Groups and Geometry

- Discrete and finite groups: Z_N, S_N

- Discrete and infinite groups: integers under addition

- Continuous groups:

non-compact: real under addition (real line)

$SO(2)$ or compact U(1): circle

compact: $SO(2) \otimes SO(2)$: torus

$SU(2)$: three-sphere S^3

Continuous Groups

Parameters (essential) $g = g(x_1, x_2 ... x_n)$ (local)

Under the group product $g(x)g(x') = g(x'')$ $x'' = F(x, x')$

Inverse element $g(x)g(x') = e = g(x')g(x)$ $x' = f(x)$

Topological Group:

$F(x, x')$ and $f(x)$ are continuous functions of its arguments

Lie Group G:

G constitute a manifold

$F(x, x')$ and $f(x)$ possess derivatives of all orders with respect to its arguments, i.e., are analytic functions

Lie Groups

Definition 2.1 *A Lie group is an analytic manifold which is also a group such that the analytic structure is compatible with the group structure, i.e. the operation $G \times G \to G$ is an analytic mapping.*

Manifold M

On every point P there is a tangent plane R^N (N the same everywhere)

One can map a neighbourhood of P into the tangent plane R^N, and such maps are continuous and differentiable.

Not

Example 2.1 *The real numbers under addition constitute a Lie group. Indeed, we can use a real variable x to parametrize the group elements. Therefore for two elements with parameters x and x' the function in (2.2) is given by*

$$x'' = F(x, x') = x + x' \tag{2.5}$$

The function given in (2.4) is just

$$f(x) = -x \tag{2.6}$$

These two functions are obviously analytic functions of the parameters.

Example 2.2 *The group of rotations on the plane, discussed in example 1.25, is a Lie group. In fact the groups of rotations on $\mathbb{R}^n$, denoted by $SO(n)$, are Lie groups. These are the groups of orthogonal $n \times n$ real matrices O with unit determinant ($O^\top O = \mathbb{1}$, $detO = 1$)*

Example 2.3 *The groups $GL(n)$ and $SL(n)$ discussed in example 1.16 are Lie groups, as well as the group $SU(n)$ discussed in example 1.17*

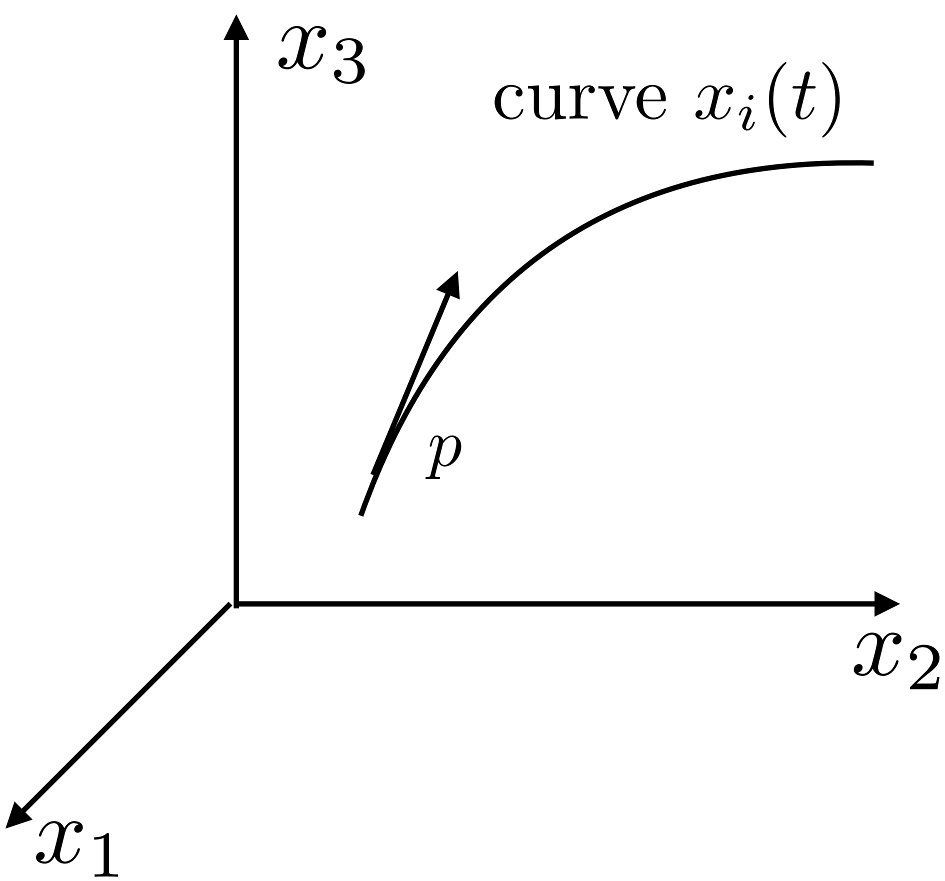
The Strategy

Lie groups are differentiable manifolds

Manifolds are locally Euclidean spaces

Do perturbation theory:
approximate the structure of the group on the tangent plane

Tangent vectors



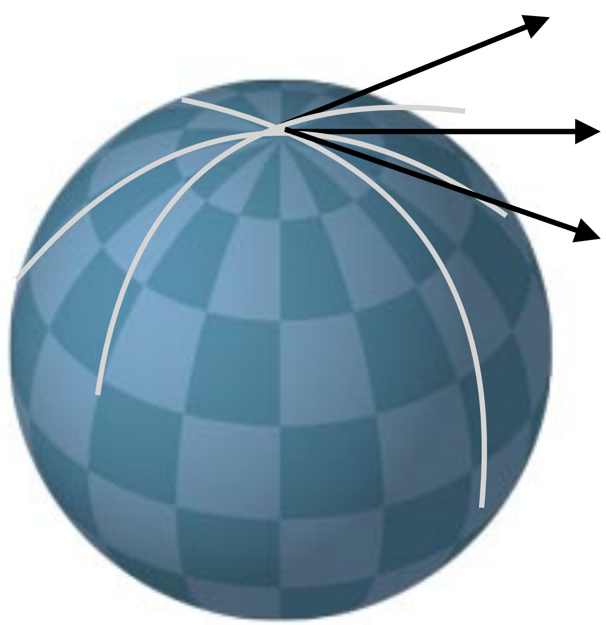
Differentiable function $f(x_i)$

Tangent vector V_p

$$V_p(f) = \frac{dx^i(t)}{dt}\,\Big|_{t=0}\,\frac{\partial f}{\partial x^i}$$

The tangent space

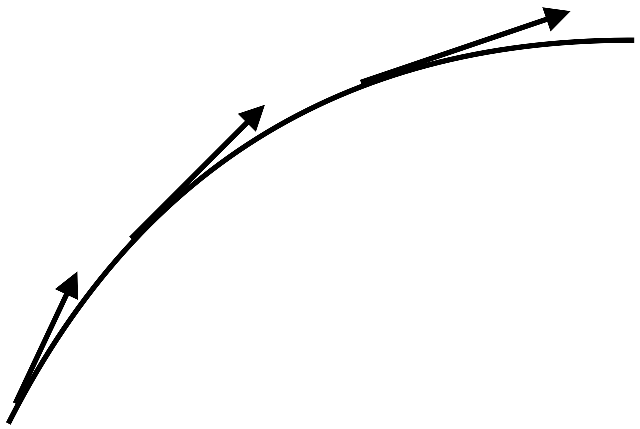
The tangent vectors at p to all differentiable curves passing through p form the *tangent space* T_pM of the manifold M at the point p. This space is a vector space since the sum of tangent vectors is again a tangent vector and the muliplication of a tangent vector by a scalar (real or complex number) is also a tangent vector.



Given a set of local coordinates x^i , $i = 1, 2, \ldots \dim M$ in a neighbourhood of a point p of M we have that the operators $\frac{\partial}{\partial x^i}$ are linearly independent and constitute a basis for the tangent space T_pM. Then, any tangent vector V_p on T_pM can be written as a linear combination of this basis

$$V_p = V_p^i \frac{\partial}{\partial x^i} \tag{2.8}$$

Vector Field



On a differentiable curve the tangent vectors are continuously and differentiably related

Vector Field: choose tangent vectors on $T_p\,M$, for every $p \in M$, such that they are related in a continuous and differentiable way

Given a set of local coordinates on M we can write a vector field V, in that coordinate neighbourhood, in terms of the basis $\frac{\partial}{\partial x^i}$, and its components V^i are differentiable functions of these coordinates

$$V = V^i(x)\frac{\partial}{\partial x^i}$$

Given two vector fields V and W in a coordinate neighbourhood we can evaluate their composite action on a function f. We have

$$W(Vf) = W^j \frac{\partial V^i}{\partial x^j} \frac{\partial f}{\partial x^i} + W^j V^i \frac{\partial^2 f}{\partial x^j \partial x^i} \tag{2.10}$$

Due to the second term on the r.h.s of (2.10) the operator WV is not a vector field and therefore the ordinary composition of vector fields is not a vector field. However if we take the commutator of the linear operators V and W we get

$$[V, W] = \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{2.11}$$

and this is again a vector field. So, the set of vector fields close under the operation of commutation and they form what is called a *Lie algebra*.

So: on any manifold we can construct a Lie algebra

Lie Algebras

Definition 2.2 *A Lie algebra $\mathcal{G}$ is a vector space over a field k with a bilinear composition law*

$$\begin{aligned} (x, y) &\rightarrow [x, y] \\ [x, ay + bz] &= a[x, y] + b[x, z] \end{aligned} \tag{2.12}$$

with x, y, $z \in L$ and a, $b \in k$, and such that

1. $[x, x] = 0$
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$*; (Jacobi identity)*

Notice that (2.12) implies that $[x, y] = -[y, x]$, since

$$\begin{aligned} [x + y, x + y] &= 0 \\ &= [x, y] + [y, x] \end{aligned}$$

Field (corpo)

Definition 2.3 *A field is a set k together with two operations*

$$(a, b) \to a + b \tag{2.14}$$

and

$$(a, b) \to ab \tag{2.15}$$

called respectively addition and multiplication such that

1. *k is an abelian group under addition*
2. *k without the identity element of addition is an abelian group under multiplication*
3. *multiplication is distributive with respect to addition, i.e.*

$$\begin{aligned} a\,(b+c) &= ab + ac \\ (a+b)\,c &= ac + bc \end{aligned}$$

The real and complex numbers are fields.

Other Fields

- Rational numbers $\frac{p}{q}$, with $p\,,\ q \in Z$

- The field F_2

$$0+0=0 \qquad 0+1=1 \qquad 1+1=0$$

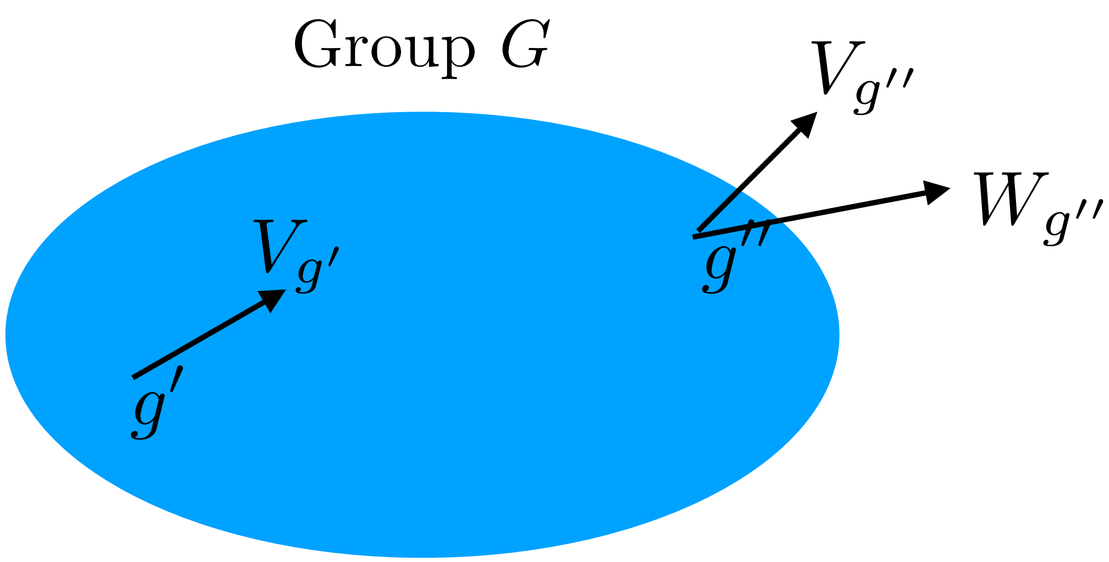
$$1\cdot 1=1 \qquad 1\cdot 0=0 \qquad 0\cdot 0=0$$

- The field $F_4 = \{0\,,\ 1\,,\ a\,,\ b\}$

$$0+a=a \qquad 0+b=b \qquad 1+a=b \qquad 1+b=a \qquad a+a=0 \qquad b+b=0 \qquad a+b=1$$

$$0\cdot a=0 \qquad 0\cdot b=0 \qquad 1\cdot a=a \qquad 1\cdot b=b \qquad a\cdot a=b \qquad b\cdot b=a \qquad a\cdot b=1$$

The Lie Algebra of a Lie Group



$$g'' = g\, g'$$

Define $$W_{g''} f \equiv V_{g'}(f \circ x'') = V^i_{g'} \frac{\partial}{\partial x'^i} f(x'') = V^i_{g'} \frac{\partial x''^j}{\partial x'^i} \frac{\partial f}{\partial x''^j}$$

If $$W_{g''} = V_{g''}$$

V is a left invariant vector field

The commutator of two left invariant vector fields is a left invariant vector field

Consider $$\tilde{V}_{g'} \equiv [V_{g'}\,,\,\bar{V}_{g'}] = \left(V^i_{g'} \frac{\partial \bar{V}^j_{g'}}{\partial x'^i} - \bar{V}^i_{g'} \frac{\partial V^j_{g'}}{\partial x'^i} \right) \frac{\partial}{\partial x'^j}$$

and so

$$\begin{aligned}
\tilde{V}_{g''} &\equiv [V_{g''}\,,\,\bar{V}_{g''}] \\
&= \left(V^i_{g''} \frac{\partial \bar{V}^j_{g''}}{\partial x''^i} - \bar{V}^i_{g''} \frac{\partial V^j_{g''}}{\partial x''^i} \right) \frac{\partial}{\partial x''^j} \\
&= \left(V^k_{g'} \frac{\partial x''^i}{\partial x'^k} \frac{\partial}{\partial x''^i} \left(\bar{V}^l_{g'} \frac{\partial x''^j}{\partial x'^l} \right) - \bar{V}^k_{g'} \frac{\partial x''^i}{\partial x'^k} \frac{\partial}{\partial x''^i} \left(V^l_{g'} \frac{\partial x''^j}{\partial x'^l} \right) \right) \frac{\partial}{\partial x''^j} \\
&= \left(V^i_{g'} \frac{\partial \bar{V}^j_{g'}}{\partial x'^i} - \bar{V}^i_{g'} \frac{\partial V^j_{g'}}{\partial x'^i} \right) \frac{\partial x''^k}{\partial x'^j} \frac{\partial}{\partial x'^k} \\
&= \tilde{V}^j_{g'} \frac{\partial x''^k}{\partial x'^j} \frac{\partial}{\partial x'^k}
\end{aligned}$$

Definition 2.4 *A vector subspace $\mathcal{H}$ of a Lie algebra $\mathcal{G}$ is said to be a* Lie subalgebra *of $\mathcal{G}$ if it closes under the Lie bracket, i.e.*

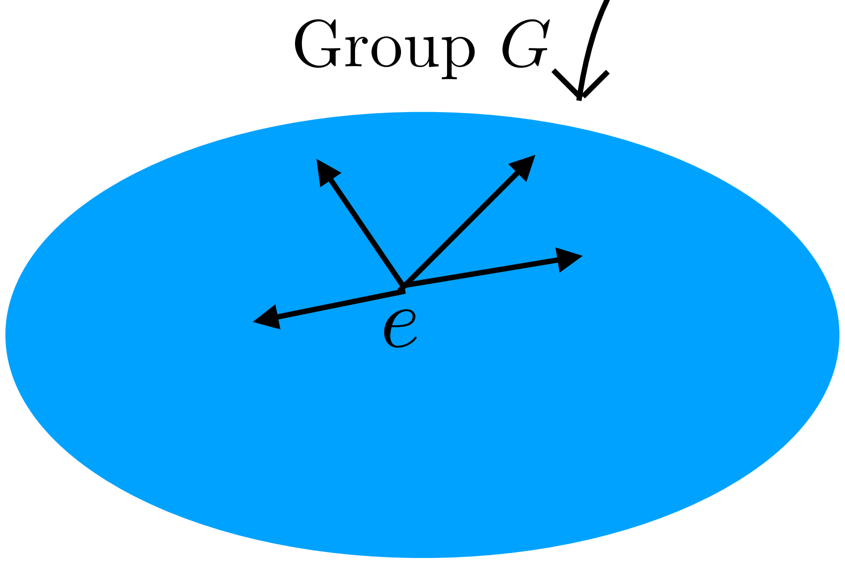
$$[\mathcal{H}\,,\,\mathcal{H}] \subset \mathcal{H} \tag{2.19}$$

and if $\mathcal{H}$ itself is a Lie algebra.

Definition 2.5 *The Lie algebra of the left invariant vector fields on a Lie group is the Lie algebra of this Lie group.*

One should notice that a left invariant vector field is completely determined by its value at any particular point of G. In particular it is determined by its value at the group identity e .

vectors of left invariant vector fields



Take a basis on the tangent plane at e

$$T_a,\ a = 1, 2, 3 \ldots \dim G$$

$$[T_a\,,\,T_b] = i f_{ab}^{c} T_c$$

structure constants

Change basis

$$T'_a = \Lambda_{ab}\, T_b$$

$$f'_{ab}{}^{c} = \Lambda_{ad}\, \Lambda_{be}\, \Lambda^{-1}_{cg}\, f^{g}_{de}$$

One-parameter subgroup

A *one parameter subgroup* of a Lie group G is a differentiable curve, i.e., a differentiable mapping from the real numbers onto G, $t \to g(t)$ such that

$$\begin{aligned} g(t)g(s) &= g(t+s) \\ g(0) &= e \end{aligned} \tag{2.21}$$

If T is the tangent vector at the identity element to a differentiable curve $g(t)$ which is a one parameter subgroup, then it is possible to show that

$$g(t) = exp(tT) \tag{2.22}$$

The exponential map

It is an analytic mapping of T_eG onto G and that it maps a neighbourhood of the zero element of T_eG in a one to one manner onto a neighbourhood of the identity element of G.

Notions on Lie Algebras

Commutation relations $$[T_a\,,\,T_b] = i f_{ab}^{c} T_c$$

Jacobi indentity

$$[T_a,[T_b,T_c]] + [T_c,[T_a,T_b]] + [T_b,[T_c,T_a]] = 0$$

$$f_{ad}^{e} f_{bc}^{d} + f_{cd}^{e} f_{ab}^{d} + f_{bd}^{e} f_{ca}^{d} = 0$$

Example: $SU(2)$ $$[\,T_a\,,\,T_b\,] = i\,\varepsilon_{abc}\,T_c \qquad \varepsilon_{123} = 1$$

$$T_a = \frac{1}{2}\,\sigma_a \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exponential of a matrix

$$e^L = 1 + L + \frac{1}{2!}\,L^2 + \frac{1}{3!}\,L^3 + \ldots$$

Take

$$L = i\,\left(x_1\,\sigma_1 + x_2\,\sigma_2 + x_3\,\sigma_3\right) = i\begin{pmatrix} x_3 & x_1 - i\,x_2 \\ x_1 + i\,x_2 & -x_3 \end{pmatrix} \qquad (L^\dagger = -L)$$

$$L^2 = -\left(x_1^2 + x_2^2 + x_3^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad L^3 = -\left(x_1^2 + x_2^2 + x_3^2\right)\,L$$

$$e^L = \left(1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \ldots\right)\mathbb{1} + \left(1 - \frac{r^2}{3!} + \frac{r^4}{5!}\ldots\right)L \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$e^L = \cos r\;\mathbb{1} + \frac{\sin r}{r}L \qquad\qquad e^{L^\dagger} = e^{-L} = \cos r\;\mathbb{1} - \frac{\sin r}{r}L$$

e^L is an unitary matrix $\rightarrow SU(2)$ group $\qquad (\det e^L = e^{\mathrm{Tr}\,L} = 1)$

Consider the matrix function

$$f\,(\lambda) \equiv \exp\,(\lambda L) T \exp\,(-\lambda L)$$

$$\begin{aligned} f' &= \exp\,(\lambda L)\,[\,L\,,\,T\,]\exp\,(-\lambda L) \\ f'' &= \exp\,(\lambda L)\,[\,L\,,\,[\,L\,,\,T\,]\,]\exp\,(-\lambda L) \\ \ldots &= \ldots \\ f^{(n)} &= \exp\,(\lambda L)\,[\,L\,,\,\ldots[\,L\,,\,[\,L\,,\,T\,]\,]\,]\exp\,(-\lambda L) \end{aligned}$$

Taylor expansion

$$f\,(\lambda) = T + [\,L\,,\,T\,]\;\lambda + [\,L\,,\,[\,L\,,\,T\,]\,]\;\frac{\lambda^2}{2!} + [\,L\,,\,[\,L\,,\,[\,L\,,\,T\,]\,]\,]\;\frac{\lambda^3}{3!} + \ldots$$

So, f is a Lie algebra element

$$f\,(1) = e^{L}\,T\,e^{-L} = T + [\,L\,,\,T\,] + \frac{1}{2!}\,[\,L\,,\,[\,L\,,\,T\,]\,] + \frac{1}{3!}\,[\,L\,,\,[\,L\,,\,[\,L\,,\,T\,]\,]\,] + \ldots$$

The adjoint representation

The Lie algebra is a vector space V

The operator $g = e^{L}$ maps V into V

$$T \to g\,T\,g^{-1} = e^{L}\,T\,e^{-L}$$

$$D(g) \mid T\rangle = \mid g\,T\,g^{-1}\rangle$$

But

$$g_1 \left(g_2\,T\,g_2^{-1}\right) g_1^{-1} = (g_1\,g_2)\ T\ (g_1\,g_2)^{-1}$$

$$D(g_1)\ D(g_2) \mid T\rangle = D(g_1\,g_2)\ \mid T\rangle$$

It is a representation

Given a basis T_a, one builds the matrix representation

$$gT_ag^{-1} = T_b d^b_{\ a}(g)$$

$$\begin{aligned} g_1g_2T_a(g_1g_2)^{-1} &= T_b d^b_{\ a}(g_1g_2) \\ &= g_1(g_2T_ag_2^{-1})g_1^{-1} \\ &= g_1T_cg_1^{-1}d^c_{\ a}(g_2) \\ &= T_b d^b_{\ c}(g_1)d^c_{\ a}(g_2) \end{aligned}$$

$$d(g_1g_2) = d(g_1)d(g_2)$$

Given an element T of a Lie algebra, it maps the Lie into itself

$$T: \; \mathcal{G} \to \mathcal{G}' = [\, T \,,\, \mathcal{G} \,] \qquad\qquad D\,(T)\;\mid \mathcal{G}\rangle = \mid [\, T \,,\, \mathcal{G} \,]\rangle$$

Jacobi identity

$$[\, T \,,\, [\, T' \,,\, \mathcal{G} \,]\,] + [\, \mathcal{G} \,,\, [\, T \,,\, T' \,]\,] + [\, T' \,,\, [\, \mathcal{G} \,,\, T \,]\,] = 0$$

or $$[\, T \,,\, [\, T' \,,\, \mathcal{G} \,]\,] - [\, T' \,,\, [\, T \,,\, \mathcal{G} \,]\,] = [\,[\, T \,,\, T' \,]\,,\, \mathcal{G} \,]$$

$$(D\,(T)\; D\,(T') - D\,(T')\; D\,(T))\;\mid \mathcal{G}\rangle = D\,([\, T \,,\, T' \,])\;\mid \mathcal{G}\rangle$$

We have a representation of the Lie algebra on the vector space of the algebra itself.

The adjoint representation

Definition 2.7 *If one can associate to every element T of a Lie algebra $\mathcal{G}$ a $n \times n$ matrix $D(t)$ such that*

1. $D(T + T') = D(T) + D(T')$
2. $D(aT) = aD(T)$
3. $D([T, T']) = [D(T), D(T')]$

for $T, T' \in \mathcal{G}$ and a being a c-number. Then we say that the matrices D define a n-dimensional matrix representation of $\mathcal{G}$.

Adjoint matrix representation: take a basis

$$[\,T\,,\,T_a\,] \equiv T_b d^b_a(T)$$

$$\begin{aligned}
[\,T\,,\,[\,T'\,,\,T_a\,]\,] - [\,T'\,,\,[\,T\,,\,T_a\,]\,] &= T_c\, d^c_b(T) d^b_a(T') - T_c\, d^c_b(T') d^b_a(T) \\
&= [\,[\,T\,,\,T'\,]\,,\,T_a\,] \\
&= T_c\, d^c_a([\,T\,,\,T'\,]) \qquad (
\end{aligned}$$

$$[\,d(T)\,,\,d(T')\,] = d([\,T\,,\,T'\,])$$

$$[\,T\,,\,T_b\,] = T_c\, d_{cb}\,(T) \longrightarrow [\,T_a\,,\,T_b\,] = T_c\, d_{cb}\,(T_a)$$

$$[\,T_a\,,\,T_b\,] = i\, f^c_{ab}\, T_c \longrightarrow d_{cb}\,(T_a) = i\, f^c_{ab}$$

The physicist's way

Take g close to the identity $\qquad g = 1 + i\varepsilon^a T_a$

$$\begin{aligned}(1+i\varepsilon^a T_a)T_b(1-i\varepsilon^c T_c) &= T_c d^c_b(1+i\varepsilon^a T_a)\\ &= T_c(\delta^c_b + i\varepsilon^a d^c_b(T_a))\\ &= T_b + i\varepsilon^a[T_a, T_b]\\ &= T_b - \varepsilon^a f^c_{ab} T_c\end{aligned}$$

$$d^c_b(T_a) = i f^c_{ab}$$

Notice that the conjugation defines a mapping of the Lie algebra $\mathcal{G}$ into itself which respects the commutation relations. Defining $\sigma : \mathcal{G} \rightarrow \mathcal{G}$

$$\sigma(T) \equiv gTg^{-1} \tag{2.34}$$

for a fixed $g \in G$ and any $T \in \mathcal{G}$, one has

$$\begin{aligned} [\sigma(T), \sigma(T')] &= [gTg^{-1}, gT'g^{-1}] \\ &= g[T, T']g^{-1} \\ &= \sigma([T, T']) \end{aligned} \tag{2.35}$$

Such mapping is called an *automorphism* of the Lie algebra.

Definition 2.6 *A mapping σ of a Lie algebra $\mathcal{G}$ into itself is an automorphism if it preserves the Lie bracket of the algebra, i.e.*

$$[\sigma(T), \sigma(T')] = \sigma([T, T']) \tag{2.36}$$

for any $T, T' \in \mathcal{G}$.

Trace forms (bilinear forms)

In a given finite dimensional representation D of a Lie algebra we define the quantity

$$\eta^D(T,T') \equiv Tr\,(D(T)D(T')) \tag{2.45}$$

which is symmetric and bilinear

1. $\eta^D(T,T') = \eta^D(T',T)$
2. $\eta^D(T, xT' + yT'') = x\eta^D(T,T') + y\eta^D(T,T'')$

It is invariant under the adjoint representation

$$\eta^D(T,T') = \eta^D(gTg^{-1}, gT'g^{-1})$$

using the cyclic property of the trace

$$Tr([D(T),D(T')]D(T'')) = Tr(D(T)[D(T'),D(T'')])$$

$$g = 1 + \varepsilon\, T''$$

$$\eta^D([T,T'],T'') + \eta^D(T,[T'',T'] = 0$$

The Killing Form

The trace form in the adjoint representation is called the Killing form

$$\eta_{ab} \equiv \eta(T_a, T_b) \equiv Tr(d(T_a)d(T_b)) = -f_{ac}^{d} f_{bd}^{c} \qquad\qquad d_{cb}\,(T_a) = i\,f_{ab}^{c}$$

Definition 2.8 *A Lie algebra is said to be* abelian *if all its elements commute with one another.*

In this case all the structure constants vanish and consequently the Killing form is zero. However there might exist some representation D of an abelian algebra for which the bilinear form (2.45) is not zero.

Definition 2.9 *A subalgebra $\mathcal{H}$ of $\mathcal{G}$ is said to be an* invariant subalgebra *(or ideal) if*

$$[\mathcal{H}, \mathcal{G}] \subset \mathcal{H} \tag{2.50}$$

Definition 2.10 *We say a Lie algebra $\mathcal{G}$ is* simple *if it has no invariant subalgebras, except zero and itself, and it is* semisimple *if it has no invariant abelian subalgebras.*

for non-simple Lie algebras, the adjoint representation is not irreducible

Theorem 2.1 (Cartan) *A Lie algebra $\mathcal{G}$ is semisimple if and only if its Killing form is non degenerated, i.e.*

$$det \mid Tr(d(T_a)d(T_b)) \mid \neq 0. \tag{2.51}$$

or in other words, there is no $T \in \mathcal{G}$ such that

$$Tr(d(T)d(T')) = 0 \tag{2.52}$$

for every $T' \in \mathcal{G}$.

Definition 2.11 *We say a semisimple Lie algebra is* compact *if its Killing form is positive definite.*

The Lie algebra of a compact semisimple Lie group is a compact semisimple Lie algebra. By choosing a suitable basis T_a we can put the Killing form of a compact semisimple Lie algebra in the form .

$$\eta_{ab} = \delta_{ab} \tag{2.53}$$

Let us define the quantity

$$f_{abc} \equiv f_{ab}^{d}\eta_{dc} \tag{2.54}$$

From (2.49) we have

$$f_{abc} = f_{ab}^{d} Tr(d(T_d)d(T_c)) = -iTr(d([T_a, T_b]T_c)) \tag{2.55}$$

Using the cyclic property of the trace one sees that f_{abc} is antisymmetric with respect to all its three indices. Notice that, in general, f_{abc} is not a structure constant.

For a compact semisimple Lie algebra we have from (2.53) that $f_{ab}^{c} = f_{abc}$, and therefore the commutation relations (2.23) can be written as

$$[T_a, T_b] = if_{abc}T_c \tag{2.56}$$

Therefore the structure constants of a compact semisimple Lie algebra can be put in a completely antisymmetric form.

2.5 su(2) and sl(2): Lie algebra prototypes

As we have seen the group $SU(2)$ is defined as the group of 2×2 complex unitary matrices with unity determinant. If an element of such group is written as $g = \exp iT$, then the matrix T has to be hemitian and traceless. Therefore the basis of the algebra $su(2)$ of this group can be taken to be (half of) the Pauli matrices ($T_i \equiv \frac{1}{2}\sigma_i$)

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.57}$$

They satisfy the following commutation relations

$$[T_i \, , \, T_j] = i\epsilon_{ijk} T_k \tag{2.58}$$

The matrices (2.57) define what is called the spinor (2-dimensional) representation of the algebra $su(2)$.

From (2.39) we obtain the adjoint representation (3-dimensional) of $su(2)$

$$d_{ij}(T_k) = i\epsilon_{kji} = i\epsilon_{ikj} \tag{2.59}$$

and so

$$d(T_1) = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad d(T_2) = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix};$$

$$d(T_3) = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.60}$$

One can easily check that they satisfy (2.58).

As we have seen the group of rotations in three dimensions $SO(3)$ is defined as the group of 3×3 real orthogonal matrices. Its elements close to the identity can be written as $g = \exp iT$, and therefore the Lie algebra $so(3)$ of this group is given by 3×3 pure imaginary, antisymmetric and traceless matrices. But the matrices (2.60) constitute a basis for such algebra. Thefore the Lie algebras $su(2)$ and $so(3)$ are isomorphic, although the Lie groups $SU(2)$ and $SO(3)$ are just homomorphic (in fact $SO(3) \sim SU(2)/Z_2$).

The Killing form of this algebra, according to (2.49), is given by

$$\eta_{ij} = Tr(d(T_iT_j)) = 2\delta_{ij} \tag{2.61}$$

So, it is non degenerate. This is in agreement with theorem 2.1, since this algebra is simple. According to the definition 2.11 this is a compact algebra.

The trace form (2.45) in the spinor representation is given by

$$\eta^s_{ij} = Tr(D(T_i T_j)) = \frac{1}{2}\delta_{ij} \tag{2.62}$$

So, it is proportional to the Killing form, $\eta^s = \frac{1}{4}\eta$. This is a particular example of a general theorem we will prove later: the trace form in any representation of a simple Lie algebra is proportional to the Killing form.

Notice that the matrices in these representations discussed above are hermitian and therefore the matrices representing the elements of the group are unitary ($g = \exp iT$). In fact this is a result which constitute a generalization of theorem 1.3 to the case of compact Lie groups: any finite dimensional representation of a compact Lie group is equivalent to a unitary representation. Since the generators are hermitian we can always choose one of them to be diagonal. Traditionally one takes T_3 to be diagonal and defines (in the spinor rep. T_3 is already diagonal)

$$T_\pm = T_1 \pm i T_2 \tag{2.63}$$

Notice that formally, these are not elements of the algebra $su(2)$ since we have taken complex linear combination of the generators. These are elements of the complex algebra denoted by A_1.

Using (2.58) one finds

$$\begin{aligned} [T_3, T_\pm] &= \pm T_\pm \\ [T_+, T_-] &= 2T_3 \end{aligned} \tag{2.64}$$

Therefore the generators of A_1 are written as eigenvectors of T_3 . The eigenvalues ± 1 are called the roots of $su(2)$. We will show later that all Lie algebras can be put in a similar form. In any representation one can check that the operator

$$C = T_1^2 + T_2^2 + T_3^2 \tag{2.65}$$

commutes with all generators of $su(2)$. It is called the *quadractic Casimir operator*. The basis of the representation space can always be chosen to be eigenstates of the operators T_3 and C simultaneously. These states can be labelled by the spin j and the weight m

$$T_3 \mid j, m\rangle = m \mid j, m\rangle \tag{2.66}$$

The operators $T_\pm$ raise and lower the eigenvalue of T_3 since using (2.64)

$$\begin{aligned} T_3 T_\pm \mid j, m\rangle &= \left([T_3, T_\pm] + T_\pm T_3\right) \mid j, m\rangle \\ &= (m \pm 1)\, T_\pm \mid j, m\rangle \end{aligned} \tag{2.67}$$

We are interested in finite representations and therefore there can only exists a finite number of eigenvalues m in a given representation. Consequently there must exist a state which possess the highest eigenvalue of T_3 which we denote j

$$T_+ \mid j, j\rangle = 0 \tag{2.68}$$

The other states of the representation are obtained from $\mid j, j\rangle$ by applying T_- successively on it. Again, since the representation is finite there must exist a positive integer l such that

$$(T_-)^{l+1} \mid j, j\rangle = 0 \tag{2.69}$$

Using (2.63) one can write the Casimir operator (2.65) as

$$C = T_3^2 + \frac{1}{2}\left(T_+ T_- + T_- T_+\right) \tag{2.70}$$

So, using (2.64), (2.66) and (2.68)

$$\begin{aligned} C \mid j, j\rangle &= \left(T_3^2 + \frac{1}{2}[T_+, T_-] + T_- T_+\right) \mid j, j\rangle \\ &= j\,(j+1) \mid j, j\rangle \end{aligned} \tag{2.71}$$

Since C commutes with all generators of the algebra, any state of the representation is an eigenstate of C with the same eigenvalue

$$C \mid j, m\rangle = j\,(j+1) \mid j, m\rangle \tag{2.72}$$

where $\mid j, m\rangle = (T_-)^n \mid j, j\rangle$ for $m = j - n$ and $n \leq l$. From Schur's lemma (see lemma1.1), in a irreducible representation, the Casimir operator has to be proportional to the unity matrix and so

$$C = j(j+1)\mathbb{1} \tag{2.73}$$

Using (2.70) one can write

$$T_+ T_- = C - T_3^2 + T_3 \tag{2.74}$$

Therefore applying T_+ on both sides of (2.69)

$$\begin{aligned} T_+T_-(T_-)^l \mid j,j\rangle &= 0 \\ &= \Big(j(j+1) - (j-l)^2 + (j-l)\Big) \mid j,j\rangle \end{aligned} \tag{2.75}$$

Since, by assumption the state $(T_-)^l \mid j,j\rangle$ does exist, one must have

$$j(j+1) - (j-l)^2 + (j-l) = (2j-l)(l+1) = 0 \tag{2.76}$$

Since l is a positive integer, the only possible solution is $l = 2j$. Therefore we conclude that

1. The lowest eigenvalue of T_3 is $-j$

2. The eigenvalues of T_3 can only be integers or half integers and in a given representation they vary from j to $-j$ in integral steps.

The group $SL(2)$, as defined in example 1.16, is the group of 2×2 real matrices with unity determinant. If one writes the elements close to the identity as $g = \exp L$ (without the i factor), then L is a real traceless 2×2 matrix. So the basis of the algebra $sl(2)$ can be taken as

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; L_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \; L_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.77}$$

This defines a 2-dimensional representation of $sl(2)$ which differ from the spinor representation of $su(2)$, given in (2.57), by a factor i in L_2. One can check the they satisfy

$$[L_1, L_2] = -L_3; \;\; [L_1, L_3] = -L_2; \;\; [L_2, L_3] = -L_1 \tag{2.78}$$

From these commutation relations one can obtain the adjoint representation of $sl(2)$, using (2.39)

$$d(L_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad ; \qquad d(L_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$d(L_3) \;\; = \;\; \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.79}$$

According to (2.49), the Killing form of $sl(2)$ is given by

$$\eta_{ij} = Tr(d(L_i L_j)) = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.80}$$

$sl(2)$ is a simple algebra and we see that its Killing form is indeed non-degenerate (see theorem 2.1). From definition 2.11 we conclude $sl(2)$ is a non-compact Lie algebra.

The trace form (2.45) in the 2-dimensional representation (2.77) of $sl(2)$ is

$$\eta_{ij}^{2-dim} = Tr(L_i L_j) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.81}$$

Similarly to the case of $su(2)$, this trace form is proportional to the Killing form, $\eta^{2-dim} = \frac{1}{4}\eta$.

The operators

$$L_\pm \equiv L_1 \pm L_2 \tag{2.82}$$

according to (2.78), satisfy commutation relations identical to (2.64)

$$[L_3, L_\pm] = \pm L_\pm; \quad [L_+, L_-] = 2L_3 \tag{2.83}$$

The quadratic Casimir operator of $sl(2)$ is

$$C = L_1^2 - L_2^2 + L_3^2 = L_3^2 + \frac{1}{2}\left(L_+L_- + L_-L_+\right) \tag{2.84}$$

The analysis we did for $su(2)$, from eqs. (2.66) to (2.76), applies also to $sl(2)$ and the conclusions are the same, i.e. , in a finite dimensional representation of $sl(2)$ with highest eigenvalue j of L_3 the lowest eigenvalue is $-j$. In addition the eigenvalues of L_3 can only be integers or half integers varying from j to $-j$ in integral steps. The striking difference however, is that the finite representations of $sl(2)$ (where these results hold) are not unitary. On the contrary, the finite dimensional representations of $su(2)$ are all equivalent to unitary representations. Indeed, the exponentiation of the matrices (2.57) and (2.60) (with the i factor) provide unitary matrices while the exponentiation of (2.77) and (2.79) do not. All unitary representations of $sl(2)$ are necessarily infinite dimensional. In fact this is true for any non compact Lie algebra.

The structures discussed in this section for the cases of $su(2)$ and $sl(2)$ are in fact the basic structures underlying all simple Lie algebras. The rest of this course will be dedicated to this study.

2.6 The structure of semisimple Lie algebras

We now start the study of the features which are common to all semisimple Lie algebras. These features are in fact a generalization of the properties of the algebra of angular momentum discussed in section 2.5. We will be mainly interested in compact semisimple algebras although several results also apply to the case of non-compact Lie algebras.

Theorem 2.2 *Given a subalgebra $\mathcal{H}$ of a compact semisimple Lie algebra $\mathcal{G}$ we can write*

$$\mathcal{G} = \mathcal{H} + \mathcal{P} \tag{2.85}$$

where

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P} \tag{2.86}$$

where $\mathcal{P}$ is the orthogonal complement of $\mathcal{H}$ in $\mathcal{G}$ w.r.t. a trace form in a given representation, i.e.

$$Tr(\mathcal{P}\mathcal{H}) = 0 \tag{2.87}$$

Proof $\mathcal{P}$ does not contain any element of $\mathcal{H}$ and contains all elements of $\mathcal{G}$ which are not in $\mathcal{H}$. Using the cyclic property of the trace

$$Tr(\mathcal{H}[\mathcal{H}, \mathcal{P}]) = Tr([\mathcal{H}, \mathcal{H}]\mathcal{P}) = Tr(\mathcal{H}\mathcal{P}) = 0 \tag{2.88}$$

Therefore

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P}. \tag{2.89}$$

□

This theorem does not apply to non compact algebras because the trace form does not provide an Euclidean type metric, i.e. there can exist null vectors which are orthogonal to themselves. As an example consider $sl(2)$.

Example 2.5 *Consider the subalgebra $\mathcal{H}$ of $sl(2)$ generated by $(L_1 + L_2)$ (see section 2.5). Its complement $\mathcal{P}$ is generated by $(L_1 - L_2)$ and L_3. However this is not an orthogonal complement since, using (2.80)*

$$Tr((L_1 + L_2)(L_1 - L_2)) = 4 \tag{2.90}$$

In addition $(L_1 \pm L_2)$ are null vectors, since

$$Tr(L_1 + L_2)^2 = Tr(L_1 - L_2)^2 = 0 \tag{2.91}$$

Using (2.78) one can check (2.86) is not satisfied. Indeed

$$\begin{aligned} [L_1 + L_2, L_1 - L_2] &= 2L_3 \\ [L_1 + L_2, L_3] &= -(L_1 + L_2) \end{aligned} \tag{2.92}$$

So

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{H} + \mathcal{P} \tag{2.93}$$

Notice $\mathcal{P}$ is a subalgebra too

$$[L_3, L_1 - L_2] = -(L_1 - L_2) \tag{2.94}$$

Theorem 2.3 *A compact semisimple Lie algebra is a direct sum of simple algebras that commute among themselves.*

Proof If $\mathcal{G}$ is not simple then it has an invariant subalgebra $\mathcal{H}$ such that

$$[\mathcal{H}, \mathcal{G}] \subset \mathcal{H} \tag{2.95}$$

But from theorem 2.2 we have that

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P} \tag{2.96}$$

and therefore, since $\mathcal{P} \cap \mathcal{H} = 0$, we must have

$$[\mathcal{H}, \mathcal{P}] = 0 \tag{2.97}$$

But $\mathcal{P}$, in this case, is a subalgebra since

$$Tr([\mathcal{P}, \mathcal{P}]\mathcal{H}) = Tr(\mathcal{P}[\mathcal{P}, \mathcal{H}]) = 0 \tag{2.98}$$

and from theorem 2.2 again

$$[\mathcal{P}, \mathcal{P}] \subset \mathcal{P} \tag{2.99}$$

If $\mathcal{P}$ and $\mathcal{H}$ are not simple we repeat the process. $\square$

Theorem 2.4 *For a simple Lie algebra the invariant bilinear trace form defined in eq. (2.45) is the same in all representations up to an overall constant. Consequentely they are all proportional to the Killing form.*

Proof Using the definition (2.31) of the adjoint representation and the invariance property (2.48) of $\eta^D(T, T')$ we have

$$\begin{aligned}
\eta^D(T_a, T_b) &= Tr(D(gT_ag^{-1}gT_bg^{-1})) \\
&= Tr(D(T_cd^c_a(g)T_dd^d_b(g))) \\
&= (d^\top)^c_a(g)\eta^D(T_c, T_d)d^d_b(g) \\
&= (d^\top \eta^D d)_{ab}
\end{aligned} \tag{2.100}$$

Therefore η^D is an invariant tensor under the adjoint representation. This is true for any representation D, in particular the adjoint itself. So, the Killing form defined in (2.49) also satisfies (2.100). From theorem 2.1 we have that for a semisimple Lie algebra, $\det\eta \neq 0$ and therefore η has an inverse. Then multiplying both sides of (2.100) by η^{-1} and using the fact that $\eta^{-1} = (d^\top \eta d)^{-1}$ we get

$$\eta^{-1}\eta^D = (d^\top \eta d)^{-1}(d^\top \eta^D d) = d^{-1}\eta^{-1}\eta^D d \tag{2.101}$$

and so

$$d(g)\eta^{-1}\eta^D = \eta^{-1}\eta^D d(g) \tag{2.102}$$

For a simple Lie algebra the adjoint representation is irreducible. Therefore using Schur's lemma (see lemma 1.1) we get

$$\eta^{-1}\eta^D = \lambda\mathbb{1} \rightarrow \eta^D = \lambda\eta \tag{2.103}$$

So, the theorem is proven. $\square$

The constant λ is representation dependent and is called the *Dynkin index* of the representation D.

We will now show that it is possible to find a set of commuting generators such that all other generators are written as eigenstates of them (under the commutator). These commuting generators are the generalization of T_3 in $su(2)$ and they generate what is called the Cartan subalgebra.

Definition 2.12 *For a semisimple Lie algebra $\mathcal{G}$, the* Cartan subalgebra *is the maximal set of commuting elements of $\mathcal{G}$ which can be diagonalized simultaneously.*

The formal definition of the Cartan subalgebra of a Lie algebra (semisimple or not) is a little bit more sophisticated and involves two concepts which we now discuss. The *normalizer* of a subalgebra $\mathcal{K}$ of $\mathcal{G}$ is defined by the set

$$N(\mathcal{K}) \equiv \{x \in \mathcal{G} \mid [x, \mathcal{K}] \subset \mathcal{K}\} \tag{2.104}$$

Using the Jacobi identity we have

$$[[x, x'], \mathcal{K}] \subset \mathcal{K} \tag{2.105}$$

with $x, x' \in N(\mathcal{K})$. Therefore the normalizer $N(\mathcal{K})$ is a subalgebra of $\mathcal{G}$ and $\mathcal{K}$ is an invariant subalgebra of $N(\mathcal{K})$. So we can say that the normalizer of $\mathcal{K}$ in $\mathcal{G}$ is the largest subalgebra of $\mathcal{G}$ which contains $\mathcal{K}$ as an invariant subalgebra.

Consider the sequence of subspaces of $\mathcal{G}$

$$\mathcal{G}_0 = \mathcal{G};\;\; \mathcal{G}_1 = [\mathcal{G}, \mathcal{G}];\;\; \mathcal{G}_2 = [\mathcal{G}, \mathcal{G}_1];\;\; ...\; \mathcal{G}_i = [\mathcal{G}, \mathcal{G}_{i-1}] \tag{2.106}$$

We have that $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset ... \supset \mathcal{G}_i$ and each $\mathcal{G}_i$ is a invariant subalgebra of $\mathcal{G}$. We say $\mathcal{G}$ is a *nilpotent algebra* if $\mathcal{G}_n = 0$ for some n. Nilpotent algebras are not semisimple.

Similarly we can define the derived series

$$\mathcal{G}_{(0)} = \mathcal{G};\;\; \mathcal{G}_{(1)} = [\mathcal{G}, \mathcal{G}];\;\; \mathcal{G}_{(2)} = [\mathcal{G}_{(1)}, \mathcal{G}_{(1)}];\;\; ...\; \mathcal{G}_{(i)} = [\mathcal{G}_{(i-1)}, \mathcal{G}_{(i-1)}] \tag{2.107}$$

If $\mathcal{G}_{(n)} = 0$ for some n then we say $\mathcal{G}$ is a *solvable algebra* . All nilpotent algebras are solvable, but the converse is not true.

Definition 2.13 *A* Cartan subalgebra *of a Lie algebra $\mathcal{G}$ is a nilpotent subalgebra which is equal to its normalizer in $\mathcal{G}$.*

Lemma 2.1 *If $\mathcal{G}$ is semisimple then a Cartan subalgebra of $\mathcal{G}$ is a maximal abelian subalgebra of $\mathcal{G}$ such that its generators can be diagonalized simultaneously.*

Definition 2.14 *The dimension of the Cartan subalgebra of $\mathcal{G}$ is the rank of $\mathcal{G}$.*

Notice that if H_1 , H_2 ... H_r are the generators of the Cartan subalgebra then $g^{-1}H_1g$, $g^{-1}H_2g$ ... $g^{-1}H_rg$ $(g \in G)$ generates an abelian subalgebra of $\mathcal{G}$ with the same dimension as that one generated by H_i, $i = 1, 2, ...r$. This is also a Cartan subalgebra. Therefore there are an infinite number of Cartan subalgebras in $\mathcal{G}$ and they are all related by conjugation by elements of the group G which algebra is $\mathcal{G}$.

By choosing suitable linear combinations one can make the basis of the Cartan subalgebra to be orthonormal with respect to the Killing form of $\mathcal{G}$, i.e.[1]

$$Tr(H_i H_j) = \delta_{ij} \tag{2.108}$$

with $i, j = 1, 2, ...$ rank $\mathcal{G}$. From the definition of Cartan subalgebra we see that these generators can be diagonalized simultaneously.

We now want to construct the generalization of the operators $T_{\pm} = T_1 + iT_2$ of $su(2)$, discussed in section 2.5, for the case of any compact semisimple Lie algebra. They are called *step operators* and their number is dim $\mathcal{G}$ - rank $\mathcal{G}$. According to theorem 2.2 they constitute the orthogonal complement of the Cartan subalgebra and therefore

$$Tr(H_i T_m) = 0 \tag{2.109}$$

with $i = 1, 2...$ rank $\mathcal{G}$, $m = 1, 2...$ (dim $\mathcal{G}$ - rank $\mathcal{G}$). In addition, since a compact semisimple Lie algebra is an Euclidean space we can make the basis T_m orthonormal, i.e.

$$Tr(T_m T_n) = \delta_{mn} \tag{2.110}$$

Again from theorem 2.2 we have that the commutator of an element of the Cartan subalgebra with T_m is an element of the subspace generated by the basis T_m . Then, since the algebra is compact we can put its structure constants in a completely antisymmetric form, and write

$$[H_i, T_m] = i f_{imn} T_n \tag{2.111}$$

or

$$[H_i, T_m] = (h_i)_{mn} T_n \tag{2.112}$$

where we have defined the matrices

$$(h_i)_{mn} = i f_{imn} \tag{2.113}$$

of dimension (dim $\mathcal{G}$ - rank $\mathcal{G}$) and which are hermitian

$$(h_i)^\dagger_{mn} = (h_i)^*_{nm} = -i f_{inm} = i f_{imn} = (h_i)_{mn} \tag{2.114}$$

Therefore we can find a unitary transformation that diagonalizes the matrices h_i without affecting the Cartan subalgebra generators H_i .

$$\begin{aligned} T_m &\rightarrow U_{mn} T_n \\ (h_i)_{mn} &\rightarrow (U h_i U^\dagger)_{mn} \end{aligned} \tag{2.115}$$

with $U^\dagger = U^{-1}$. We shall denote by E_α the new basis of the subspace orthogonal to the Cartan subalgebra. The indices stand for the eigenvalues of the

matrix h_i (or of the generators H_i). The commutation relations (2.112) can now be written as

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.116}$$

The eigenvalues α_i are the components of a vector of dimension rank $\mathcal{G}$ and they are called the *roots* of the algebra $\mathcal{G}$. The generators E_α are called *step operators* and they are complex linear combinations of the hermitian generators T_m. Notice that the roots α are real since they are the eigenvalues of the hermitian matrices h_i.

From (2.113) we see that the matrices h_i are antisymmetric, and their off diagonal elements are purely imaginary. So

$$h_i^\dagger = h_i; \quad h_i^* = -h_i \tag{2.117}$$

Therefore if v is an eigenstate of the matrix h_i then since the eigenvalue α_i is real we have

$$h_i v = \alpha_i v \tag{2.118}$$

and then

$$h_i^* v^* = -h_i v^* = \alpha_i v^* \tag{2.119}$$

Consenquently if α is a root its negative $(-\alpha\ )$ is also a root. Thus the roots always occur in pairs.

We have shown that we can decompose a compact semisimple algebra L as

$$\mathcal{G} = \mathcal{H} + \sum_\alpha \mathcal{G}_\alpha \tag{2.120}$$

where $\mathcal{H}$ is generated by the commuting generators H_i and constitute the Cartan subalgebra of $\mathcal{G}$. The subspace $\mathcal{G}_\alpha$ is generated by the step operators E_α. This is called the *root space decomposition* of $\mathcal{G}$.In addition one can show that for a semisimple Lie algebra

$$dim\ \mathcal{G}_\alpha = 1; \quad for\ any\ root\ \alpha \tag{2.121}$$

and consequently the roots are not degenerated. So, there are not two step operators E_α and E'_α corresponding to the same root α. Therefore for a semisimple Lie algebra one has

$$\text{dim } \mathcal{G} \text{ - rank } \mathcal{G} = \textstyle\sum_\alpha \text{dim } \mathcal{G}_\alpha = \text{number of roots} = \text{even number}$$

Using the Jacobi identity and the commutation relations (2.116) we have that if α and β are roots then

$$\begin{aligned}
[H_i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \\
&= (\alpha_i + \beta_i)\,[E_\alpha, E_\beta] \qquad (2.122)
\end{aligned}$$

Since the algebra is closed under the commutator we have that $[E_\alpha, E_\beta]$ must be an element of the algebra. We have then three possibilities

1. $\alpha + \beta$ is a root of the algebra and then $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$
2. $\alpha + \beta$ is not a root and then $[E_\alpha, E_\beta] = 0$
3. $\alpha + \beta = 0$ and consequently $[E_\alpha, E_\beta]$ must be an element of the Cartan subalgebra since it commutes with all H_i .

Since in a semisimple Lie algebra the roots are not degenerated (see (2.121)), we conclude from (2.122) that 2α is never a root.

We then see that the knowlegde of the roots of the algebra provides all the information about the commutation relations and consequently about the structure of the algebra. From what we have learned so far, we can write the commutation relations of a semisimple Lie algebra $\mathcal{G}$ as

$$[H_i, H_j] = 0 \qquad (2.123)$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad (2.124)$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2.125)$$

where $H_\alpha \equiv 2\alpha.H/\alpha^2$, $i, j = 1, 2, ...$ rank $\mathcal{G}$ (see discussion leading to (2.129) and (2.130)). The structure constants $N_{\alpha\beta}$ will be determined later. The basis $\{H_i, E_\alpha\}$ is called the *Weyl-Cartan basis* of a semisimple Lie algebra.