

$$p_{30} = a^{1}_{1,30}U_1 + a^{2}_{1,30}U_2 + a^{1}_{2,30}\dot{U}_1 + a^{2}_{2,30}\dot{U}_2 + a^{11}_{3,30}\left(U_1\right)^2 + a^{22}_{3,30}\left(U_1\right)^2$$

$$+a^{11}_{4,30}U_1\dot{U}_1 + a^{22}_{4,30}U_2\dot{U}_2 + a^{11}_{5,30}\left(\dot{U}_1\right)^2 + a^{22}_{5,30}\left(\dot{U}_2\right)^2 + a^{111}_{6,30}\left(U_1\right)^3 + a^{222}_{6,30}\left(U_2\right)^3$$

$$+a^{111}_{7,30}\left(U_1\right)^2\dot{U}_1 + a^{222}_{7,30}\left(U_2\right)^2\dot{U}_2 + a^{111}_{8,30}U_1\left(\dot{U}_1\right)^2 + a^{222}_{8,30}U_2\left(\dot{U}_2\right)^2 + a^{111}_{9,30}\left(\dot{U}_1\right)^3 + a^{222}_{9,30}\left(\dot{U}_2\right)^3$$

$$\dot{p}_{30} = b^{1}_{1,30}U_1 + b^{2}_{1,30}U_2 + b^{1}_{2,30}\dot{U}_1 + b^{2}_{2,30}\dot{U}_2 + b^{11}_{3,30}\left(U_1\right)^2 + b^{22}_{3,30}\left(U_1\right)^2$$

$$+b^{11}_{4,30}U_1\dot{U}_1 + b^{22}_{4,30}U_2\dot{U}_2 + b^{11}_{5,30}\left(\dot{U}_1\right)^2 + b^{22}_{5,30}\left(\dot{U}_2\right)^2 + b^{111}_{6,30}\left(U_1\right)^3 + b^{222}_{6,30}\left(U_2\right)^3$$

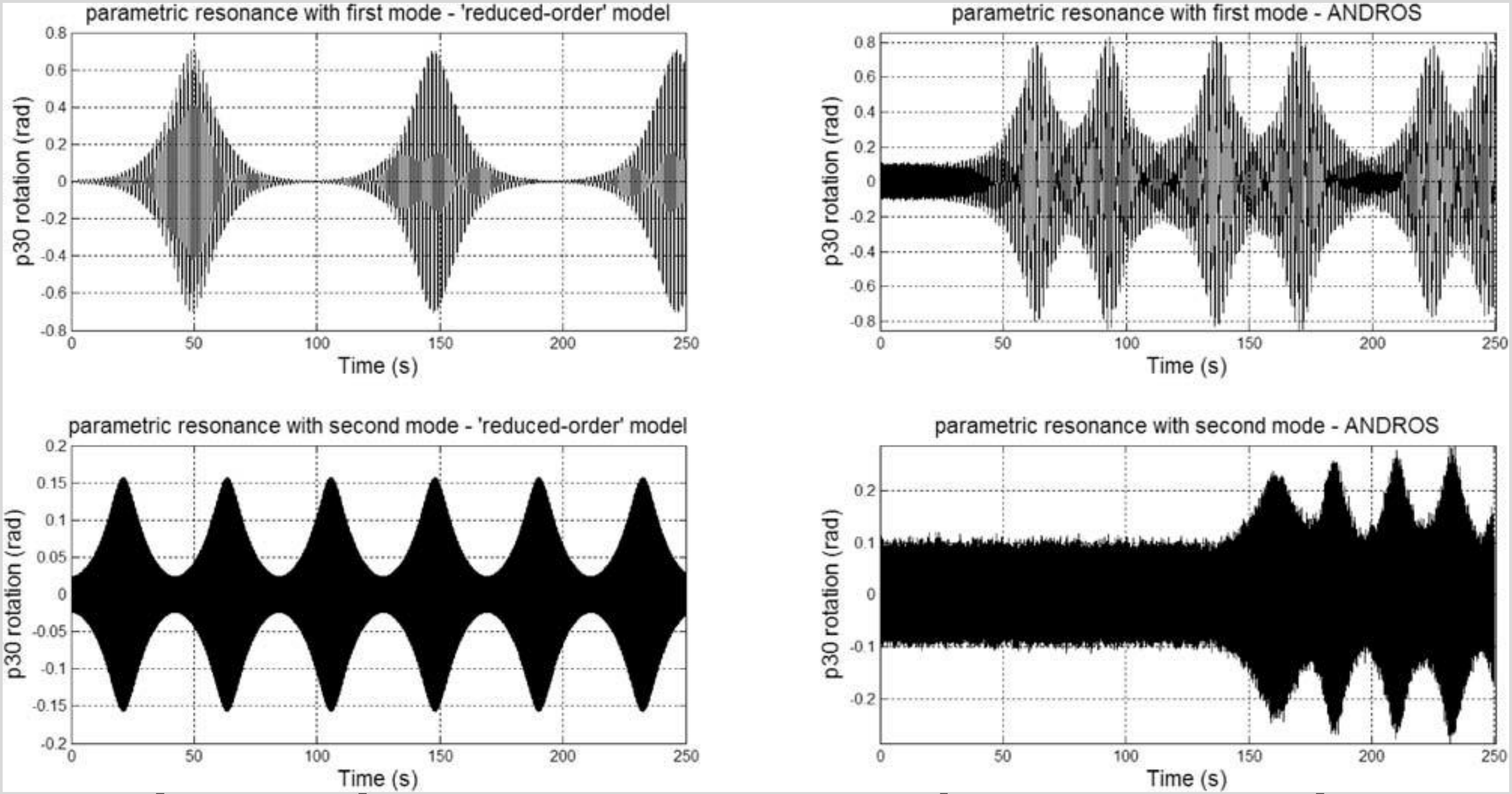
$$+b^{111}_{7,30}\left(U_1\right)^2\dot{U}_1 + b^{222}_{7,30}\left(U_2\right)^2\dot{U}_2 + b^{111}_{8,30}U_1\left(\dot{U}_1\right)^2 + b^{222}_{8,30}U_2\left(\dot{U}_2\right)^2 + b^{111}_{9,30}\left(\dot{U}_1\right)^3 + b^{222}_{9,30}\left(\dot{U}_2\right)^3$$

Reduced-Order Modelling

Example 2 (parametric resonance)

Use of modal relationships to recover other generalized displacements and velocities

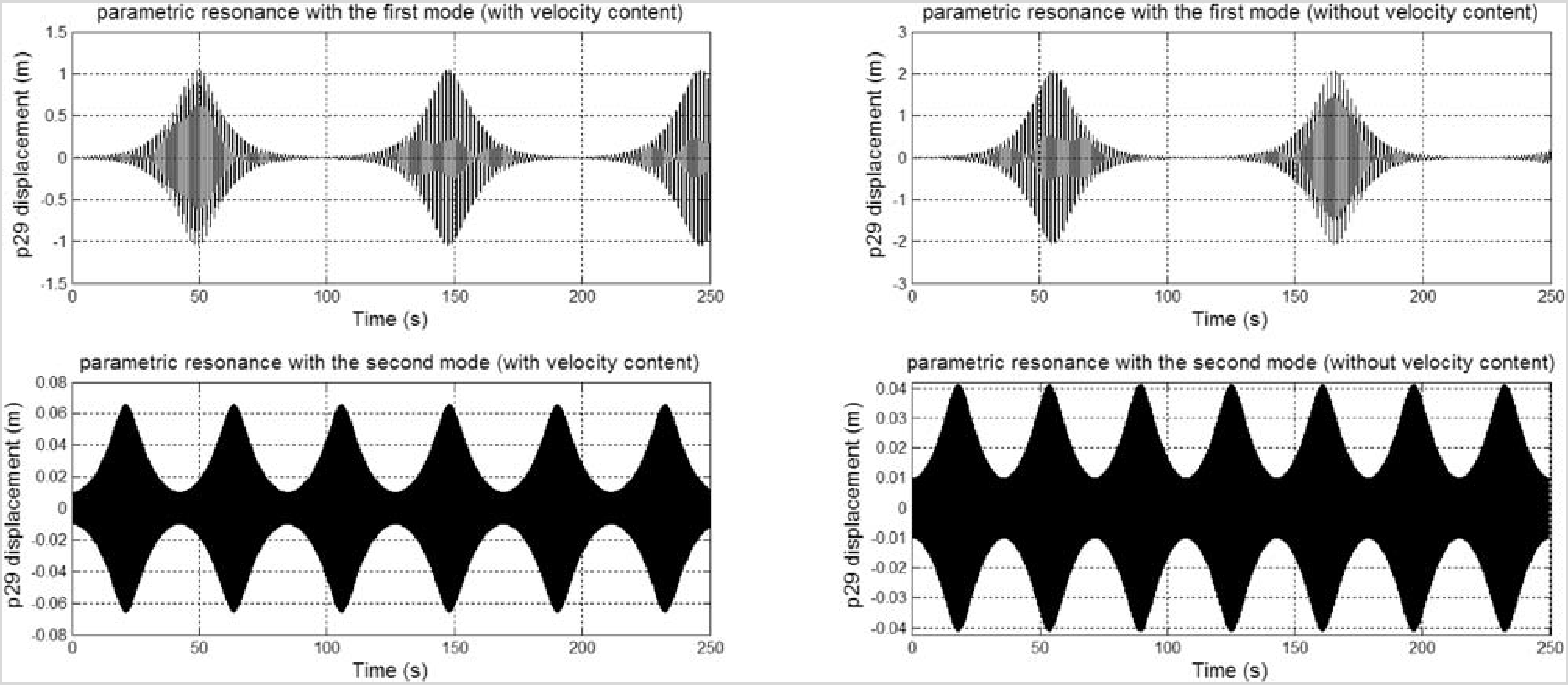
Example: p_{30} and $\dot{p}_{30}$



Slave coordinate response: comparison between ROM and FE

Reduced-Order Modelling

Example 2 (parametric resonance)

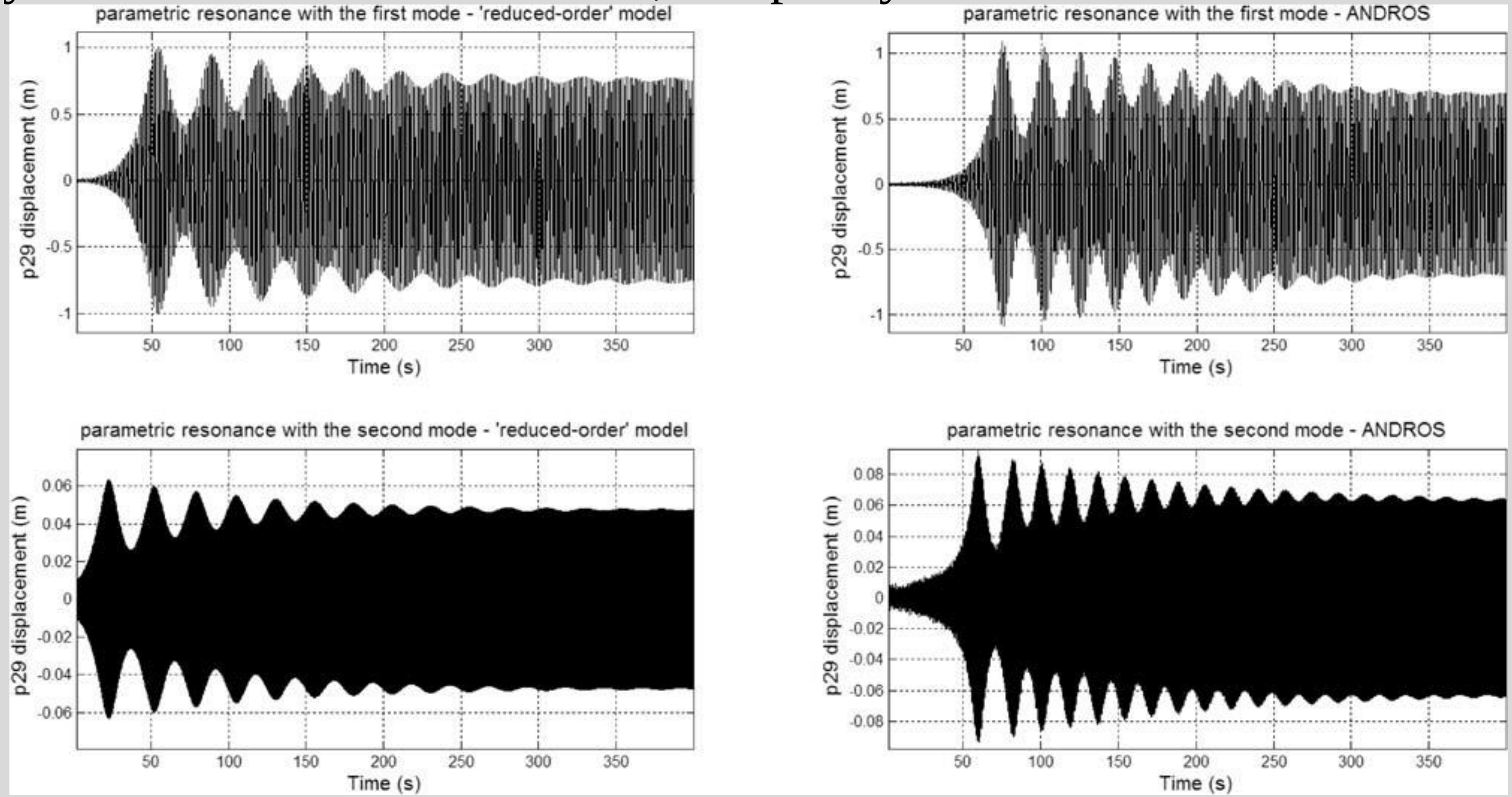


Master (modal) coordinate response with/without velocity contents within the non-linear mode

Reduced-Order Modelling

Example 2 (parametric resonance)

Generalized modal forces and modal oscillators are strongly influenced by non-linearities in both modes; damped system $c = 0.01 Nsm^{-1}$

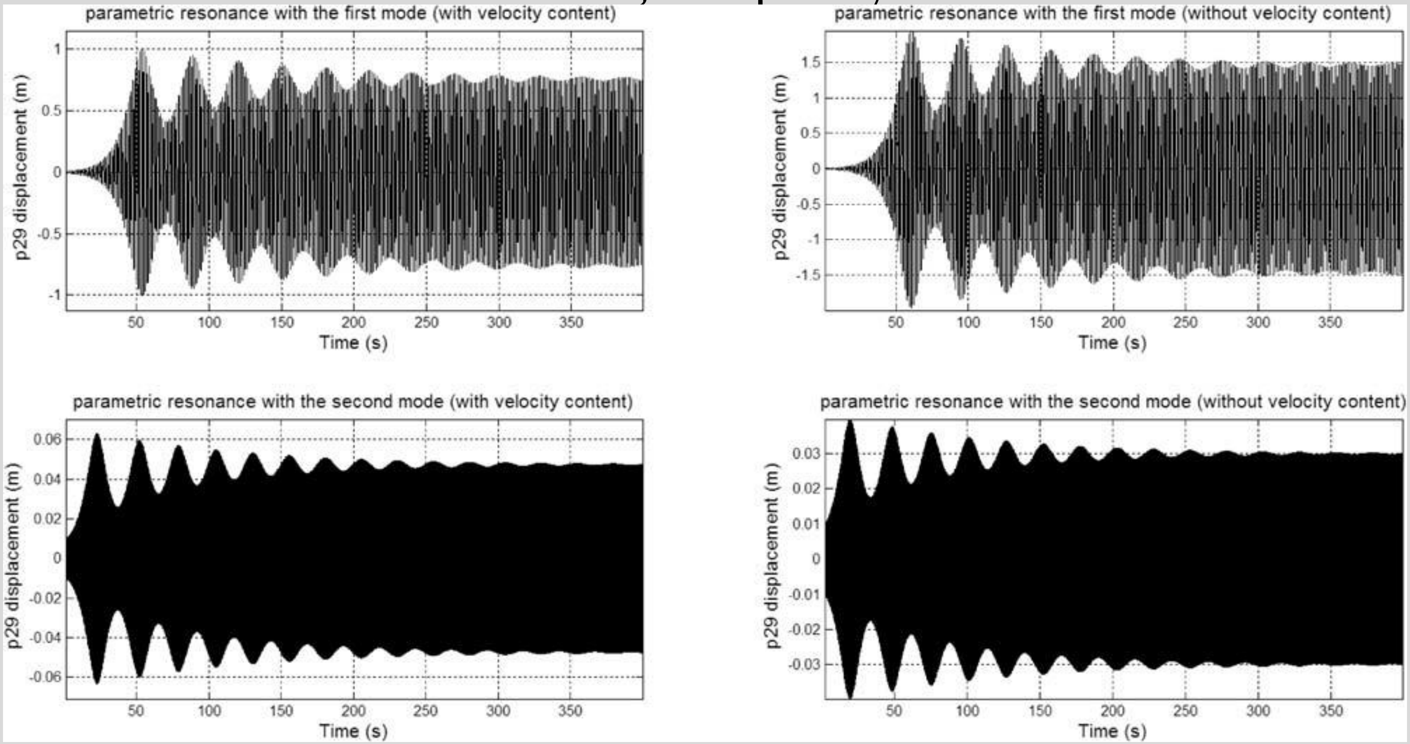


Master (modal) coordinate response: comparison between ROM and FE

Reduced-Order Modelling

Example 2 (parametric resonance)

Generalized modal forces and modal oscillators are strongly influenced by non-linearities in both modes; damped system $c = 0.01 Nsm^{-1}$



Master (modal) coordinate response with and without velocity contents in the non-linear mode