Poularikas A. D. “Legendre Polynomials”
*The Handbook of Formulas and Tables for Signal Processing.*
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC,1999

# 21
# Legendre Polynomials


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## 21.1 Legendre Polynomials

### 21.1.1 Definition

$$P_n(t)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!\,t^{n-2k}}{2^n\,k!(n-k)!(n-2k)!}$$

$$[n/2]=\begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd}\end{cases}$$

### 21.1.2 Generating Function

$$w(t,s)=\frac{1}{\sqrt{1-2st+s^2}}=\begin{cases}\sum_{n=0}^{\infty}P_n(t)s^n & |s|<1 \\ \sum_{n=0}^{\infty}P_n(t)s^{-n-1} & |s|>1\end{cases} \qquad \text{generating function}$$

$$w(-t,-s) = w(t,s)$$

### 21.1.3 Rodrigues Formula

$$P_n(t)=\frac{1}{2^n\,n!}\frac{d^n}{dt^n}(t^2-1)^n \qquad n=0,1,2\cdots$$

### 21.1.4 Recursive Formulas

1. $(n+1)P_{n+1}(t)-(2n+1)t\,P_n(t)+n\,P_{n-1}(t)=0 \qquad n=1,2,\cdots$

2. $P'_{n+1}(t)-t\,P'_n(t)=(n+1)P_n(t) \qquad (P'(t)\doteq \text{derivative of } P(t) \qquad n=0,1,2,\cdots$

3. $t\,P_n'(t) - P_{n-1}'(t) = n\,P_n(t)$ $\qquad n = 1,2,\cdots$

4. $P_{n+1}'(t) - P_{n-1}'(t) = (2n+1)\,P_n(t)$ $\qquad n = 1,2,\cdots$

5. $(t^2-1)\,P_n'(t) = n\,t\,P_n(t) - n\,P_{n-1}(t)$

6. $P_0(t) = 1 \qquad P_1(t) = t$

**TABLE 21.1** Legendre Polynomials

| |
|---|
| $P_0 = 1$ |
| $P_1 = t$ |
| $P_2 = \frac{3}{2}t^2 - \frac{1}{2}$ |
| $P_3 = \frac{5}{2}t^3 - \frac{3}{2}t$ |
| $P_4 = \frac{35}{8}t^4 - \frac{30}{8}t^2 + \frac{3}{8}$ |
| $P_5 = \frac{63}{8}t^5 - \frac{70}{8}t^3 + \frac{15}{8}t$ |
| $P_6 = \frac{231}{16}t^6 - \frac{315}{16}t^4 + \frac{105}{16}t^2 - \frac{5}{16}$ |
| $P_7 = \frac{429}{16}t^7 - \frac{693}{16}t^5 + \frac{315}{16}t^3 - \frac{35}{16}t$ |

Figure 21.1 shows a few Legendre functions.

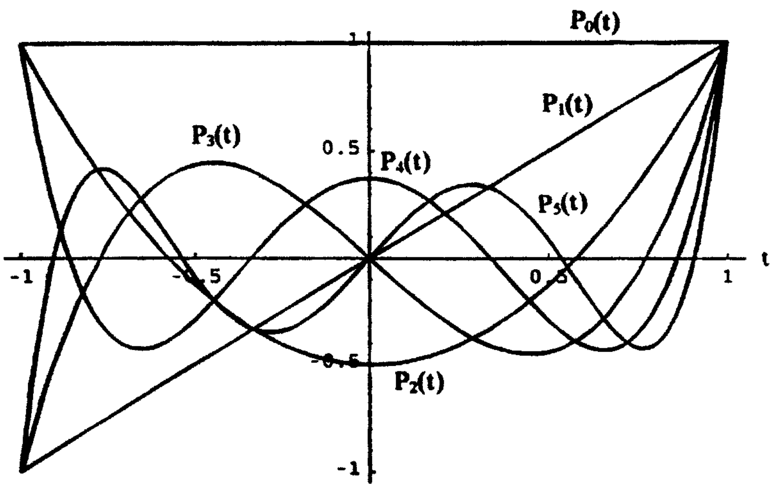


**FIGURE 21.1**

### 21.1.5 Legendre Differential Equation

If $y = P_n(x)$ $(n = 0,1,2,\cdots)$ is a solution to the second-order DE

$$(1-t^2)\,y'' - 2ty' + n(n+1)\,y = 0$$

For $t = \cos\varphi$: $\quad \dfrac{1}{\sin\varphi}\dfrac{d}{d\varphi}\left(\sin\varphi\,\dfrac{dy}{d\varphi}\right) + n(n+1)\,y = 0$

**Example**

From (21.1.4.4) and $t$ = 1 implies $0 = nP_n(1) - nP_{n-1}(1)$ or $P_n(1) = P_{n-1}(1)$. For $n = 1$, $P_1(1) = P_0(1) = 1$. For $n = 2$, $P_2(1) = P_1(1) = 1$ and so forth. Hence $P_n(1) = 1$.

## 21.1.6 Integral Representation

1. Laplace integral: $$P_n(t) = \frac{1}{\pi}\int_0^{\pi} [t + \sqrt{t^2 - 1}\cos\varphi]^n \, d\varphi$$

2. Mehler-Dirichlet formula: $$P_n(\cos\theta) = \frac{2}{\pi}\int_0^{\theta} \frac{\cos(n+\frac{1}{2})\psi}{\sqrt{2\cos\psi - \cos\theta}}\, d\psi \qquad 0 < \theta < \pi,\ n = 0,1,2,\ldots$$

3. Schläfli integral: $$P_n(t) = \frac{1}{2\pi j}\int_C \frac{(z^2 - 1)^n}{2^n (z - t)^{n+1}}\, dz$$

   where C is any regular, simple, closed curve surrounding $t$.

## 21.1.7 Complete Orthonormal System

$$\{[\tfrac{1}{2}(2n+1)]^{1/2} P_n(t)\}$$

The Legendre polynomials are orthogonal in [–1,1]

$$\int_{-1}^{1} P_n(t) P_m(t)\, dt = 0$$

$$\int_{-1}^{1} [P_n(t)]^2\, dt = \frac{2}{2n+1} \qquad n = 0,1,2\cdots$$

and therefore the set

$$\varphi_n(t) = \sqrt{\frac{2n+1}{2}}\, P_n(t) \qquad n = 0,1,2\cdots$$

is orthonormal.

## 21.1.8 Asymptotic Representation:

$$P_n(\cos\theta) \cong \sqrt{\frac{2}{\pi n \sin\theta}}\, \sin\left[\left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right], \qquad n \to \infty, \quad \delta \le \theta \le \pi - \delta$$

$\delta$ = fixed positive number

## 21.1.9 Series Expansion

If $f(t)$ is integrable in [–1,1] then

$$f(t) = \sum_{n=0}^{\infty} a_n P_n(t) \qquad -1 < t < 1$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(t)P_n(t)\,dt \qquad n = 0,1,2\cdots$$

For even $f(t)$, the series will contain term $P_n(t)$ of even index; if $f(t)$ is odd, the term of odd index only.

If the real function $f(t)$ is piecewise smooth in (–1,1) and if it is square integrable in (–1,1), then the series converges to $f(t)$ at every continuity point of $f(t)$. If there is a discontinuity at t then the series converges at $[f(t+0)+f(t-0)]/2$.

## 21.1.10 Change of Range

If a function $f(t)$ is defined in $[a,b]$, it is sometimes necessary in the applications to expand the function in a series in the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$t = \frac{2}{b-a}\left[x - \frac{b+a}{2}\right], \qquad a < b, \qquad \left[x = \frac{b-a}{2}t + \frac{b+a}{2}\right]$$

transform the interval $[a,b]$ of the x-axis into the interval [–1,1] of the t-axis. It is, therefore, sufficient to consider

$$f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right] = \sum_{n=0}^{\infty} a_n P_n(t)$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right]P_n(t)\,dt$$

The above equation can also be accomplished as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n X_n(t)$$

$$X_n(t) = \frac{1}{n!(b-a)^n}\,\frac{d^n (t-a)^n (t-b)^n}{dt^n}$$

$$a_n = \frac{2n+1}{b-a}\int_{b}^{a} f(t)X_n(t)\,dt$$

**Example**

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} 0 & -1 \le t < a \\ 1 & a < t \le 1 \end{cases}$$

Then

$$a_n = \frac{2n+1}{2}\int_{a}^{1} P_n(t)\,dt$$

Using (21.1.4.4), and noting that $P_n(1) = 1$ we obtain

$$a_n = -\tfrac{1}{2}[P_{n+1}(a) - P_{n-1}(a)], \qquad a_0 = \tfrac{1}{2}(1-a)$$

which leads to the expansion

$$f(t) \cong \frac{1}{2}(1-a) - \frac{1}{2}\sum_{n=1}^{\infty}[P_{n+1}(a) - P_{n-1}(a)]P_n(t), \qquad -1 < t < 1$$

**Example**

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 < t \le 1 \end{cases}$$

The function is an odd function and, therefore, $f(t)P_n(t)$ is an odd function of $P_n(t)$ with even index. Hence $a_n$ are zero for $n = 0,2,4,\ldots$. For odd index $n$, the product $f(t)P_n(t)$ is even and hence

$$a_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(t)P_n(t)\,dt = 2\left(n + \frac{1}{2}\right)\int_{0}^{1} P_n(t)\,dt \qquad n = 1,3,5,\cdots$$

Using (21.1.4.4) and setting $n = 2k + 1$, $k = 0,1,2\ldots$ we obtain

$$a_{2k+1} = (4k+3)\int_0^1 P_{2k+1}(t)\,dt = \int_0^1 [P'_{2k+2}(t) - P'_{2k}(t)]\,dt$$

$$= [P_{2k+2}(t) - P_{2k}(t)]\Big|_0^1 = P_{2k}(0) - P_{2k+2}(0)$$

where we have used the property $P_n(1) = 1$ for all $n$. But

$$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}$$

and, thus, we have

$$a_{2k+1} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)}{2^{2k+2}[(k+1)!]^2} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[1 + \frac{2k+1}{2k+2}\right]$$

$$= \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!}$$

The expansion is

$$f(t) = \sum_{n=0}^{\infty}\frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!}P_{2k+1}(t) \qquad -1 \le t \le 1$$

### 21.1.11 Expansion of Polynomials

If $q_m(t) = \sum_{k=0}^{m} c_k x^k$ is an arbitrary polynomial, then $q_m(t) = c_0 P_0(t) + c_1 P_1(t) + \cdots + c_m P_m(t)$ where $c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} q_m(t) P_n(t)\, dt = 0,\ n = 0,1,2\cdots.$ If $q_m(t)$ is a polynomial of degree m and $m < r$, then $\int_{-1}^{1} q_m(t) P_r(t)\, dt = 0,\ m < r.$

**Example**

To find $P_{2n}(0)$ we use the summation

$$P_{2n}(t) = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} \frac{(-1)^k (2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!} t^{2k}$$

with $k = 0$. Hence

$$P_{2n}(0) = \frac{(-1)^n (2n-1)!}{2^{2n-1}(n-1)!n!} = \frac{(-1)^n 2n[(2n-1)!]}{2^{2n} n[(n-1)!]n!} = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

**Example**

To evaluate $\int_0^1 P_m(t)\, dt$ for $m \neq 0$ we must consider the two cases: $m$ = odd and $m$ = even.

(a) $m$ = even and $m \neq 0$

$$\int_0^1 P_m(t)\, dt = \frac{1}{2}\int_{-1}^{1} P_m(t)\, dt = \frac{1}{2}\int_{-1}^{1} P_m(t)\cdot 1\, dt = \frac{1}{2}\int_{-1}^{1} P_m(t) P_0(t)\, dt = 0$$

The result is due to the orthogonality principle.

(b) $m$ = odd and $m \neq 0$. From the relation (see Table 21.2)

$$\int_{-1}^{1} P_m(t)\, dt = \frac{1}{2m+1}[P_{m-1}(t) - P_{m+1}(t)]$$

with $t = 0$ we obtain

$$\int_0^1 P_m(t)\, dt = \frac{1}{2m+1}[P_{m-1}(0) - P_{m+1}(0)]$$

Using the results of the previous example, we obtain

$$\int_0^1 P_m(t)\, dt = \frac{1}{2m+1}\left[\frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^{m-1}\left[\left(\frac{m-1}{2}\right)!\right]^2} - \frac{(-1)^{\frac{m+1}{2}}(m+1)!}{2^{m+1}\left[\left(\frac{m+1}{2}\right)!\right]^2}\right]$$

$$= \frac{(-1)^{\frac{m-1}{2}}(m-1)!(2m+1)(m+1)}{(2m+1)2^{m+1}\left(\frac{m+1}{2}\right)!\left(\frac{m+1}{2}\right)\left(\frac{m-1}{2}\right)!} = \frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^m\left(\frac{m+1}{2}\right)!\left(\frac{m-1}{2}\right)!} \qquad m = \text{odd}$$

## 21.2 Legendre Functions of the Second Kind (Second Solution)

### 21.2.1 Second Kind:

1. $Q_0 = \frac{1}{2}\ln\frac{1+t}{1-t}, \quad |t| < 1;$
2. $Q_1(t) = \frac{1}{2}t\ln\frac{1+t}{1-t} - 1, \quad |t| < 1;$
3. $Q_{n+1}(t) = \frac{2n+1}{n+1}tQ_n(t) - \frac{n}{n+1}Q_{n-1}(t), \quad n = 1,2,\cdots$
4. $Q_n(t) = P_n(t)Q_0(t) - \sum_{k=0}^{[\frac{1}{2}(n-1)]}\frac{2n-4k-1}{(2k+1)(n-k)}P_{n-2k-1}(t), \quad |t| < 1, \quad n = 1,2,\cdots$

   for $[\frac{1}{2}(n-1)]$ see 21.1.1.

### 21.2.2 Recursions

$Q_n(t)$ satisfies all the recurrence relations of $P_n(t)$.

### 21.2.3 Property

$$\frac{1}{x-t} = \sum_{n=0}^{\infty}(2n+1)P_n(t)Q_n(x)$$

### 21.2.4 Newman Formula

$$Q_n(t) = \frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{t-x}dx, \qquad n = 0,1,2\cdots$$

## 21.3 Associated Legendre Polynomials

### 21.3.1 Definition

If $m$ is a positive integer and $-1 \le t \le 1$, then

$$P_n^m(t) = (1-t^2)^{m/2}\frac{d^m P_n(t)}{dt^m} \qquad m = 1,2,\cdots,n$$

where $P_n^m(t)$ is known as the *associated Legendre function* or *Ferrers' functions*.

### 21.3.2 Rodrigues Formula

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n, \qquad m = 1,2,\cdots,n; \quad n+m \ge 0$$

### 21.3.3 Properties

1. $P_n^{-m}(t) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(t)$

2. $P_n^0(t) = P_n(t)$

3. $(n-m+1)P_{n+1}^m(t) - (2n+1)t\,P_n^m(t) + (n+m)P_{n-1}^m(t) = 0$

4. $(1-t^2)^{1/2}\,P_n^m(t) = \dfrac{1}{2n+1}[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$

5. $(1-t^2)^{1/2}\,P_n^m(t) = \dfrac{1}{2n+1}[(n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(t)]$

6. $P_n^m(t) = 2mt(1-t^2)^{-1/2}\,P_n^m(t) - [n(n+1) - m(m-1)]P_n^{m-1}(t)$

7. $P_n^{m+1}(t) = (t^2-1)^{-1/2}[(n-m)\,tP_n^m(t) - (n+m)P_{n-1}^m(t)]$

8. $P_{n+1}^m(t) = P_{n-1}^m(t) + (2n+1)\,(t^2-1)^{1/2}\;P_n^{m-1}(t)$

9. $P_n^m(t) = (t^2-1)^{m/2}\,\dfrac{d^m P_n(t)}{dt^m}$

10. $\displaystyle\int_{-1}^{1} P_n^m(t)P_k^m(t)\,dt = 0 \qquad k \neq n$

11. $\displaystyle\int_{-1}^{1} [P_n^m(t)]^2\,dt = \frac{2(n+m)!}{(2n+1)(n-m)!}$

### 21.3.4 Differential Equation

$$(1-t^2)\frac{d^2 P_n^m(t)}{dt^2} - 2t\frac{d\,P_n^m(t)}{dt} + \left[n(n+1) - \frac{m^2}{(1-t^2)}\right]P_n^m(t) = 0$$

### 21.3.5 Schlafli Formula

$$P_n^m(t) = \frac{(n+m)!}{2\pi\, jn!}(1-t^2)^{m/2}\oint_C \frac{(x^2-1)^n}{2^n(x-t)^{n+m+1}}\,dx$$

where C is any regular closed curve surrounding the point $t$ and taking it counterclockwise.

## 21.4 Bounds for Legendre Polynomials

### 21.4.1 Stieltjes Theorem

$$\left|P_n(\cos\gamma)\right| \leq \sqrt{2}\,\frac{4}{\sqrt{\pi}}\,\frac{1}{\sqrt{n}\,\sqrt{\sin\gamma}}, \qquad 0 < \gamma < \pi, \quad n = 1,2,\cdots$$

### 21.4.2 Second Stieltjes Theorem

$$\left|P_n(t) - P_{n+2}(t)\right| < \frac{4}{\sqrt{\pi}\,\sqrt{n+2}}$$

**21.4.3** $$\left|\frac{dP_n(t)}{dt}\right| < \frac{2}{\sqrt{\pi}}\frac{\sqrt{n}}{1-t^2}, \qquad |t|<1, \quad n=1,2,\cdots$$

**21.4.4** $$|P_{n+1}(t)+P_n(t)| < \frac{6\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{1-t}}, \qquad |t|<1$$

## 21.5 Table of Legendre and Associate Legendre Functions

**TABLE 21.2** Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind]

| | | |
|---|---|---|
| 1. | $\dfrac{1}{\sqrt{1-2tx+x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n \qquad \lvert t\rvert \le 1 \qquad \lvert x\rvert < 1$ | |
| 2. | $P_n(t) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k(2n-2k)!\,t^{n-2k}}{2^n k!(n-k)!(n-2k)!} \qquad [n/2]=\dfrac{n}{2} \quad n = \text{even}; \quad [n/2]=(n-1)/2 \quad n = \text{odd}$ | |
| 3. | $P_0(t)=1$ | |
| 4. | $P_{2n}(0) = \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}$ | $n=1,2,\cdots$ |
| 5. | $P_{2n+1}(0)=0$ | $n=0,1,2\cdots$ |
| 6. | $P_{2n}(-t)=P_{2n}(t) \qquad P_{2n+1}(-t)=-P_{2n+1}(t)$ | $n=0,1,2\cdots$ |
| 7. | $P_n(-t)=(-1)^n P_n(t)$ | $n=0,1,2\cdots$ |
| 8. | $P_n(1)=1 \quad n=0,1,2\cdots; \quad P_n(-1)=(-1)^n$ | $n=0,1,2\cdots$ |
| 9. | $P_n(t)=\dfrac{1}{2^n n!}\dfrac{d^n}{dt^n}(t^2-1)^n$ = Rodrigues formula, | $n=0,1,2\cdots$ |
| 10. | $(n+1)P_{n+1}(t)-(2n+1)tP_n(t)+nP_{n-1}(t)=0$ | $n=1,2,\cdots$ |
| 11. | $P'_{n+1}(t)-2tP'_n(t)+P'_{n-1}(t)-P_n(t)=0$ | $n=1,2,\cdots$ |
| 12. | $P'_{n-1}(t)=P_n(t)+2tP'_n(t)-P'_{n+1}(t)$ | |
| 13. | $P'_{n+1}(t)=P_n(t)+2tP'_n(t)-P'_{n-1}(t)$ | |
| 14. | $P'_{n+1}(t)-tP'_n(t)=(n+1)P_n(t)$ | |
| 15. | $tP'_n(t)-P'_{n-1}(t)=nP_n(t)$ | $n=1,2,\cdots$ |
| 16. | $P'_{n+1}(t)-P'_{n-1}(t)=(2n+1)P_n(t)$ | $n=1,2,\cdots$ |
| 17. | $(1-t^2)P'_n(t)=nP_{n-1}(t)-ntP_n(t)$ | |
| 18. | $\lvert P_n(t)\rvert<1, \qquad -1<t<1$ | |
| 19. | $P_{2n}(t)=\dfrac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n}\dfrac{(-1)^k(2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!}t^{2k}$ | $n=0,1,2\cdots$ |
| 20. | $(1-t^2)P'_n(t)=(n+1)[tP_n(t)-P_{n+1}(t)]$ | $n=0,1,2\cdots$ |
| 21. | $\int_{-1}^{1}P_n(t)\,dt=0$ | $n=1,2,\cdots$ |
| 22. | $\lvert P_n(t)\rvert \le 1$ | $\lvert t\rvert \le 1$ |

**TABLE 21.2** Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

| | | |
|---|---|---|
| 23. | $\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0$ | $n \neq m$ |
| 24. | $\int_{-1}^{1} [P_n(t)]^2\,dt = \frac{2}{2n+1}$ | $n = 0,1,2\cdots$ |
| 25. | $\frac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \frac{m(m-2)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+1)}$ | $m, s =$ even |
| 26. | $\frac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \frac{(m-1)(m-3)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+2)}$ | $m, s =$ odd |
| 27. | $\int_{-1}^{1} t\,P_n(t)P_{n-1}(t)\,dt = \frac{2n}{4n^2-1}$ | $n = 1,2,\cdots$ |
| 28. | $\int_{-1}^{1} P_n(t)P'_{n+1}(t)\,dt = 2$ | $n = 0,1,2\cdots$ |
| 29. | $\int_{-1}^{1} t\,P'_n(t)P_n(t)\,dt = \frac{2n}{2n+1}$ | $n = 0,1,2\cdots$ |
| 30. | $\int_{-1}^{1} (1-t^2)P'_n(t)P'_k(t)\,dt = 0$ | $k \neq n$ |
| 31. | $\int_{-1}^{1} (1-t)^{-1/2} P_n(t)\,dt = \frac{2\sqrt{2}}{2n+1}$ | $n = 0,1,2\cdots$ |
| 32. | $\int_{-1}^{1} t^2 P_{n+1}(t)P_{n-1}(t)\,dt = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$ | $n = 1,2,\cdots$ |
| 33. | $\int_{-1}^{1} (t^2-1)P_{n+1}(t)P'_n(t)\,dt = \frac{2n(n+1)}{(2n+1)(2n+3)}$ | $n = 1,2,\cdots$ |
| 34. | $\int_{-1}^{1} t^n P_n(t)\,dt = \frac{2^{n+1}(n!)^2}{(2n+1)!}$ | $n = 0,1,2\cdots$ |
| 35. | $\int_{-1}^{1} t^2[P_n(t)]^2\,dt = \frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\right]$ | $n = 0,1,2\cdots$ |
| 36. | $P_n^m(t) = (1-t^2)^{m/2}\frac{d^m}{dt^m}P_n(t)$ | $m > 0$ |
| 37. | $P_n^m(t) = \frac{1}{2^n n!}(1-t^2)^{m/2}\frac{d^{n+m}}{dt^{n+m}}[(t^2-1)^n]$ | $m+n \geq 0$ |
| 38. | $P_n^{-m}(t) = (-1)^m \frac{(n-m)!}{(n+m)!}P_n^m(t)$ | |
| 39. | $P_n^0(t) = P_n(t), \quad P_n^m(t) = 0$ for $m > n$ | |
| 40. | $(n-m+1)P_{n+1}^m(t) - (2n+1)t\,P_n^m(t) + (n+m)P_{n-1}^m(t) = 0$ | |
| 41. | $(1-t^2)^{1/2}P_n^m(t) = \frac{1}{2n+1}[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$ | |

**TABLE 21.2** Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

| | | |
|---|---|---|
| 42. | $(1-t^2)^{1/2} P_n^m(t) = \frac{1}{2n+1}[(n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+2)P_{n+1}^{m-1}(t)]$ | |
| 43. | $P_n^{m+1}(t) = 2mt(1-t^2)^{-1/2} P_n^m(t) - [n(n+1)-m(m-1)]P_n^{m-1}(t)$ | |
| 44. | $\int_{-1}^{1} P_n^m(t) P_k^m(t)\,dt = 0$ | $k \neq n$ |
| 45. | $\int_{-1}^{1} [P_n^m(t)]^2\,dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$ | |
| 46. | $P_n^m(-t) = (-1)^{n+m} P_n^m(t)$ | |
| 47. | $P_n^m(\pm 1) = 0$ | $m > 0$ |
| 48. | $P_{2n}^1(0) = 0 \qquad P_{2n+1}^1(0) = \frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$ | |
| 49. | $P_n^m(0) = 0$ | $n+m = \text{odd}$ |
| | $P_n^m(0) = (-1)^{(n-m)/2}\frac{(n+m)!}{2^n[(n-m)/2]![(n+m)/2]!}$ | $n+m = \text{even}$ |
| 50. | $\int_{-1}^{1} P_n^m(t) P_n^k(t)\,(1-t^2)^{-1}\,dt = 0$ | $k \neq m$ |
| 51. | $\int_{-1}^{1} (1-t^2)^{-1/2} P_{2m}(t)\,dt = \left[\frac{\Gamma(\frac{1}{2}+m)}{m!}\right]^2$ | |
| 52. | $\int_{-1}^{1} t(1-t^2)^{-1/2} P_{2m+1}(t)\,dt = \frac{\Gamma(\frac{1}{2}+m)\Gamma(\frac{3}{2}+m)}{m!(m+1)!}$ | |
| 53. | $\int_{t}^{1} P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(t) - P_{n+1}(t)]$ | |
| 54. | $\int_{0}^{1} t^q P_n(t)\,dt = \Gamma(q+1)\sum_{k=0}^{n}\frac{(-1)^k\,\Gamma(n+k+1)}{2^k k!\,\Gamma(n-k+1)(q+k+2)}$ | $q > -1$ |
| 55. | $\int_{0}^{1} t^{-1/2} P_n(t)\,dt = \begin{cases} \frac{2(-1)^{n/2}}{2n+1} & n = \text{even} \\ \frac{2(-1)^{(n-1)/2}}{2n+1} & n = \text{odd} \end{cases}$ | |
| 56. | $\int_{0}^{1} t^{1/2} P_n(t)\,dt = \begin{cases} \frac{2(-1)^{(n+2)/2}}{(2n-1)(2n+3)} & n = \text{even} \\ \frac{2(-1)^{(n+3)/2}}{(2n-1)(2n+3)} & n = \text{odd} \end{cases}$ | |
| 57. | $Q_0 = \frac{1}{2}\ln\frac{1+t}{1-t}, \qquad \lvert t\rvert < 1$ | |

**TABLE 21.2** Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

| | |
|---|---|
| 58. | $Q_1(t)=\frac{1}{2}t\ln\frac{1+t}{1-t}-1=tQ_0(t)-1, \quad \lvert t\rvert<1$ |
| 59. | $Q_{n+1}(t)=\frac{2n+1}{n+1}tQ_n(t)-\frac{n}{n+1}Q_{n-1}(t), \quad n=1,2,\cdots$ |
| 60. | $Q_n(t)=P_n(t)Q_0(t)-\sum_{k=0}^{[\frac{1}{2}(n-1)]}\frac{2n-4k-1}{(2k+1)(n-k)}P_{n-2k-1}(t), \quad \lvert t\rvert<1$ |
| 61. | $Q_n(t)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(x)}{t-x}dx \qquad n=0,1,2\cdots$ |
| 62. | $Q'_{n+1}(t)-2tQ'_n(t)+Q'_{n-1}(t)-Q_n(t)=0$ |
| 63. | $Q'_{n+1}(t)-tQ'_n(t)-(n+1)Q_n(t)=0$ |
| 64. | $Q'_{n+1}(t)-Q'_{n-1}(t)=(2n+1)Q_n(t)$ |
| 65. | $Q_0(-t)=-Q_0(t), \quad Q_n(-t)=(-1)^{n+1}Q_n(t), \quad n=1,2,\cdots$ |

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