O que vimos até agora

- O conceito de grupo
- Homomorfismo e Isomorfismo
- Automorfismo (inner and outer), Grupo de automorfismo
- Subgrupos
- Teoremas: Cayley e Lagrange

Continuando ...

• Elemento conjugado

We say an element g of a group G is *conjugate* to an element $g' \in G$ if there exists $\bar{g} \in G$ such that

$$g = \bar{g} g' \bar{g}^{-1} \tag{1.12}$$

This concept of conjugate elements establishes an equivalence relation on the group. Indeed, g is conjugate to itself (just take $\bar{g} = e$), and if g is conjugate to g', so is g' conjugate to g (since $g' = \bar{g}^{-1} g \bar{g}$). In addition, if g is conjugate to g' and g' to g'', i.e. $g' = \tilde{g} g'' \tilde{g}^{-1}$, then g is conjugate to g'', since $g = \bar{g}\tilde{g} g'' \tilde{g}^{-1} \bar{g}^{-1}$. One can use such equivalence relation to divide the group G into classes.

• Classe de conjugação (classe de equivalência)

Definition 1.6 *The set of elements of a group G which are conjugate to each other constitute a* conjugacy class *of G.*

Obviously different conjugacy classes have no common elements. The indentity element e constitute a conjugacy class by itself in any group. Indeed, if g' is conjugate to the identity e, $e = g g' g^{-1}$, then $g' = e$.

• Subgrupo conjugado

Given a subgroup H of a group G we can form the set of elements $g^{-1}Hg$ where g is any fixed element of G and H stands for any element of the subgroup H. This set is also a subgroup of G and is said to be a *conjugate subgroup* of H in G. In fact the conjugate subgroups of H are all isomorphic to H, since if $h_1, h_2 \in H$ and $h_1 h_2 = h_3$ we have that $h_1' = g^{-1}h_1 g$ and $h_2' = g^{-1}h_2 g$ satisfy

$$h_1' h_2' = g^{-1}h_1 g g^{-1} h_2 g = g^{-1} h_1 h_2 g = g^{-1} h_3 g = h_3' \tag{1.13}$$

Notice that the images of two different elements of H, under conjugation by $g \in G$, can not be the same. Because if they were the same we would have

$$g^{-1}h_1 g = g^{-1}h_2 g \rightarrow g(g^{-1}h_1 g)g^{-1} = h_2 \rightarrow h_1 = h_2 \tag{1.14}$$

and that is a contradiction.

• Subgrupo invariante

By choosing various elements $g \in$ G we can form different conjugate subgroups of H in G. However it may happen that for all $g \in G$ we have

$$g^{-1}Hg = H \tag{1.15}$$

This means that all conjugate subgroups of H in G are not only isomorphic to H but are identical to H. In this case we say that the subgroup H is an *invariant subgroup* of G. This implies that, given an element $h_1 \in H$ we can find, for any element $g \in G$, an element $h_2 \in H$ such that

$$g^{-1}h_1 g = h_2 \rightarrow h_1 g = g h_2 \tag{1.16}$$

• Simple and Semisimple groups

Definition 1.7 *We say a group G is simple if its only invariant subgroups are the identity element and the group G itself. In other words, G is simple if it has no invariant proper subgroups. We say G is semisimple if none of its invariant subgroups is abelian.*

Example 1.16 *Consider the group of the non-singular real $n \times n$ matrices, which is generally denoted by $GL(n)$. The matrices of this group with unit determinant form a subgroup since if $det M = det N = 1$ we have $det(M.N) = 1$ and $det M^{-1} = det M = 1$. This subgroup of $GL(n)$ is denoted by $SL(n)$. If $g \in GL(n)$ and $M \in SL(n)$ we have that $g^{-1}Mg \in SL(n)$ since $det(g^{-1}Mg) = det M = 1$. Therefore $SL(n)$ is an invariant subgroup of $GL(n)$ and consequently the latter is not simple. Consider now the matrices of the form $R \equiv x\, \mathbb{1}_{n\times n}$, with x being a non-zero real number, and $\mathbb{1}_{n\times n}$ being the $n \times n$ identity matrix. Notice, that such set of matrices constitute a subgroup of $GL(n)$, since the identity belongs to it, the product of any two of them belongs to the set, and the inverse of $R \equiv x\, \mathbb{1}_{n\times n}$ is $R^{-1} \equiv (1/x)\, \mathbb{1}_{n\times n}$, which is also an element of the set. In addition, such subgroup is invariant since any matrix R commutes with any element of $GL(n)$ and so it is invariant under conjugation. Since that subgroup is abelian, it follows that $GL(n)$ is not semisimple.*

• Centralizer of an element

Definition 1.8 *Given an element g of a group G we can form the set of all elements of G which commute with g, i.e., all $x \in G$ such that $xg = gx$. This set is called the* centralizer *of g and it is a subgroup of G.*

In order to see it is a subgroup of G, take two elements x_1 and x_2 of the centralizer of g, i.e., $x_1 g = g x_1$ and $x_2 g = g x_2$. Then it follows that $(x_1 x_2) g = x_1 (x_2 g) = x_1 (g x_2) = g (x_1 x_2)$. Therefore $x_1 x_2$ is also in the centralizer. On the other hand, we have that

$$x_1^{-1}(x_1 g) x_1^{-1} = x_1^{-1}(g x_1) x_1^{-1} \rightarrow g x_1^{-1} = x_1^{-1} g. \qquad (1.18)$$

So the inverse of an element of the centralizer is also in the centralizer. Therefore the centralizer of an element $g \in G$ is a subgroup of G. Notice that although all elements of the centralizer commute with a given element g they do not have to commute among themselves and therefore it is not necessarily an abelian subgroup of G.

• The center of a group

Definition 1.9 *The* center of a group G *is the set of all elements of G which commute with all elements of G.*

We could say that the center of G is the intersection of the centralizers of all elements of G. The center of a group G is a subgroup of G and it is abelian , since by definition its elements have to commute with one another. In addition, it is an (abelian) invariant subgroup.

• Example: $U(N)$ and $SU(N)$

Example 1.17 *The set of all unitary $n \times n$ matrices form a group, called $U(n)$, under matrix multiplicaton. That is because if U_1 and U_2 are unitary ($U_1^\dagger = U_1^{-1}$ and $U_2^\dagger = U_2^{-1}$) then $U_3 \equiv U_1 U_2$ is also unitary. In addition the inverse of U is just $U^\dagger$ and the identity is the unity $n \times n$ matrix. The unitary matrices with unity determinant constitute a subgroup, because the product of two of them, as well as their inverses, have unity determinant. That subgroup is denoted $SU(n)$. It is an invariant subgroup of $U(n)$ because the conjugation of a matrix of unity determinant by any unitary matrix gives a matrix of unity determinat, i.e.* $\det\left(UMU^\dagger\right) = \det M = 1$, *with $U \in U(n)$ and $M \in SU(n)$. Therefore, $U(n)$ is not simple. However, it is not semisimple either, because it has an abelian subgroup constituted by the matrices $R \equiv e^{i\theta}\, 1\!\!1_{n\times n}$, with θ being real. Indeed, the multiplication of any two R's is again in the set of matrices R, the inverse of R is $R^{-1} = e^{-i\theta}\, 1\!\!1_{n\times n}$, and so a matrix in the set. Notice the subgroup constituted by the matrices R is isomorphic to $U(1)$, the group of 1×1 unitary matrices, i.e. phases $e^{i\theta}$. Since, the matrices R commute with any unitary matrix, it follows they are invariant under conjugation by elements of $U(n)$. Therefore, the subgroup $U(1)$ is an abelian invariant subgroup of $U(n)$, and so $U(n)$ is not semisimple. The subgroup $U(1)$ is in fact the* center *of $U(n)$, i.e. the set of matrices commuting with all unitary matrices. Notice, that such $U(1)$ is not a subgroup of $SU(n)$, since their elements do not have unity deteminant. However, the discrete subset of matrices $e^{2\pi i m/n}\, 1\!\!1_{n\times n}$ with $m = 0, 1, 2...(n-1)$ have unity determinant and belong to $SU(n)$. They certainly commute with all $n \times n$ matrices, and constitute the center of $SU(n)$. Those matrices form an abelian invariant subgroup of $SU(n)$, which is isomorphic to Z_n. Therefore, $SU(n)$ is not semisimple.*

• Direct products

We say a group G is the *direct product* of its subgroups $H_1, H_2 ... H_n$, denoted by $G = H_1 \otimes H_2 \otimes H_3 ... \otimes H_n$, if

1. the elements of different subgroups commute

2. Every element $g \in G$ can be expressed in one and only one way as

$$g = h_1 h_2 ... h_n \tag{1.19}$$

   where h_i is an element of the subgroup H_i , $i = 1, 2, ..., n$.

From these requirements it follows that the subgroups H_i have only the identity e in common. Because if $f \neq e$ is a common element to H_2 and H_5 say, then the element $g = h_1 f h_3 h_4 f^{-1} h_6 ... h_n$ could be also written as $g = h_1 f^{-1} h_3 h_4 f h_6 ... h_n$. Every subgroup H_i is an invariant subgroup of G, because if $h_i' \in H_i$ then

$$g^{-1} h_i' g = (h_1 h_2 ... h_n)^{-1} h_i' (h_1 h_2 ... h_n) = h_i^{-1} h_i' h_i \in H_i \tag{1.20}$$

Example 1.18 *Consider the cyclic group Z_6 with elements e, a, a^2, a^3, a^4 and a^5 (and $a^6 = e$). It can be written as the direct product of its subgroups $H_1 = \{e, a^2, a^4\}$ and and $H_2 = \{e, a^3\}$ since*

$$e = ee, a = a^4 a^3, a^2 = a^2 e, a^3 = ea^3, a^4 = a^4 e, a^5 = a^2 a^3 \tag{1.21}$$

Therefore we write $Z_6 = H_1 \otimes H_2$ (or $Z_6 = Z_3 \otimes Z_2$).

- ## Another way

Given two groups G and G' we can construct another group by taking the direct product of G and G' as follows: the elements of $G'' = G \otimes G'$ are formed by the pairs (g, g') where $g \in G$ and $g' \in G'$. The composition law for G'' is defined by

$$(g_1, g_1')(g_2, g_2') = (g_1 g_2, g_1' g_2') \tag{1.22}$$

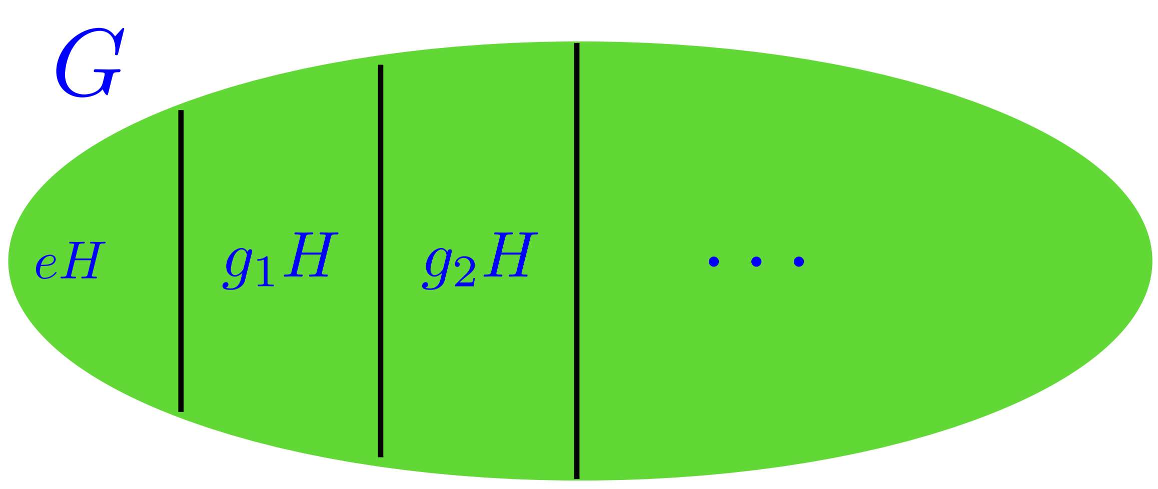
where $g_1 g_2$, $(g_1' g_2')$ is the product of g_1 by g_2, (g_1' by g_2') according to the composition law of G (G'). If e and e' are respectively the identity elements of G and G', then the sets $G \otimes 1 = \{(g, e') \mid g \in G\}$ and $1 \otimes G' = \{(e, g') \mid g' \in G'\}$ are subgroups of $G'' = G \otimes G'$ and are isomorphic respectively to G and G'. Obviously $G \otimes 1$ and $1 \otimes G'$ are invariant subgroups of $G'' = G \otimes G'$.

• Cosets

Given a group G and a subgroup H of G we can divide the group G into disjoint sets such that any two elements of a given set differ by an element of H multiplied from the right. That is, we construct the sets

$$gH \equiv \{ \text{ all elements } gh \text{ of } G \text{ such that } h \text{ is any element of } H \text{ and } g \text{ is a fixed element of } G\}$$

If $g = e$ the set eH is the subgroup H itself. All elements in a set gH are different, because if $gh_1 = gh_2$ then $h_1 = h_2$. Therefore the numbers of elements of a given set gH is the same as the number of elements of the subgroup H. Also an element of a set gH is not contained by any other set $g'H$ with $g' \neq g$. Because if $gh_1 = g'h_2$ then $g = g'h_2h_1^{-1}$ and therefore g would be contained in $g'H$ and consequently $gH \equiv g'H^1$. Thus we have split the group G into disjoint sets, each with the same number of elements, and a given element $g \in G$ belongs to one and only one of these sets.



The set of elements gH are called *left cosets* of H in G. They are certainly not subgroups of G since they do not contain the identity element, except for the set $eH = H$.

Analogously we could have split G into sets Hg which are formed by elements of G which differ by an element of H multiplied from the left. The same results would be true for these sets. They are called *right cosets* of H in G.

The set of left cosets of H in G is denoted by G/H and is called the *left coset space*. An element of G/H is a set of elements of G, namely gH. Analogously the set of right cosets of H in G is denoted by $H \setminus G$ and it is called the *right coset space*.

• Lagrange's Theorem

Proof of Lagrange's theorem(section 1.2).
From the considerations above we see that for a finite group G of order m with a proper subgroup H of order n, we can write

$$m = kn \tag{1.23}$$

where k is the number of disjoint sets gH.$\square$

• Quocient or Factor Group

If the subgroup H of G is an invariant subgroup then the left and right cosets are the same since $g^{-1}Hg = H$ implies $gH = Hg$. In addition, the coset space G/H, for the case in which H is invariant, has the structure of a group and it is called the *factor group or the quocient group.* In order to show this we consider the product of two elements of two different cosets. We get

$$gh_1g'h_2 = gg'g'^{-1}h_1g'h_2 = gg'h_3h_2 \tag{1.24}$$

where we have used the fact that H is invariant, and therefore there exists $h_3 \in H$ such that $g'^{-1}h_1g' = h_3$. Thus we have obtained an element of a third coset, namely $gg'H$. If we had taken any other elements of the cosets gH and $g'H$, their product would produce an element of the same coset $gg'H$. Consequently we can introduce, in a well defined way, the product of elements of the coset space G/H, namely

$$gHg'H \equiv gg'H \tag{1.25}$$

The invariant subgroup H plays the role of the identity element since

$$(gH)H = H(gH) = gH \tag{1.26}$$

The inverse element is $g^{-1}H$ since

$$g^{-1}HgH = g^{-1}gH = H = gHg^{-1}H \tag{1.27}$$

The associativity is guaranteed by the associativity of the composition law of the group G. Therefore the coset space $G/H \equiv H \setminus G$ is a group in the case where H is an invariant subgroup. Notice that such group is not necessarily a subgroup of G or H.

• Examples

Example 1.19 *The real numbers without the zero,* $R-0$ *, form a group under multiplication. The positive real numbers,* R^+*, close under multiplication and the inverse of a positive real number* x *is also positive* $(1/x)$ *. Therefore* R^+ *is a subgroup of* $R-0$ *. In addition we have that the conjugation of a real* x *by another real* y *is equal to* x *,* $(y^{-1}xy = x)$ *. Therefore* R^+ *is an invariant subgroup of* $R-0$ *. The coset space* $(R-0)/R^+$ *has two elements, namely* R^+ *and* R^- *(the negative real numbers). This coset space is a group and it is isomorphic to the cyclic group of order 2,* Z_2 *(see example 1.10), since its elements satisfy* $R^+.R^+ \subset R^+$ *,* $R^+.R^- \subset R^-$*,* $R^-.R^- \subset R^+$*.*

Example 1.20 *Any subgroup of an abelian group is an invariant subgroup.*

Example 1.21 *Consider the cyclic group Z_6 with elements e, a, a^2, ... a^5 and $a^6 = e$ and the subgroup Z_2 with elements e and a^3. Then the cosets are given by*

$$c_0 = \{e, a^3\} \ , \ c_1 = \{a, a^4\} \ , \ c_2 = \{a^2, a^5\} \tag{1.28}$$

Since Z_2 is an invariant subgroup of Z_6 the coset space Z_6/Z_2 is a group. Following the definition of the product law on the coset given above one easily sees it is isomorphic to Z_3 since

$$\begin{aligned} c_0.c_0 = c_0 \ , \ c_0.c_1 = c_1 \ , \ c_0.c_2 = c_2 \\ c_1.c_1 = c_2 \ , \ c_1.c_2 = c_0 \ , \ c_2.c_2 = c_1 \end{aligned} \tag{1.29}$$

If we now take the subgroup Z_3 of Z_6 with elements e, a^2 and a^4 we get the cosets

$$d_0 = \{e, a^2, a^4\} \ , \ d_1 = \{a, a^3, a^5\} \tag{1.30}$$

Again the coset space Z_6/Z_3 is a group and it is isomorphic to Z_2 since

$$d_0.d_0 = d_0 \ , \ d_0.d_1 = d_1 \ , \ d_1.d_1 = d_0 \tag{1.31}$$

• Representation Theory

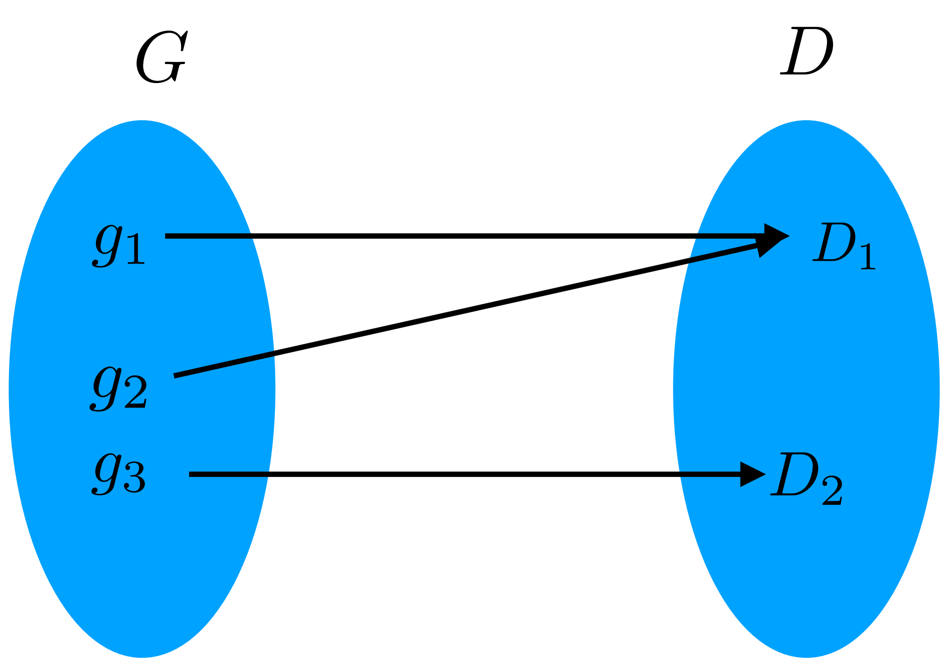
We need:

A vector space V, with vectors $| \, v \rangle$

Operators D acting on V

$$D: \qquad V \to V \qquad\qquad D \,|\, v\rangle = |\, v' \rangle$$

An homomorphism $G \to D$



$$D\,(g)\;D\,(g')\;|\,v\rangle = D\,(g\,g')\;|\,v\rangle$$

for all $|\,v\rangle \in V$

- Faithful Representation

When $G \rightarrow D$ is an isomorphism (one-to-one)

- Dimension of the Representation = Dimension of V

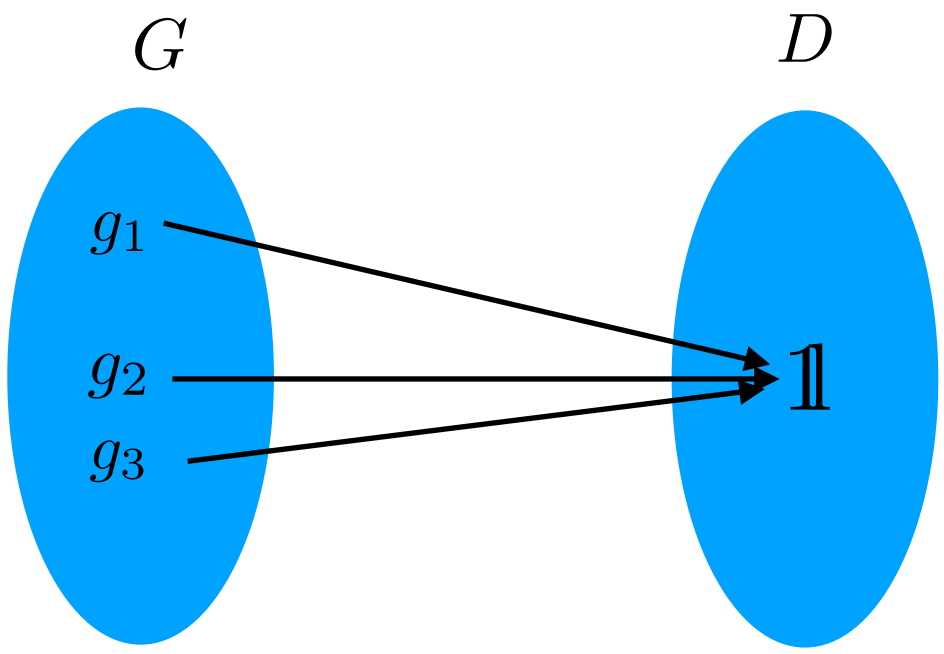
- Linear Representation

When the operators D are linear operators, i.e.,

$$\begin{aligned} D(|\, v\rangle + |\, v'\rangle) &= D \,|\, v\rangle + D \,|\, v'\rangle \\ D(a \,|\, v\rangle) &= aD \,|\, v\rangle \end{aligned} \tag{1.37}$$

with $|\, v\rangle, |\, v'\rangle \in V$ and a being a c-number, we say they form a *linear representation* of G.

- Scalar Representation



$$D(g)\ D(g') = \mathbb{1}\,\mathbb{1} = \mathbb{1} = D(g\,g')$$

Dimension of $V = 1$

$$D_{kj}(g')D_{ji}(g) = D_{ki}(g'g)$$

- Matrix Representation

Given a basis $\mid v_i\rangle$, $i = 1, 2, 3 \ldots \dim V$

$$D(g) \mid v_i\rangle = \mid v\rangle$$

$$D(g) \mid v_i\rangle = \mid v_j\rangle\, c_{ji} \equiv \mid v_j\rangle\, D_{ji}(g)$$

$$D(g')\, D(g) \mid v_i\rangle = D(g') \mid v_j\rangle\, D_{ji}(g) = \mid v_k\rangle D_{kj}(g')\ D_{ji}(g)$$

$$= D(g'g) \mid v_i\rangle = \mid v_k\rangle D_{ki}(g'g)$$

Example 1.23 *In example 1.9 we have defined the group S_n . We can construct a representation for this group in terms of $n \times n$ matrices as follows: take a vector space V_n and let $\mid v_i\rangle$, $i = 1, 2, ...n$, be a basis of V_n. One can define $n!$ operators that acting on the basis permute them, reproducing the $n!$ permutations of n elements. Using (1.38) one then obtains the matrices. For instance, in the case of S_3, consider the matrices*

$$D(a_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad D(a_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$D(a_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad D(a_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$
$$D(a_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \qquad D(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.40}$$

where a_m, $m = 0, 1, 2, 3, 4, 5$, are the 6 elements of S_3. One can check that the action

$$D(a_m) \mid v_i\rangle =\mid v_j\rangle D_{ji}(a_m) \tag{1.41}$$

gives the 6 permutations of the three basis vectors $\mid v_i\rangle$, $i = 1, 2, 3$, of V_3. In addition the product of these matrices reproduces the composition law of permutations in S_3.

By considering V_3 as the space of column vectors 3×1 , and taking the canonical basis

$$\mid v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \ \mid v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \ \mid v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.42}$$

one can check that the matrices given above play the role of the operators permuting the basis too

$$D_{ij}(a_m) \mid v_k\rangle_j =\mid v_l\rangle_i D_{lk}(a_m) \tag{1.43}$$

Example 1.24 *As an example of a non-linear representation consider the transformation on the complex plane z as*

$D(g)(z_1+z_2) \neq D(g)z_1 + D(g)z_2$

$$z \to z' = \frac{a_1\, z + a_2}{a_3\, z + a_4}\,; \qquad z\,, a_i \in C\,; \qquad a_1\, a_4 - a_2\, a_3 \neq 0 \tag{1.44}$$

Now consider a second transformation composed with the first $(b_i \in C)$

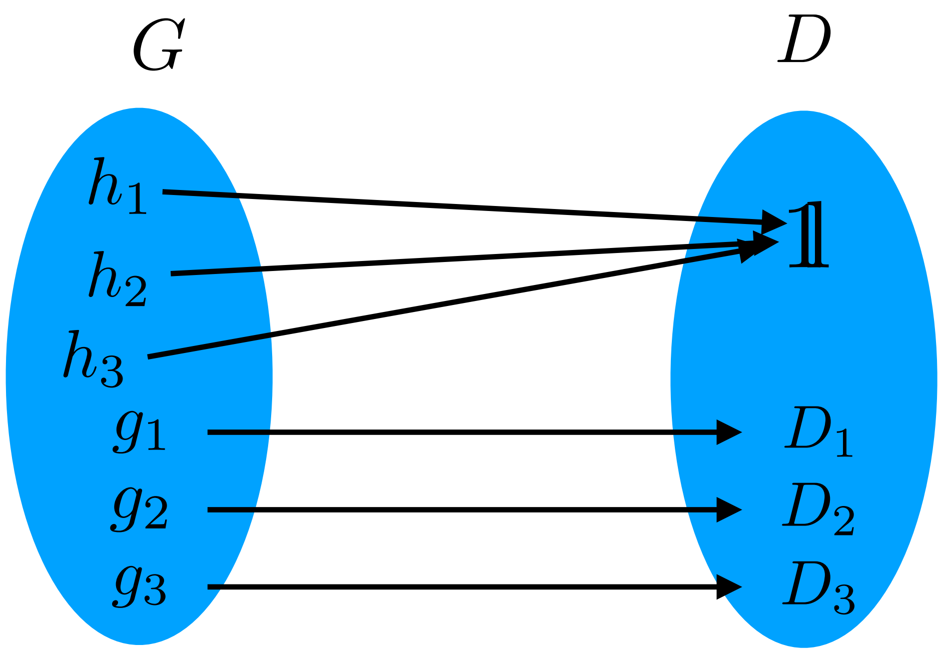
$$\begin{aligned} z' \quad \to \quad z'' = \frac{b_1\, z' + b_2}{b_3\, z' + b_4} &= \frac{b_1\, \frac{a_1\, z + a_2}{a_3\, z + a_4} + b_2}{b_3\, \frac{a_1\, z + a_2}{a_3\, z + a_4} + b_4} \\ = \quad & \frac{(b_1\, a_1 + b_2\, a_3)\ z + (b_1\, a_2 + b_2\, a_4)}{(b_3\, a_1 + b_4\, a_3)\ z + (b_3\, a_2 + b_4\, a_4)} \end{aligned} \tag{1.45}$$

Note that such transformations compose in the same way as the product of 2×2 *complex matrices, i.e.*

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} b_1\, a_1 + b_2\, a_3 & b_1\, a_2 + b_2\, a_4 \\ b_3\, a_1 + b_4\, a_3 & b_3\, a_2 + b_4\, a_4 \end{pmatrix} \tag{1.46}$$

Since the determinant of the matrices are not zero, the transformations (1.44) constitute a group isomorphic to $GL(2\,,\,C)$*. By imposing* $\mid a_1 \mid^2 + \mid a_2 \mid^2 = \mid a_3 \mid^2 + \mid a_4 \mid^2 = 1$*, and* $(a_1\, a_3^* + a_2\, a_4^*) = 0$*, one gets a group isomorphic to* $U(2)$*. If in addition one imposes* $(a_1\, a_4 - a_2\, a_3) = 1$*, one gets a group isomorphic to* $SU(2)$*. The transformations (1.44) are called the Möbius transformations of the complex plane.*

The elements of a group mapped into the identity operator constitute an invariant subgroup



$$D(h_1)\ D(h_2) = \mathbb{1}\,\mathbb{1} = \mathbb{1} = D(h_1 h_2)$$

$$D(h_1)\ D\left(h_1^{-1}\right) = \mathbb{1}\,D\left(h_1^{-1}\right) = D(e) = \mathbb{1}$$

$$D(g)\,D(h_1)\ D\left(g^{-1}\right) = D(g)\ \mathbb{1}\,D\left(g^{-1}\right) = \mathbb{1}$$

Note that that all elements in a given coset gH of the coset space G/H are mapped into the same operator

$$D(gh) = D(g)\,D(h) = D(g)\,\mathbb{1} = D(g)$$

So we have a faithful representation of the factor group G/H

• Equivalent Representations

Two representations D and D' of an abstract group G are said to be *equivalent representations* if there exists an operator C such that

$$D'(g) = CD(g)C^{-1} \tag{1.49}$$

with C being the same for every $g \in G$. Such thing happens, for instance, when one changes the basis of the representation

$$| v'_i\rangle =| v_j\rangle \Lambda_{ji} \tag{1.50}$$

Then

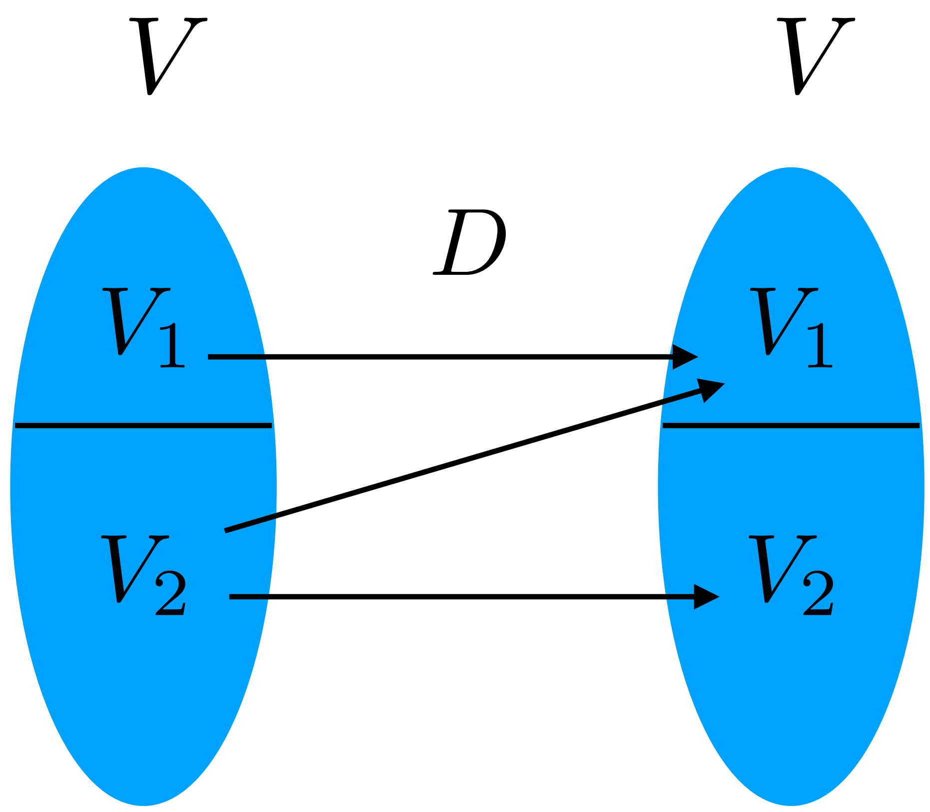
$$\begin{aligned} D(g) \mid v'_i\rangle &\equiv \mid v'_j\rangle D'_{ji}(g) \\ &= \mid v_k\rangle D_{kl}(g)\Lambda_{li} \\ &= \mid v_n\rangle \Lambda_{nj}\Lambda^{-1}_{jk}D_{kl}(g)\Lambda_{li} \\ &= \mid v'_j\rangle \Lambda^{-1}_{jk}D_{kl}(g)\Lambda_{li} \end{aligned} \tag{1.51}$$

Therefore the new matrix representatives are

$$D'_{ji}(g) = \Lambda^{-1}_{jk}D_{kl}(g)\Lambda_{li} \tag{1.52}$$

So, the matrix representatives change as in (1.49) with $C = \Lambda^{-1}$. Although the structure of the representation does not change the matrices look different.

- Reducible Representations



$$D(g) \mid v_1\rangle \in V_1$$

$$\text{any } g \in G \qquad \text{any } \mid v_1\rangle \in V_1$$

$$V_1 \equiv \text{invariant subspace}$$

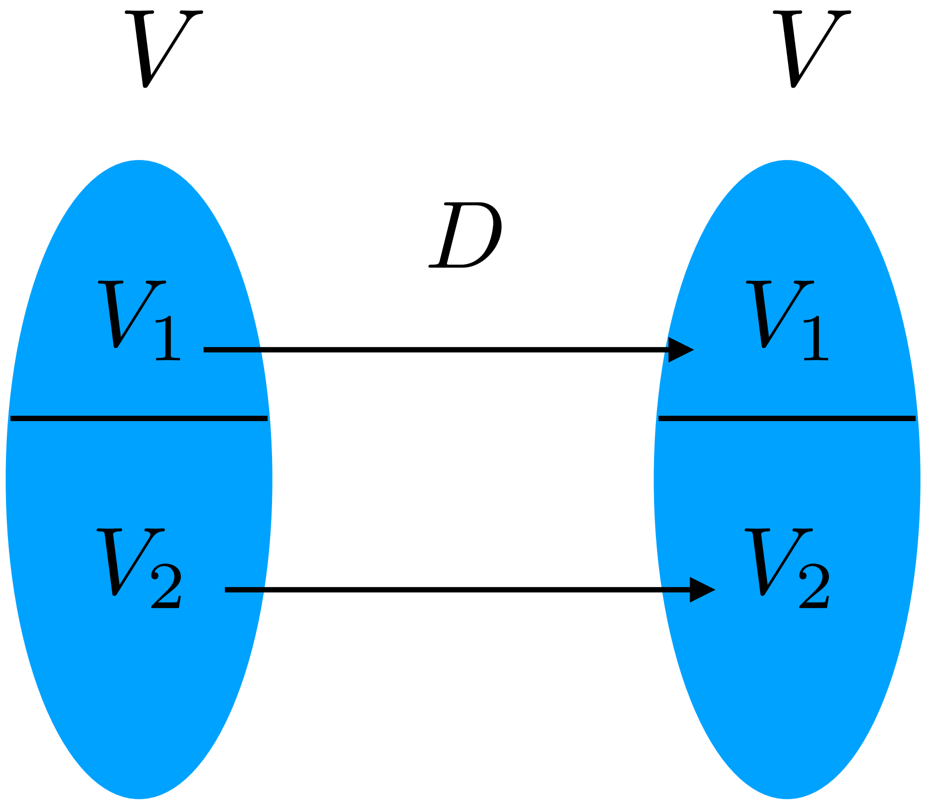
$$V \equiv \text{Reducible Representation}$$

$$V_1 \text{ alone is a representation of } G$$

In matrix notation

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} Av_1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} Cv_2 \\ Bv_2 \end{pmatrix}$$

- Completely Reducible Representations



V_1 and V_2 are invariant subspaces

V_1 and V_2 are each one representations of G

In matrix notation

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Av_1 \\ Bv_2 \end{pmatrix}$$

• Schur's Lemma

Lemma 1.1 (Schur) *Any matrix which commutes with all matrices of a given irreducible representation of a group G must be a multiple of the unit matrix.*

Proof Let A be a matrix that commutes will all matrices $D(g)$ of a given irreducible representation of G, i.e.

$$AD(g) = D(g)A \tag{1.58}$$

for any $g \in G$. Consider the eigenvalue equation

$$A \mid v\rangle = \lambda \mid v\rangle \tag{1.59}$$

where $\mid v\rangle$ is some vector in the representation space V. Notice that, if v is an eigenvector with eigenvalue λ, then $D(g) \mid v\rangle$ has also eigenvalue λ since

$$AD(g) \mid v\rangle = D(g)A \mid v\rangle = \lambda D(g) \mid v\rangle. \tag{1.60}$$

Therefore the subspace of V generated by all eigenvectors of A with eigenvalue λ is an invariant subspace of V. But if the representation is irreducible that means this subspace is the zero vector or is the entire V. In the first case we get that $A = 0$, and in the second we get that A has only one eigenvalue and therefore $A = \lambda 1$. $\square$

Algebraically closed field

Corollary 1.2 *Every irreducible representation of an abelian group is one dimensional.*

Proof Since the group is abelian any matrix has to commute with all other matrices of the representation. According to Schur's lemma they have to be proportional to the identity matrix. So, any vector of the representation space V generates an invariant subspace. Therefore V has to be unidimensional if the representation is irreducible. $\square$

Example: Rotation on the plane ($SO(2)$)

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad R(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$$

Abelian: $\quad R(\theta)R(\varphi) = R(\theta + \varphi)$

$$M R(\theta) M^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \qquad M = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + iy \\ ix + y \end{pmatrix}$$

Definition 1.11 *A representation D is said to be* unitary *if the matrices D_{ij} of the operators are unitary, i.e. $D^\dagger = D^{-1}$.*

Theorem 1.3 *Any representation of a finite group is equivalent to a unitary representation*

compact

Definition 1.12 *Given two representations D and D' of a given group G, one can construct what is called the* tensor product representation *of D and D'. Denoting by $\mid v_i\rangle$, $i = 1, 2, \ldots \dim D$, and $\mid v'_l\rangle$, $l = 1, 2, \ldots \dim D'$, the basis of D and D' respectively, one constructs the basis of $D \otimes D'$ as*

$$\mid w_{il}\rangle = \mid v_i\rangle \otimes \mid v'_l\rangle \tag{1.69}$$

The operators representing the group elements act as

$$D^{\otimes}(g) \mid w_{il}\rangle = D(g) \otimes D'(g) \mid w_{il}\rangle = D(g) \mid v_i\rangle \otimes D'(g) \mid v'_l\rangle \tag{1.70}$$

The dimension of the representation $D \otimes D'$ is the product of the dimensions of D and D', i.e. $\dim D \otimes D' = \dim D \dim D'$.

Produto direto de matrizes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} e & f \\ g & h \end{pmatrix} & b\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ c\begin{pmatrix} e & f \\ g & h \end{pmatrix} & d\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix}$$

- Example: $\mathbb{Z}_2 \sim S_2$

$$e \cdot e = e \qquad e \cdot a = a \qquad a \cdot a = e$$

Two dimensional rep.

$$D\left(e\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D\left(a\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |\, v\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$$

Tensor product rep. $D^{\otimes} \equiv D \otimes D$

$$D^{\otimes}\left(e\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad D^{\otimes}\left(a\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad |\, v\rangle^{\otimes} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

- Characters

Definition 1.13 *In a given representation D of a group G we define the character $\chi^D(g)$ of a group element $g \in G$ as the trace of the matrix representing it, i.e.*

$$\chi^D(g) \equiv Tr(D(g)) = \sum_{i=1}^{dimD} D_{ii}(g) \tag{1.71}$$

Equivalent representations

$$Tr(D'(g)) = Tr(CD(g)C^{-1}) = Tr(D(g)) \rightarrow \chi^D(g) = \chi^{D'}(g)$$

Example

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad MR(\theta)M^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$\mathrm{Tr}R(\theta) = \mathrm{Tr}M\,R(\theta)\,M^{-1} = 2\,\cos\theta$$

Note that: $\chi^D(e) = dim\ D$

Six theorems in a row

Theorem 1.4 *Let D and D' be two irreducible representations of a finite group G and χ^D and $\chi^{D'}$ the corresponding characters. Then*

$$\frac{1}{N(G)} \sum_{g \in G} (\chi^D(g))^* \chi^{D'}(g) = \delta_{DD'} \tag{1.74}$$

where $N(G)$ is the order of G, $\delta_{DD'} = 1$ if D and D' are equivalent representations and $\delta_{DD'} = 0$ otherwise.

Theorem 1.5 *A sufficient conditions for two representations of a finite group G to be equivalent is the equality of their character systems.*

Theorem 1.6 *The number of times n_D that an irreducible representation D appears in a given reducible representation D' of a finite group G is given by*

$$n_D = \frac{1}{N(G)} \sum_{g \in G} \chi^{D'}(g)(\chi^D(g))^* \tag{1.75}$$

where χ^D and $\chi^{D'}$ are the characters of D and D' respectively, and $N(G)$ is the order of G.

Theorem 1.7 *A necessary and suficient condition for a representation D of a finite group G to be irreducible is*

$$\frac{1}{N(G)} \sum_{g \in G} \mid \chi^D(g) \mid^2 = 1 \tag{1.76}$$

where χ^D are the characters of D and $N(G)$ the order of G.

Valid for compact groups as well $\quad \frac{1}{N(G)} \sum_{g \in G} \rightarrow \quad \int_G \mathcal{D}g$

Example

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad\qquad M R(\theta) M^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$\chi(\theta) = \mathrm{Tr} R(\theta) = 2\,\cos\theta$$

Volume $$\int_0^{2\,\pi} d\theta = 2\,\pi$$

$$\frac{1}{2\,\pi} \int_0^{2\,\pi} d\theta \,\mid \chi(\theta \mid^2 = \frac{1}{2\,\pi} \int_0^{2\,\pi} d\theta\, 4\,\cos^2\theta$$

$$= \frac{1}{2\,\pi} \int_0^{2\,\pi} d\theta\, 2\,(\cos^2\theta + \sin^2\theta) = 2$$

It is a reducible representation

Theorem 1.8 *The sum of the squares of the dimensions of the inequivalent irreducible representations of a finite group G is equal to the order of G.*

Theorem 1.9 *The number of inequivalent irreducible representations of a finite group G is equal to the number of conjugacy classes of G.*

Example: The tri-dimensional representation of S_3

$$D(a_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad D(a_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(a_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad D(a_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D(a_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \qquad D(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \chi^D(a_0) &= 3 \\ \chi^D(a_1) &= \chi^D(a_2) = \chi^D(a_3) = 1 \\ \chi^D(a_4) &= \chi^D(a_5) = 0 \end{aligned}$$

$$\frac{1}{6} \sum_{i=0}^{5} \mid \chi^D(a_i) \mid^2 = 2$$

It is a reducible representation

Invariant subspace

$$| \, w_3 \rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Complement

$$| \, w_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} ; \quad | \, w_2 \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Relation to canonical basis

$$| \, w_i \rangle = | \, v_j \rangle \Lambda_{ji}$$

$$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$D'(a_m) = \Lambda^{-1} D(a_m) \Lambda$$

$$D'(a_m) = \begin{pmatrix} D''(a_m) & 0 \\ 0 & 1 \end{pmatrix}$$

$$D''(a_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad D''(a_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$D''(a_2) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}; \qquad D''(a_3) = \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix};$$

$$D''(a_4) = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}; \qquad D''(a_5) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$$

$$\begin{aligned} \chi^{D''}(a_0) &= 2 \\ \chi^{D''}(a_1) &= \chi^{D''}(a_2) = \chi^{D''}(a_3) = 0 \\ \chi^{D''}(a_4) &= \chi^{D''}(a_5) = -1 \end{aligned}$$

$$\frac{1}{6}\sum_{i=0}^{5} \mid \chi^{D''}(a_i) \mid^2 = 1$$

It is an irreducible representation

Scalar Representation

$$D_s(a_i) = 1 \qquad\qquad i = 0, 1, 2, 3, 4, 5$$

But

$$\frac{1}{6}(1^2 + 2^2) \neq 1$$

Missing representation

$$\begin{aligned} D'''(a_0) &= D'''(a_4) = D'''(a_5) = 1 \\ D'''(a_1) &= D'''(a_2) = D'''(a_3) = -1 \end{aligned}$$

Now

$$\frac{1}{6}(1^2 + 2^2 + 1^2) = 1$$

• Real and complex representations

Definition 1.14 *If all the matrices of a representation are real the representation is said to be* real.

Notice that if D is a matrix representation of a group G, then the matrices $D^*(g)$, $g \in G$, also constitute a representation of G of the same dimension as D, since

$$D(g)D(g') = D(gg') \to D^*(g)D^*(g') = D^*(gg') \tag{1.77}$$

If D is equivalent to a real representation D_R, then D is equivalent to D^*. The reason is that there exists a matrix C such that

$$D_R(g) = CD(g)C^{-1} \tag{1.78}$$

and so

$$D_R(g) = C^* D^*(g)(C^*)^{-1} \tag{1.79}$$

Therefore

$$D^*(g) = (C^{-1}C^*)^{-1} D(g) C^{-1}C^* \tag{1.80}$$

and D is equivalent to D^*. However the converse is not always true, i.e. , if D is equivalent to D^* it does not means D is equivalent to a real representation. So we classify the representations into three classes regarding the relation between D and D^*.

Definition 1.15 1. *If D is equivalent to a real representation it is said* potentially real.

2. *If D is equivalent to D^* but not equivalent to a real representation it is said* pseudo real.

3. *If D is not equivalent to D^* then it is said* essentially complex.

Notice that if D is potentially real or pseudo real then its characters are real.

- Example: Dublet rep. of $SU(2)$ (defining rep.)

$$D(g) = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix} \qquad |\, Z_1 \,|^2 + |\, Z_2 \,|^2 = 1$$

$$D\left(g^{-1}\right) = D^\dagger(g) = \begin{pmatrix} Z_1^* & -Z_2 \\ Z_2^* & Z_1 \end{pmatrix}$$

$$Z_1 = x_1 + i\, x_2 \qquad Z_2 = x_3 + i\, x_4$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \qquad SU(2) \sim S^3$$

$$Z_1 = \cos\theta\, e^{i\varphi_1} = \sqrt{\alpha}\, e^{i\varphi_1} \qquad Z_2 = \sin\theta\, e^{i\varphi_1} = \sqrt{1-\alpha}\, e^{i\varphi_1}$$

$$0 \leq \alpha \leq 1$$

- Example: Dublet rep. of $SU(2)$

$$D(g) = \begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{pmatrix}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le \varphi_a \le 2\pi$$

$$D(g^{-1}) = D^{\dagger}(g) = \begin{pmatrix} e^{-i\varphi_1}\cos\theta & -e^{i\varphi_2}\sin\theta \\ e^{-i\varphi_2}\sin\theta & e^{i\varphi_1}\cos\theta \end{pmatrix}$$

anti-dublet rep. of $SU(2)$

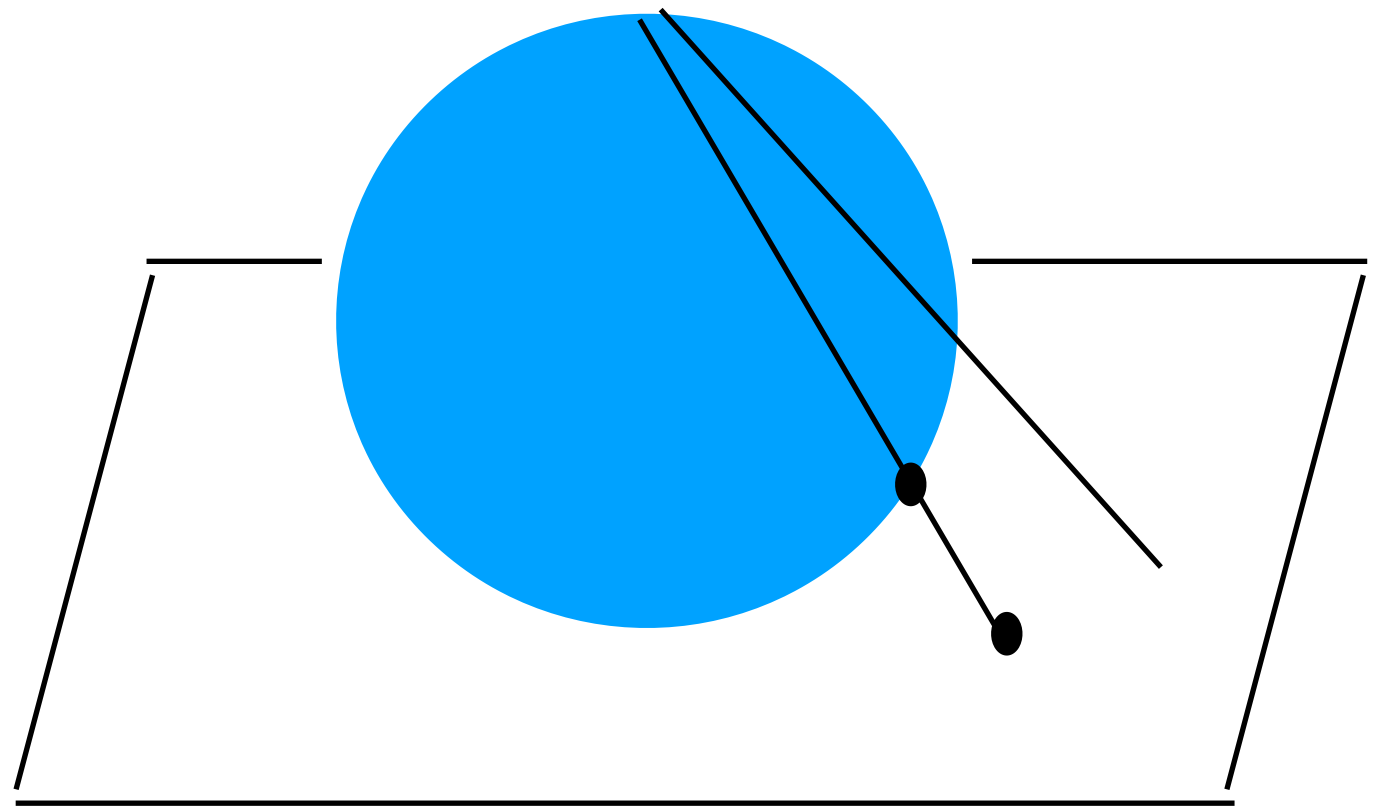
$$D^{*}(g) = \begin{pmatrix} e^{-i\varphi_1}\cos\theta & e^{-i\varphi_2}\sin\theta \\ -e^{i\varphi_2}\sin\theta & e^{i\varphi_1}\cos\theta \end{pmatrix}$$

They are equivalent:

Not true for $SU(N)$ with $N \ge 3$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{-i\varphi_2}\sin\theta & e^{-i\varphi_1}\cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -e^{i\varphi_2}\sin\theta & e^{i\varphi_1}\cos\theta \\ -e^{-i\varphi_1}\cos\theta & -e^{-i\varphi_2}\sin\theta \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_1}\cos\theta & e^{-i\varphi_2}\sin\theta \\ -e^{i\varphi_2}\sin\theta & e^{i\varphi_1}\cos\theta \end{pmatrix} = D^{*}(g)$$



$$
\begin{tikzcd}
e \arrow[d] & a^2 \arrow[d] & a \arrow[d] \\
1 & e^{i\,2\pi/3} & e^{i\,4\pi/3} \\
{} \arrow[u] & a \arrow[u] & a^2 \arrow[u]
\end{tikzcd}
$$

$$
z e^{i2\pi/3} z^{-1} \neq e^{i4\pi/3}
$$