Coriolis force, $-2m\boldsymbol{\omega}\wedge\dot{\boldsymbol{r}}$. Because the magnetic force on a charged particle tends to produce rotation about the direction of the magnetic field, rotating frames are also useful in many problems involving a magnetic field.

Problems

1. Find the centrifugal acceleration at the equator of the planet Jupiter and of the Sun. In each case, express your answer also as a fraction of the surface gravity. (The rotation periods are 10 hours and 27 days, respectively, the radii 7.1×10^4 km and 7.0×10^5 km, and the masses 1.9×10^{27} kg and 2.0×10^{30} kg.)
2. Water in a rotating container of radius 50 mm is 30 mm lower in the centre than at the edge. Find the angular velocity of the container.
3. The water in a circular lake of radius 1 km in latitude 60° is at rest relative to the Earth. Find the depth by which the centre is depressed relative to the shore by the centrifugal force. For comparison, find the height by which the centre is *raised* by the curvature of the Earth's surface. (Earth radius = 6400 km.)
4. Find the velocity relative to an inertial frame (in which the centre of the Earth is at rest) of a point on the Earth's equator. An aircraft is flying above the equator at $1000\,\mathrm{km\,h^{-1}}$. Assuming that it flies straight and level (*i.e.*, at a constant altitude above the surface) what is its velocity relative to the inertial frame (a) if it flies north, (b) if it flies west, and (c) if it flies east?
5. The apparent weight of the aircraft in Problem 4 when on the ground at the equator is 100 t weight. What is its apparent weight in each of the three cases (a)–(c)?
6. A bird of mass 2 kg is flying at $10\,\mathrm{m\,s^{-1}}$ in latitude 60°N, heading due east. Find the horizontal and vertical components of the Coriolis force acting on it.
7. The wind speed in colatitude θ is v. By considering the forces on a small volume of air, show that the pressure gradient required to balance the horizontal component of the Coriolis force, and thus to maintain a constant wind direction, is $\mathrm{d}p/\mathrm{d}x = 2\omega\rho v\cos\theta$, where ρ is the density of the air. Evaluate this gradient in $\mathrm{mbar\,km^{-1}}$ for a wind speed of $50\,\mathrm{km\,h^{-1}}$ in latitude 30°N. (1 bar = 10^5 Pa; density of air = $1.3\,\mathrm{kg\,m^{-3}}$.)

8. An aircraft is flying at $800\,\mathrm{km\,h^{-1}}$ in latitude $55°$N. Find the angle through which it must tilt its wings to compensate for the horizontal component of the Coriolis force.
9. An orbiting space station may be made to rotate to provide an artificial gravity. Given that the radius is $25\,\mathrm{m}$, find the rotation period required to produce an apparent gravity equal to $0.7g$. A man whose normal weight is $75\,\mathrm{kg}$ weight runs around the station in one direction and then the other (*i.e.*, on a circle on the inside of the cylindrical wall) at $5\,\mathrm{m\,s^{-1}}$. Find his apparent weight in each case. What effects will he experience if he climbs a ladder to a higher level (*i.e.*, closer to the axis), climbing at $1\,\mathrm{m\,s^{-1}}$?
10. A beam of particles of charge q and velocity v is emitted from a point source, roughly parallel with a magnetic field $\boldsymbol{B}$, but with a small angular dispersion. Show that the effect of the field is to focus the beam to a point at a distance $z = 2\pi mv/|q|B$ from the source. Calculate the focal distance for electrons of kinetic energy $500\,\mathrm{eV}$ in a magnetic field of $0.01\,\mathrm{T}$. (Charge on electron $= -1.6 \times 10^{-19}\,\mathrm{C}$, mass $= 9.1 \times 10^{-31}\,\mathrm{kg}$, $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$.)
11. *Write down the equation of motion for a charged particle in uniform, *parallel* electric and magnetic fields, both in the z-direction, and solve it, given that the particle starts from the origin with velocity $(v, 0, 0)$. A screen is placed at $x = a$, where $a \ll mv/qB$. Show that the locus of points of arrival of particles with given m and q, but different speeds v, is approximately a parabola. How does this locus depend on m and q?
12. A beam of particles with velocity $(v, 0, 0)$ enters a region containing crossed electric and magnetic fields, as in the example at the end of §5.2. Show that if the ratio E/B is correctly chosen the particles are undeviated, while particles with other speeds follow curved trajectories. Suppose the particles have velocities equal to v in magnitude, but with a small angular dispersion. Show that if the path length l is correctly chosen, all such particles are focussed onto a line parallel to the z-axis. (Thus a slit at that point can be used to select particles with a given speed.) For electrons of velocity $10^8\,\mathrm{m\,s^{-1}}$ in a magnetic field of $0.02\,\mathrm{T}$, find the required electric field, and the correct (smallest possible) choice for l.
13. The angular velocity of the electron in the lowest Bohr orbit of the hydrogen atom is approximately $4 \times 10^{16}\,\mathrm{s^{-1}}$. What is the largest

magnetic field which may be regarded as small in this case, in the sense of §5.5? Determine the Larmor frequency in a field of 2 T.

14. *The orbit of an electron (charge $-e$) around a nucleus (charge Ze) is a circle of radius a in a plane perpendicular to a uniform magnetic field $\boldsymbol{B}$. By writing the equation of motion in a frame rotating with the electron, show that the angular velocity ω is given by one of the roots of the equation

$$m\omega^2 - eB\omega - Ze^2/4\pi\epsilon_0 a^3 = 0.$$

Verify that for small values of B, this agrees with §5.5. Evaluate the two roots if $B = 10^5$ T, $Z = 1$ and $a = 5.3 \times 10^{-11}$ m. (Note, however, that in reality a would be changed by the field.)

15. *A projectile is launched due north from a point in colatitude θ at an angle $\pi/4$ to the horizontal, and aimed at a target whose distance is y (small compared to Earth's radius R). Show that if no allowance is made for the effects of the Coriolis force, the projectile will miss its target by a distance

$$x = \omega \left(\frac{2y^3}{g}\right)^{1/2} (\cos\theta - \tfrac{1}{3}\sin\theta).$$

Evaluate this distance if $\theta = 45°$ and $y = 40$ km. Why is it that the deviation is to the east near the north pole, but to the west both on the equator and near the south pole? (Neglect atmospheric resistance.)

16. *Solve the problem of a particle falling from height h above the equator by using an inertial frame, and verify that the answer agrees with that found using a rotating frame. (*Hint*: Use equations (3.48). Recall Fig. 5.8.)

17. Find the equations of motion for a particle in a frame rotating with *variable* angular velocity $\boldsymbol{\omega}$, and show that there is another apparent force of the form $-m\dot{\boldsymbol{\omega}} \wedge \boldsymbol{r}$. Discuss the physical origin of this force.

18. Find the equation of motion for a particle in a *uniformly accelerated* frame, with acceleration $\boldsymbol{a}$. Show that for a particle moving in a uniform gravitational field, and subject to other forces, the gravitational field may be eliminated by a suitable choice of $\boldsymbol{a}$.

19. *The co-ordinates (x, y, z) of a particle with respect to a uniformly rotating frame may be related to those with respect to a fixed inertial

frame, (x^*, y^*, z^*), by the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix}.$$

(Here, we use matrix notation: this stands for three separate equations,

$$x = \cos\omega t \cdot x^* + \sin\omega t \cdot y^*,$$

etc.) Write down the inverse relation giving (x^*, y^*, z^*) in terms of (x, y, z). By differentiating with respect to t, rederive the relation (5.15) between $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$ and $\ddot{\boldsymbol{r}}$. [*Hint*: Note that $\ddot{\boldsymbol{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$, while $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$ is the vector obtained by applying the above transformation to $(\ddot{x}^*, \ddot{y}^*, \ddot{z}^*)$.]

20. Another way of deriving the equation of motion (5.16) is to use Lagrange's equations. Express the kinetic energy $\frac{1}{2}m(\mathrm{d}\boldsymbol{r}/\mathrm{d}t)^2$ in terms of (x, y, z), and show that Lagrange's equations (3.44) reproduce (5.16) for the case where the force is conservative.

Problems

1. A double star is formed of two components, each with mass equal to that of the Sun. The distance between them is 1 AU (see Chapter 4, Problem 2). What is the orbital period?
2. Where is the centre of mass of the Sun–Jupiter system? (The mass ratio is $M_S/M_J = 1047$. See Chapter 4, Problems 2 and 3.) Through what angle does the Sun's position as seen from the Earth oscillate because of the gravitational attraction of Jupiter?
3. The parallax of a star (the angle subtended at the star by the radius of the Earth's orbit) is ϖ. The star's position is observed to oscillate with angular amplitude α and period τ. If the oscillation is interpreted as being due to the existence of a planet moving in a circular orbit around the star, show that its mass m_1 is given by

$$\frac{m_1}{M_S} = \frac{\alpha}{\varpi}\left(\frac{M\tau_E}{M_S\tau}\right)^{2/3},$$

   where M is the total mass of star plus planet, M_S is the Sun's mass, and $\tau_E = 1$ year. Evaluate the mass m_1 if $M = 0.25M_S$, $\tau = 16$ years, $\varpi = 0.5''$ and $\alpha = 0.01''$. What conclusion can be drawn without making the assumption that the orbit is circular?
4. Two particles of masses m_1 and m_2 are attached to the ends of a light spring. The natural length of the spring is l, and its tension is k times its extension. Initially, the particles are at rest, with m_1 at a height l above m_2. At $t = 0$, m_1 is projected vertically upward with velocity v. Find the positions of the particles at any subsequent time (assuming that v is not so large that the spring is expanded or compressed beyond its elastic limit).
5. *Prove that in an elastic scattering process the angle $\theta + \alpha$ between the emerging particles is related to the recoil angle α by

$$\frac{\tan(\theta+\alpha)}{\tan\alpha} = \frac{m_1 + m_2}{m_1 - m_2}.$$

   (*Hint*: Express both tangents in terms of $\tan\frac{1}{2}\theta^*$.) What is the mass ratio if the particles emerge at right angles to each other?
6. A proton is elastically scattered through an angle of 56° by a nucleus, which recoils at an angle $\alpha = 60°$. Find the atomic mass of the nucleus, and the fraction of the kinetic energy transferred to it.

7. An experiment is to be designed to measure the differential cross-section for elastic pion–proton scattering at a CM scattering angle of 70° and a pion CM kinetic energy of 490 keV. (The electron-volt (eV) is the atomic unit of energy.) Find the angles in the Lab at which the scattered pions, and the recoiling protons, should be detected, and the required Lab kinetic energy of the pion beam. (The ratio of pion to proton mass is 1/7.)
8. *An unstable particle of mass $M = m_1 + m_2$ decays into two particles of masses m_1 and m_2, releasing an amount of energy Q. Determine the kinetic energies of the two particles in the CM frame. Given that $m_1/m_2 = 4$, $Q = 1\,\text{MeV}$, and that the unstable particle is moving in the Lab with kinetic energy 2.25 MeV, find the maximum and minimum Lab kinetic energies of the particle of mass m_1.
9. The molecules in a gas may be treated as identical hard spheres. Find the average loss of kinetic energy of a molecule with kinetic energy T in a collision with a stationary molecule. (*Hint*: Use the fact that the collisions are isotropic in the CM frame, so that all values of $\cos\theta^*$ between ± 1 are equally probable.) How many collisions are required, on average, to reduce the velocity of an exceptionally fast molecule by a factor of 1000?
10. Two identical charged particles, each of mass m and charge e, are initially far apart. One of the particles is at rest at the origin, and the other is approaching it with velocity v along the line $x = b, y = 0$, where $b = e^2/2\pi\epsilon_0 mv^2$. Find the scattering angle in the CM frame, and the directions in which the two particles emerge in the Lab. (See §4.7.)
11. *Find the distance of closest approach for the particles in Problem 10, and the velocity of each at the moment of closest approach.
12. Obtain the relation between the total kinetic energy in the CM and Lab frames. Discuss the limiting cases of very large and very small mass for the target.
13. *Suppose that the asteroid of Chapter 4, Problem 17, has a mass of $6 \times 10^{20}\,\text{kg}$. Find the proportional change in the kinetic energy of the Earth in this encounter. What is the resulting change in the semi-major axis of the Earth's orbit? By how much is its orbital period lengthened? (*Note* that the postulated event is exceedingly improbable.)
14. Calculate the differential cross-section for the scattering of identical hard spheres directly in the Lab frame.

15. Find the Lab differential cross-section for the scattering of identical particles of charge e and mass m, if the incident velocity is v. (See (4.50).)
16. At low energies, protons and neutrons behave roughly like hard spheres of equal mass and radius about $1.3 \times 10^{-14}\,\mathrm{m}$. A parallel beam of neutrons, with a flux of 3×10^{10} neutrons $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$, strikes a target containing 4×10^{22} protons. A circular detector of radius 20 mm is placed 0.7 m from the target, in a direction making an angle of 30° to the beam direction. Calculate the rate of detection of neutrons, and of protons.
17. *Write down the equations of motion for a pair of charged particles of equal masses m, and of charges q and $-q$, in a uniform electric field $\boldsymbol{E}$. Show that the field does not affect the motion of the centre of mass. Suppose that the particles are moving in circular orbits with angular velocity ω in planes parallel to the xy-plane, with $\boldsymbol{E}$ in the z-direction. Write the equations in a frame rotating with angular velocity ω, and hence find the separation of the planes.