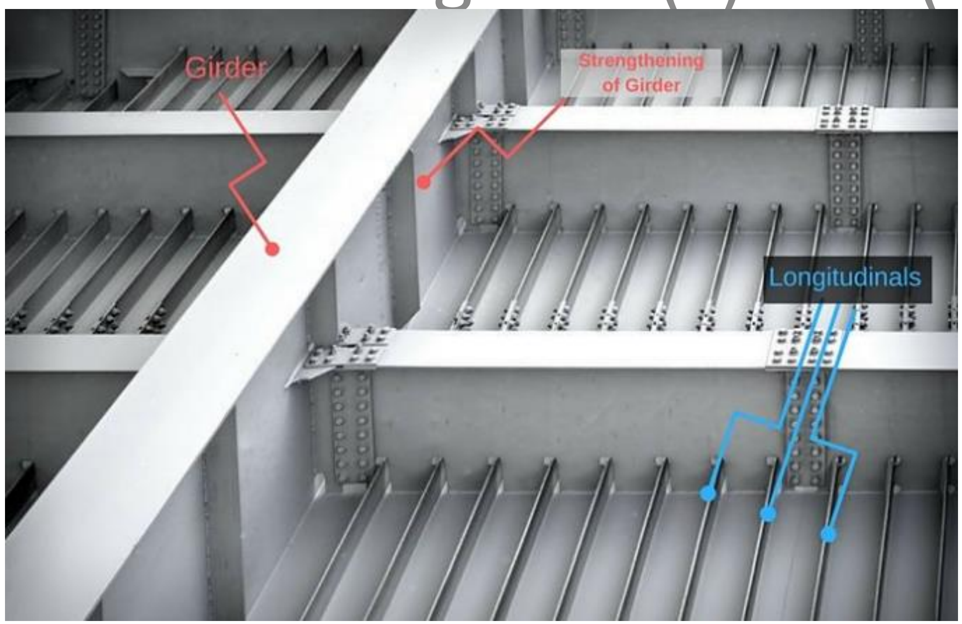
DEPARTAMENTO DE ENGEHARIA NAVAL E OCEÂNICA
ESCOLA POLITÉCNICA DA USP

Análise de Vigas : V(x) & M(x)



PNV 3212 – Mecânica Dos Sólidos I

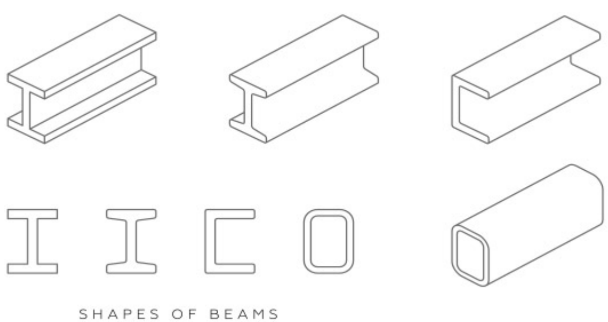
2020

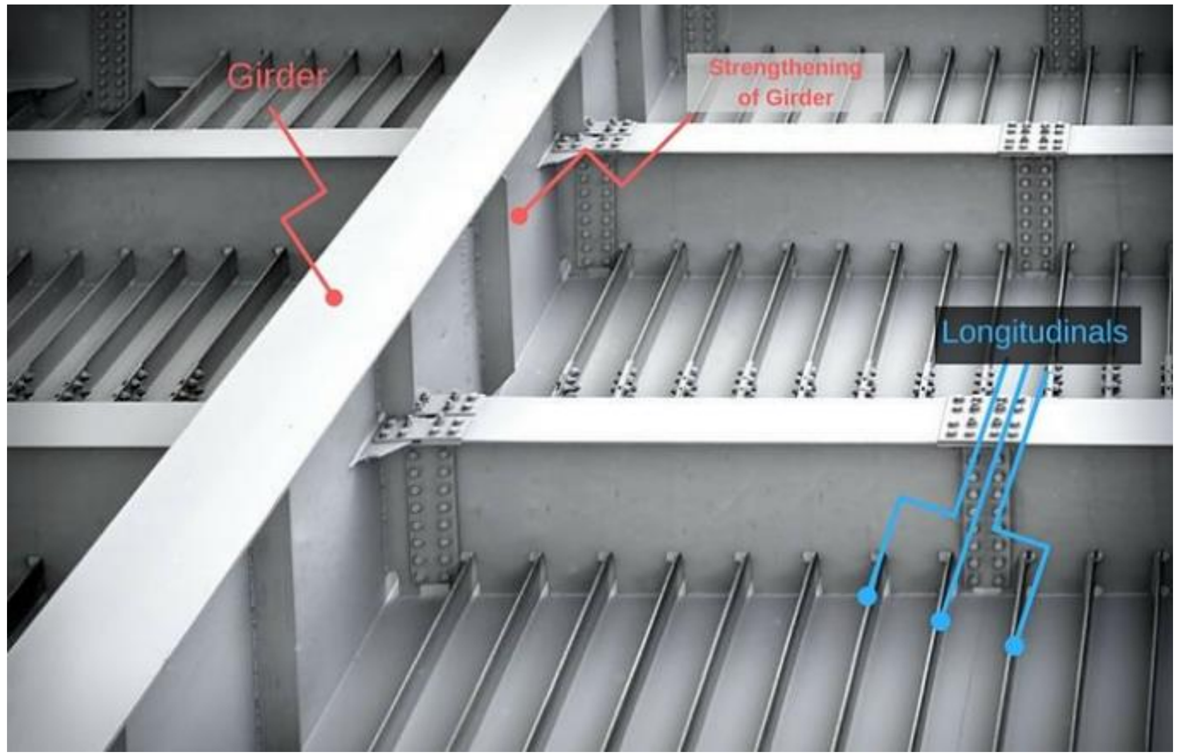
Agenda

- Motivação
- Cálculo de V(x) e M(x)
  - Método de Integração

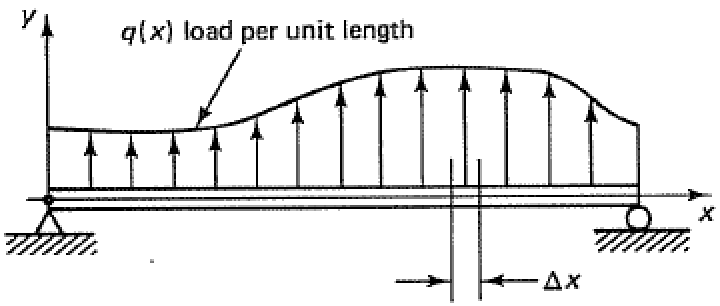
Motivação

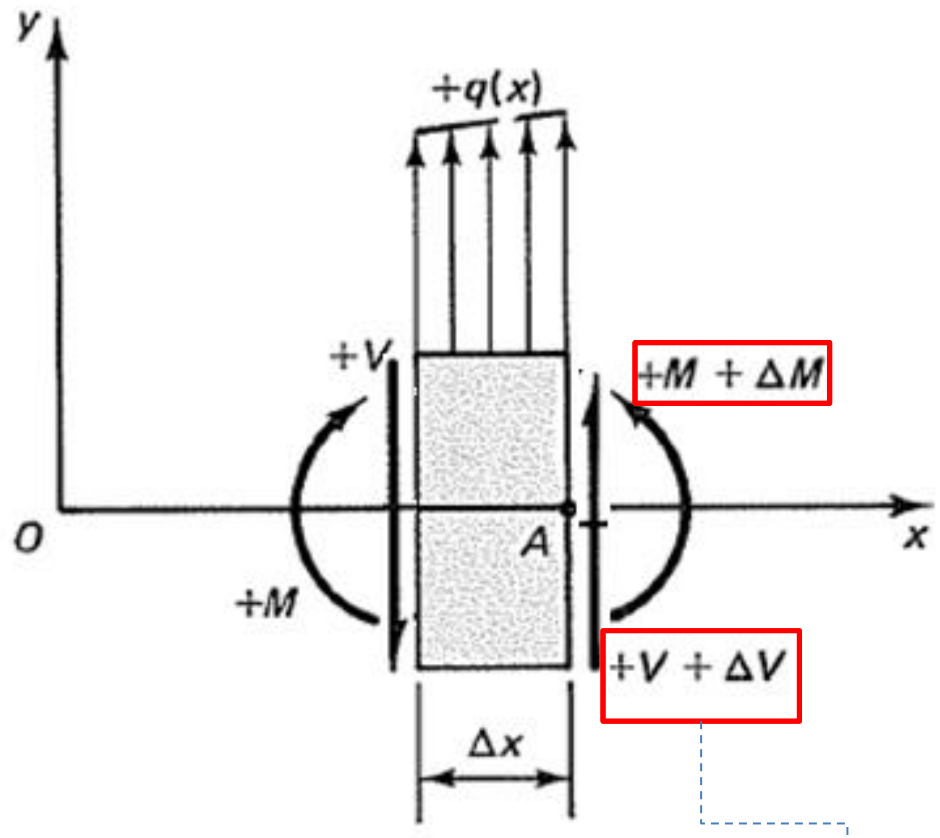
- Projeto/Análise dos elementos estruturais (Vigas)
  - Esforços Internos (F. Cortante, M. Fletor)





Método Somatório



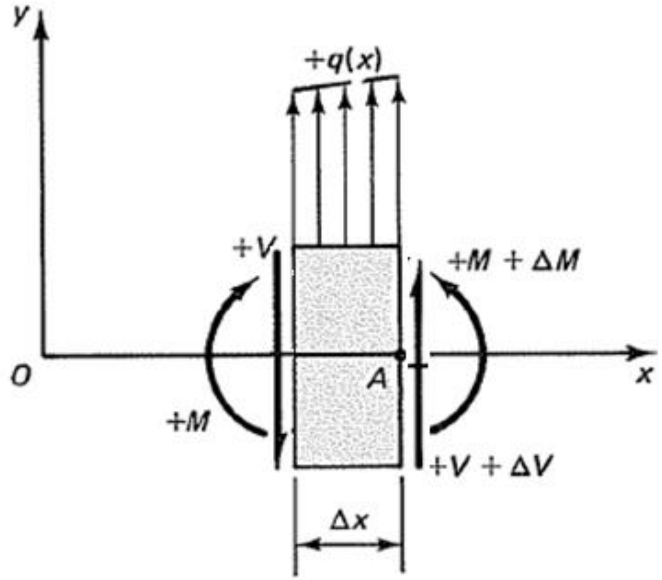


- Eq. Diferencial de Equilíbrio

Variação com a posição dx

Método Somatório

- Equilíbrio



Forças Verticais

$$\sum F_y = 0 \ (+) \uparrow$$

Momentos

$$\left(\sum M_z = 0\right)_A \ (+) \circlearrowleft$$

$$-V + q(x) \times \Delta x + (V + \Delta V) = 0$$

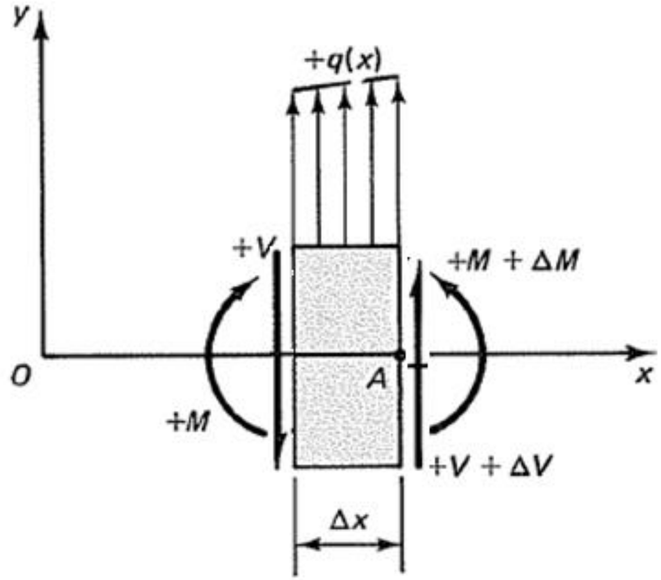
$$\frac{\Delta V}{\Delta x} = -q(x)$$

$$V \times \Delta x - M(x) + (M + \Delta M) - [q(x) \times \Delta x] \times \frac{\Delta x}{2} = 0$$

$$\frac{\Delta M}{\Delta x} = -V(x) + \frac{q(x) \times \Delta x}{2}$$

Método Somatório

- Equilíbrio



Forças Verticais

$$\sum F_y = 0 \;\; (+)\uparrow$$

$$\frac{\Delta V}{\Delta x} = -q(x)$$

$$\Delta\, x \rightarrow 0$$

$$\frac{dV}{dx} = -q(x)$$ **Eq. (1)**

Momentos

$$\left(\sum M_z = 0\right)_A \;\; (+)$$

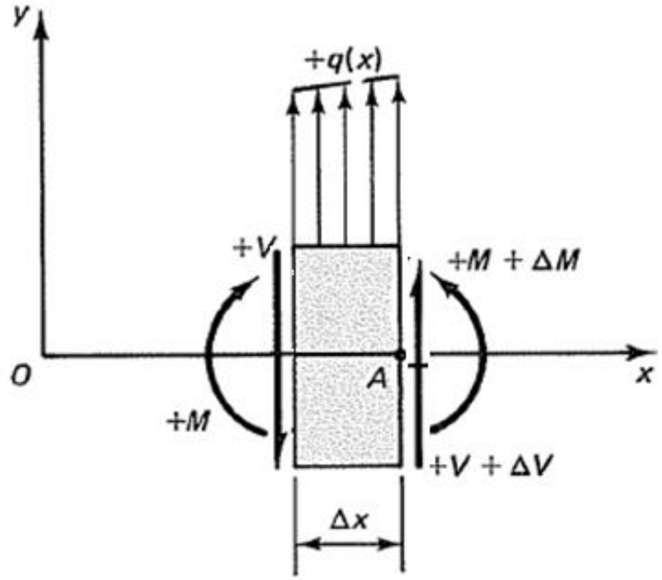
$$\frac{\Delta M}{\Delta x} = -V(x) + \frac{q(x)\times\Delta x}{2}$$

$$\Delta\, x \rightarrow 0$$

$$\frac{dM}{dx} = -V(x)$$ **Eq. (2)**

Método Somatório

- Equilíbrio



Forças Verticais

$$\sum F_y = 0 \ \ (+) \uparrow$$

Momentos

$$\left(\sum M_z = 0\right)_A \ (+)$$

$$\frac{d}{dx}[V(x)] = -q(x)$$

$$\frac{d}{dx}[M(x)] = -V(x)$$

$$\frac{d^2}{dx^2}[M(x)] = q(x)$$

Eq. (3)

DIAGRAMA DE FORÇA CORTANTE

$$\frac{d}{dx}[V(x)] \equiv \frac{dV}{dx} = -q(x)$$

$$dV = -q(x)\, dx$$

- Integral indefinida (valor depende de x)

$$\int dV = -\int_0^x q(x)\, dx + C_1$$

$$V(x) = -\int_0^x q(x)\, dx + C_1$$ **Eq. (4)**

DIAGRAMA DE MOMENTO FLETOR

$$\frac{d}{dx}[M(x)] \equiv \frac{dM}{dx} = -V(x)$$

$$\text{dM} = -V(x)\,dx$$

- Integral indefinida (valor depende de x)

$$\int \text{d}M = -\int_0^x V(x)\,dx + C_1$$

$$M(x) = -\int_0^x V(x)\,dx + C_2 \qquad \textbf{Eq. (5)}$$

Resumo

$$V(x) = -\int_0^x q(x)\,dx + C_1 \longrightarrow x = 0 \longrightarrow V(0) = C_1$$

$$\boxed{\frac{dV}{dx} = -q(x)}$$

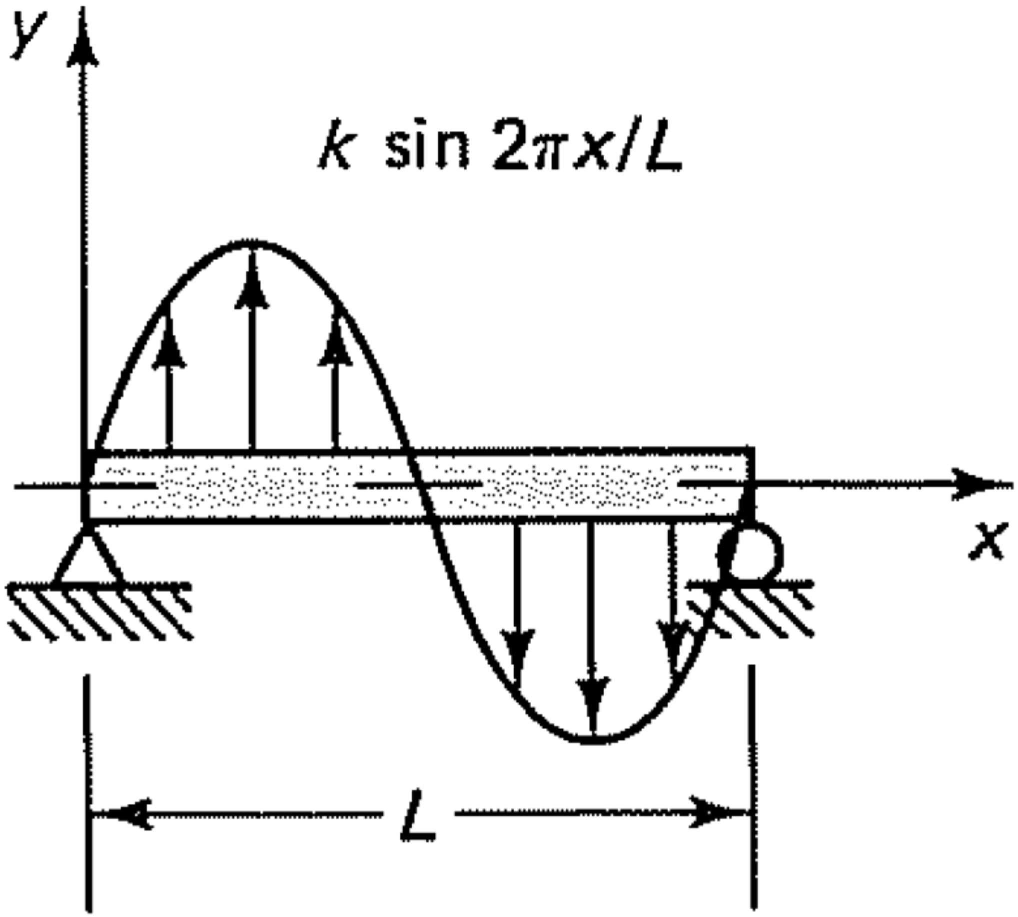
Condição de contorno

$$M(x) = -\int_0^x V(x)\,dx + C_2 \longrightarrow x = 0 \longrightarrow \mathrm{M}(0) = C_2$$

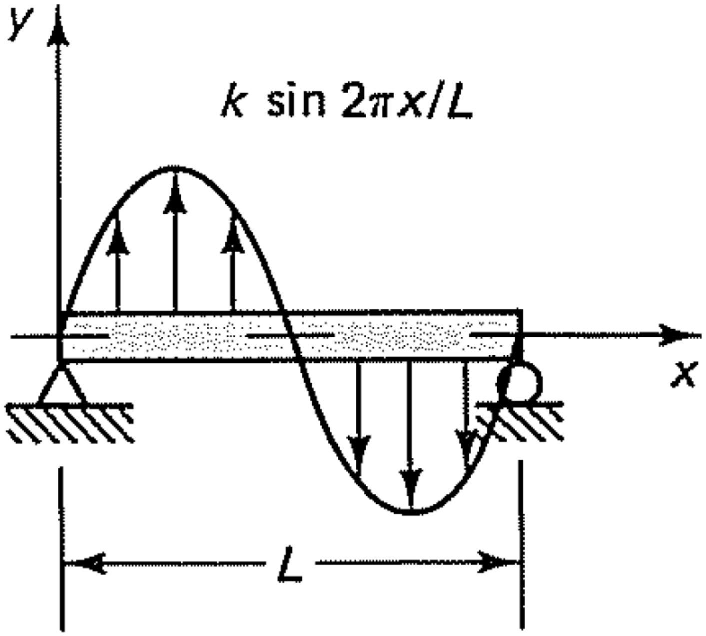
$$\boxed{\frac{dM}{dx} = -V(x)}$$

Exemplo

- **Determine os diagramas V(x) e M(x) para a viga mostrada**



Exemplo



- 2 caminhos

✓ Calcular reações
✓ Integrar q(x)
✓ Integrar V(x)

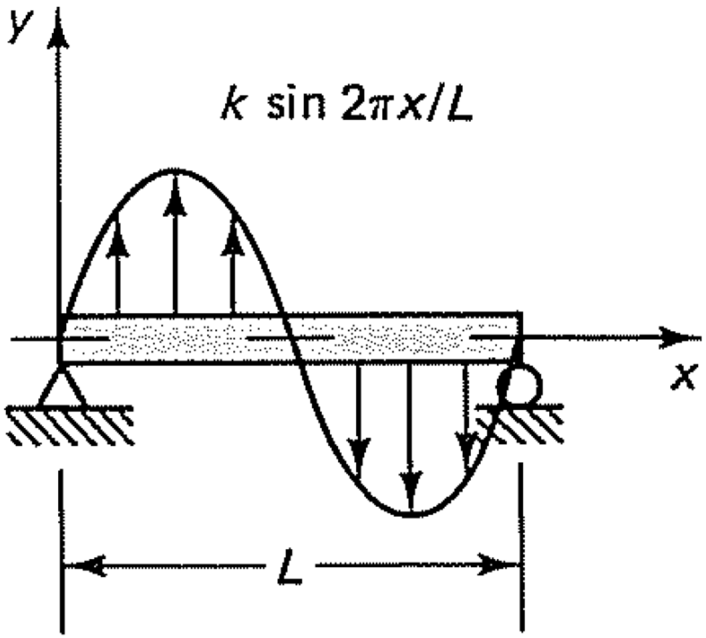
(Homework)

- ***27/03/2019***

✓ Integrar diretamente Eq. 3

$$\frac{d^2}{dx^2}[M(x)] = q(x)$$

Exemplo



$$\frac{d^2}{dx^2}[M(x)] = q(x)$$

$$\frac{d}{dx}M(x) = \int_0^x k\sin\left(2\pi\tfrac{x}{L}\right)\,dx + C_1$$

$$\frac{d}{dx}M(x) = -\frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right)\right]_0^x + C_1$$

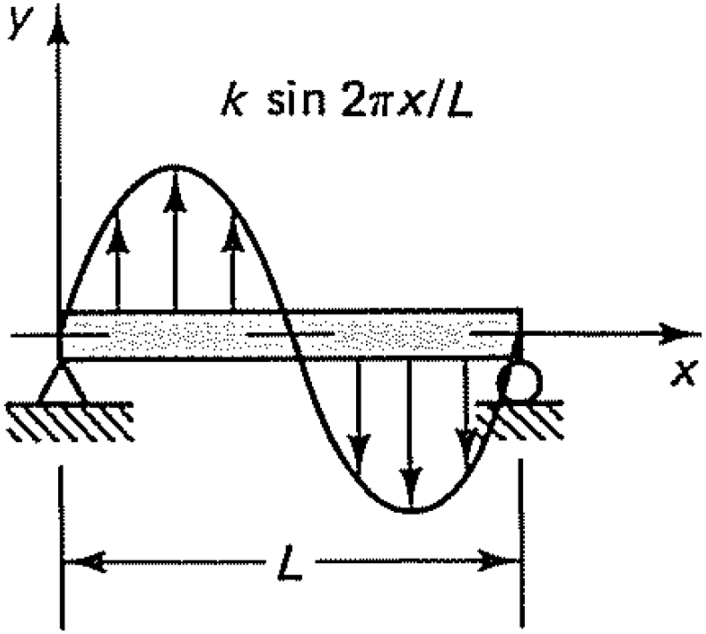
$$\frac{dM(x)}{dx} = -\frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right) - 1\right] + C_1$$

$$M(x) = \int_0^x -\left\{\frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right) - 1\right] + C_1\right\}dx + C_2$$

- $M(0) = 0$
- $M(L) = 0$

$$M(x) = -\frac{kL^2}{4\pi^2}\left[\sin\left(2\pi\tfrac{x}{L}\right)\right]_0^x + \frac{kL}{2\pi}[x]_0^x + C_1 x + C_2$$

Exemplo



$$\frac{d^2}{dx^2}[M(x)] = q(x)$$

$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{x}{L}\right)+\frac{kL}{2\pi}x+C_1\text{x}+C_2$$

- $\text{M}(0) = 0$

$$M(0) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{0}{L}\right)+\frac{kL}{2\pi}0+C_1 0+C_2\text{=0}$$

$$C_2\text{=0}$$

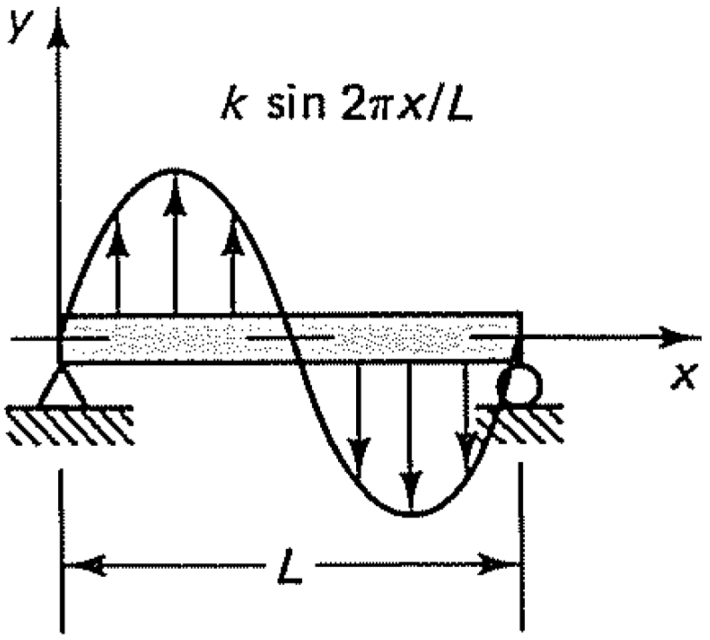
- $\text{M}(L) = 0$

$$M(0) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{L}{L}\right)+\frac{kL}{2\pi}\text{L}+C_1\text{L=0}$$

$$-\frac{kL^2}{4\pi^2}\sin(2\pi)+\frac{kL}{2\pi}\text{L}+C_1\text{L=0}$$

$$\boldsymbol{C_1= -\frac{kL}{2\pi}}$$

Exemplo



$$\frac{d^2}{dx^2}[M(x)] = q(x)$$

$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{x}{L}\right)+\frac{kL}{2\pi}x+C_1x+C_2$$

↓

- $M(0) = 0$

$$M(0) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{0}{L}\right)+\frac{kL}{2\pi}0+C_1 0+C_2=0$$

↓

$$C_2=0$$

- $M(L) = 0$

↓

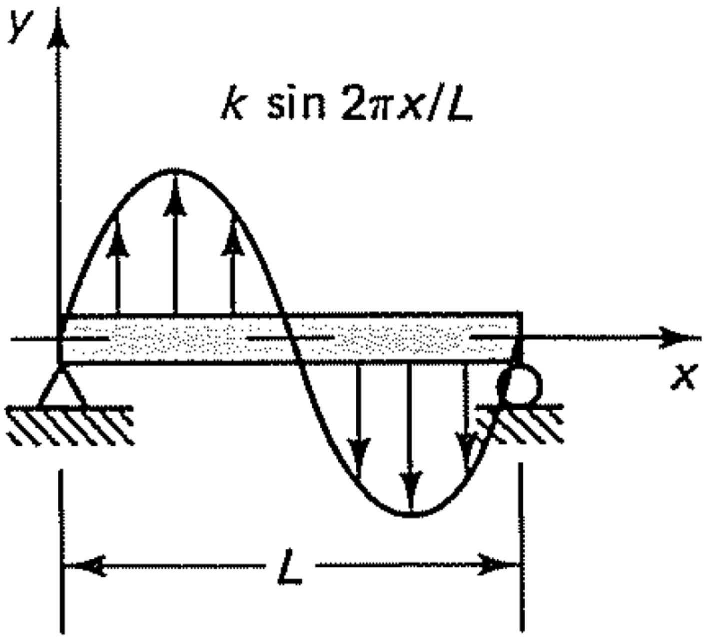
$$M(0) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{L}{L}\right)+\frac{kL}{2\pi}L+C_1L=0$$

$$-\frac{kL^2}{4\pi^2}\sin(2\pi)+\frac{kL}{2\pi}L+C_1L=0$$

↓

$$C_1= -\frac{kL}{2\pi}$$

Exemplo



$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{x}{L}\right) + \frac{kL}{2\pi}x + -\frac{\mathbf{kL}}{\mathbf{2\pi}_1}\, x + 0$$

$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{x}{L}\right)$$

Amplitude!

V(x) =?

$$\frac{d}{dx}M(x) = -\frac{kL}{2\pi}\left[\cos\left(2\pi\frac{x}{L}\right)\right]_0^x + C_1$$

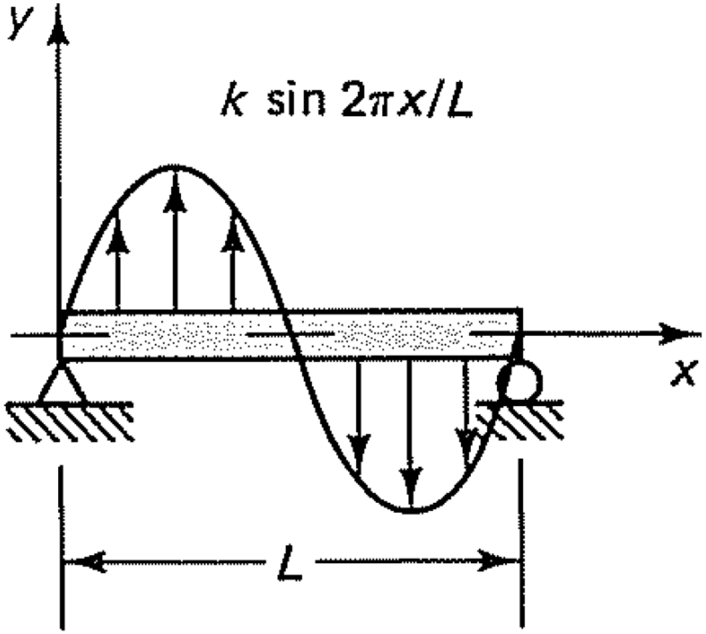
$$V(x) = \frac{kL}{2\pi}\left[\cos\left(2\pi\frac{x}{L}\right)\right]_0^x + C_1$$

$$V(x) = \frac{kL}{2\pi}\left[\cos\left(2\pi\frac{x}{L}\right) - 1\right] + \frac{kL}{2\pi}$$

$$V(x) = \frac{kL}{2\pi}\cos\left(2\pi\frac{x}{L}\right)$$

Amplitude!

Exemplo



$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\tfrac{x}{L}\right) + \frac{kL}{2\pi}x + -\frac{\mathbf{kL}}{\mathbf{2\pi}_1}\,\mathrm{x}+0$$

$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\tfrac{x}{L}\right)$$

Amplitude!

V(x) =?

$$\frac{d}{dx}M(x) = -\frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right)\right]_0^x + C_1$$

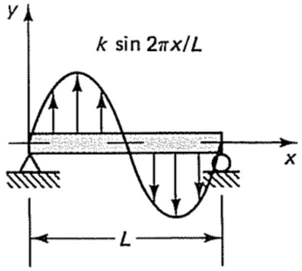
$$V(x) = \frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right)\right]_0^x + C_1$$

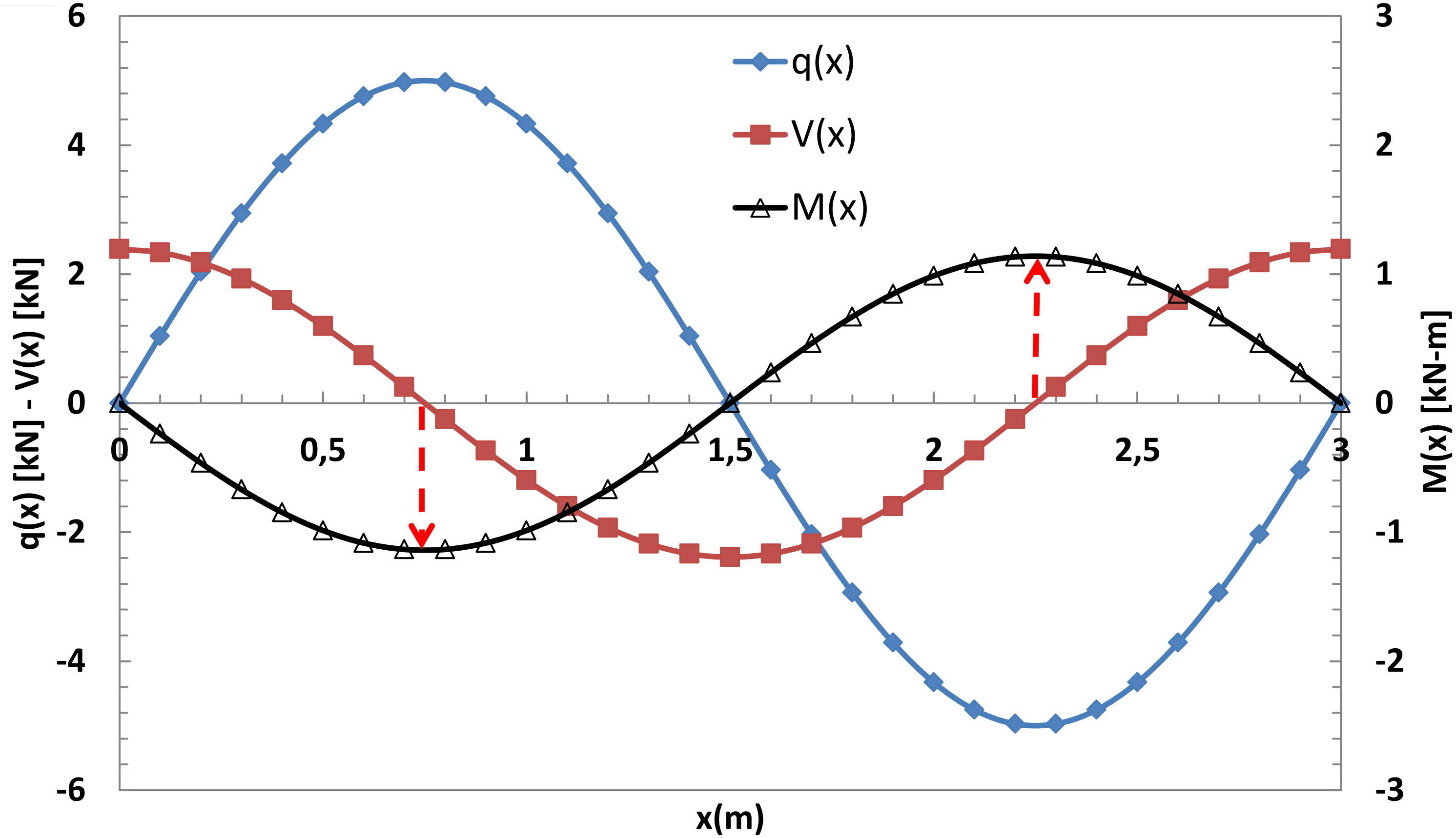
$$V(x) = \frac{kL}{2\pi}\left[\cos\left(2\pi\tfrac{x}{L}\right) - 1\right] + \frac{kL}{2\pi}$$

$$V(x) = \frac{kL}{2\pi}\cos\left(2\pi\tfrac{x}{L}\right)$$

Amplitude!

Exemplo





$$q(x) = \mathrm{k}\sin\left(2\pi\frac{x}{L}\right)$$

$$V(x) = \frac{kL}{2\pi}\cos\left(2\pi\frac{x}{L}\right)$$

$$M(x) = -\frac{kL^2}{4\pi^2}\sin\left(2\pi\frac{x}{L}\right)$$

Conclusões

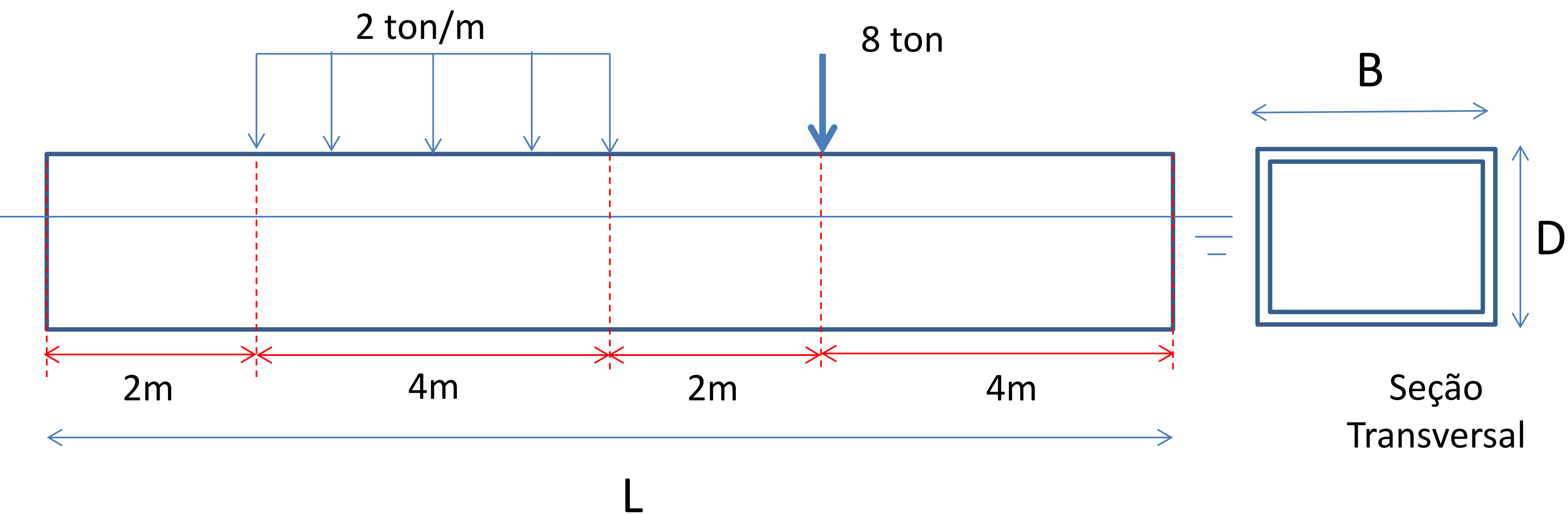
- Inclinação da curva V(x) é igual ao negativo da curva de carga q(x)
- Inclinação da curva M(x) é igual ao negativo da curva de V(x)

$$V(x) = -\int_0^x q(x)\,dx + C_1 \qquad M(x) = -\int_0^x V(x)\,dx + C_2$$

$$M(x) = \iint_0^x q(x)\,dx + C_1 x + C_2$$

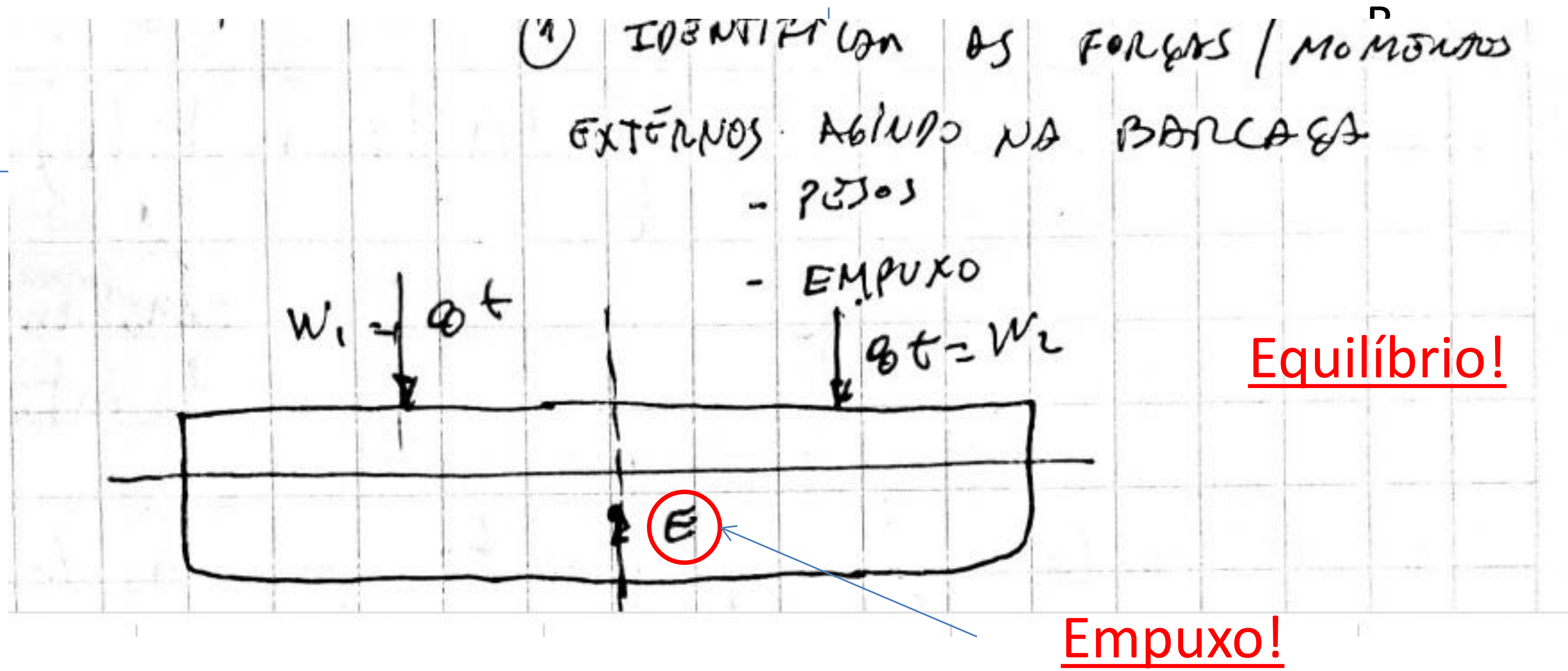
Exemplo 2

- **Uma barcaça é carregada como mostra a figura. Determine os diagramas de força cortante V(x) e momento fletor M(x)**



Exemplo 2

Solução

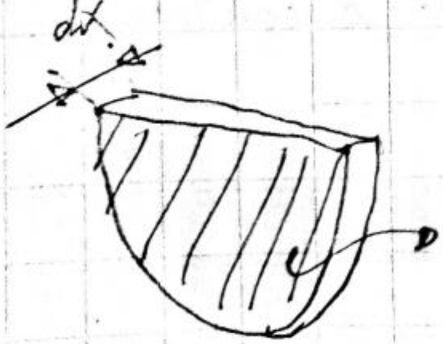


Exemplo 2

EQUILÍBRIO

$W_1 + W_2 = E \Rightarrow E = 16$ ton.

Distribuição da carga de empuxo:



barcaça → seção constante → distribuição constante

e(x) : empuxo/comprimento

↓

$e(x) = A(x) \cdot dx \cdot \gamma_{água}$

↳ peso específico da água.

A(x): constante.

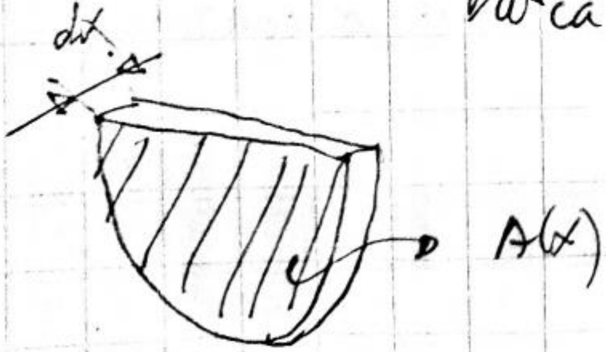
e(x): constante $= \frac{16}{12} \frac{ton}{m}$

Exemplo 2

EQUILÍBRIO

$W_1 + W_2 = E \Rightarrow E = 16$ ton.

Distribuição da carga de empuxo:



barcaça → seção constante → distribuição constante

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$e(x) = A(x) \cdot dx \cdot \gamma_{água}$

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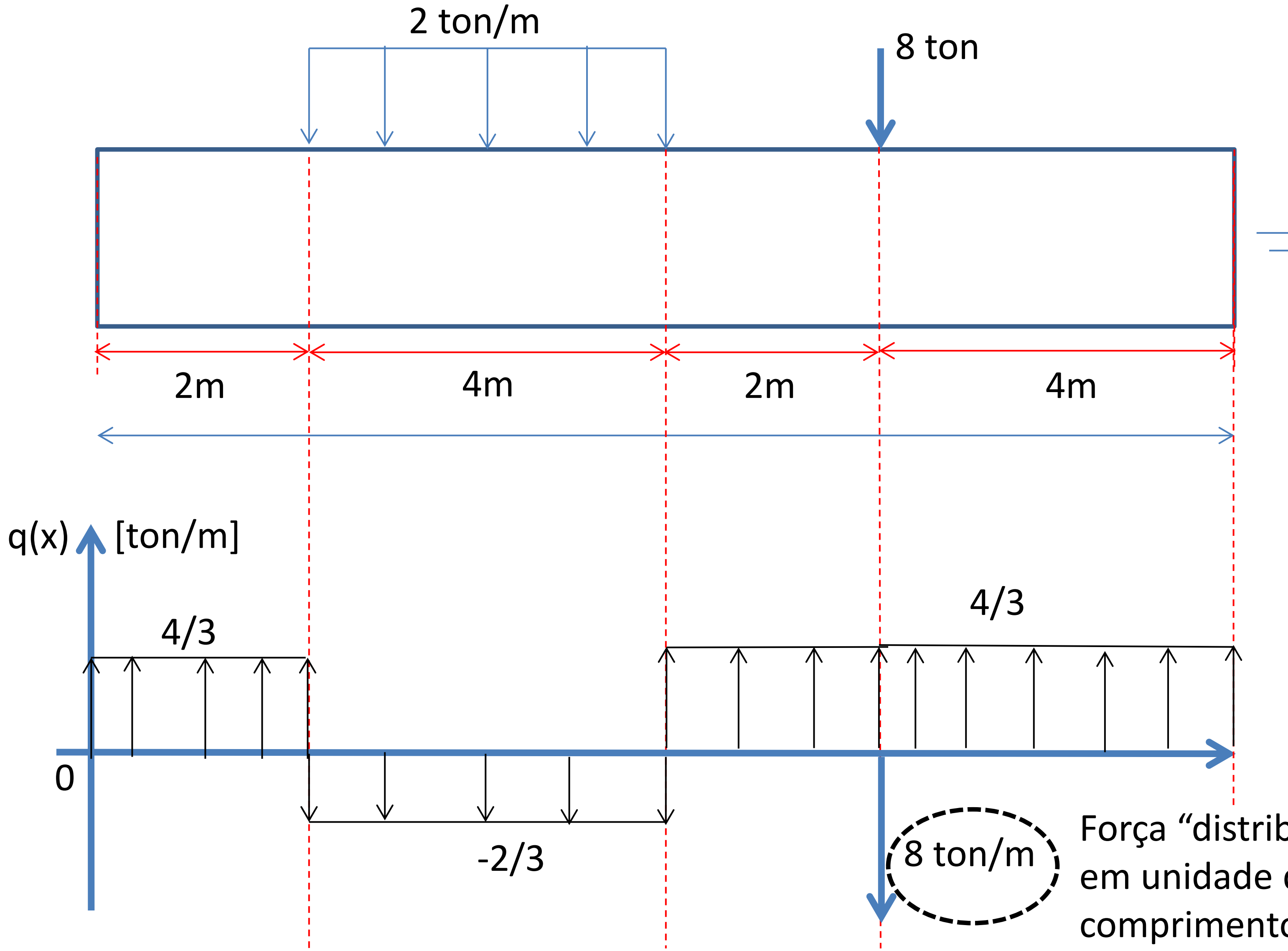
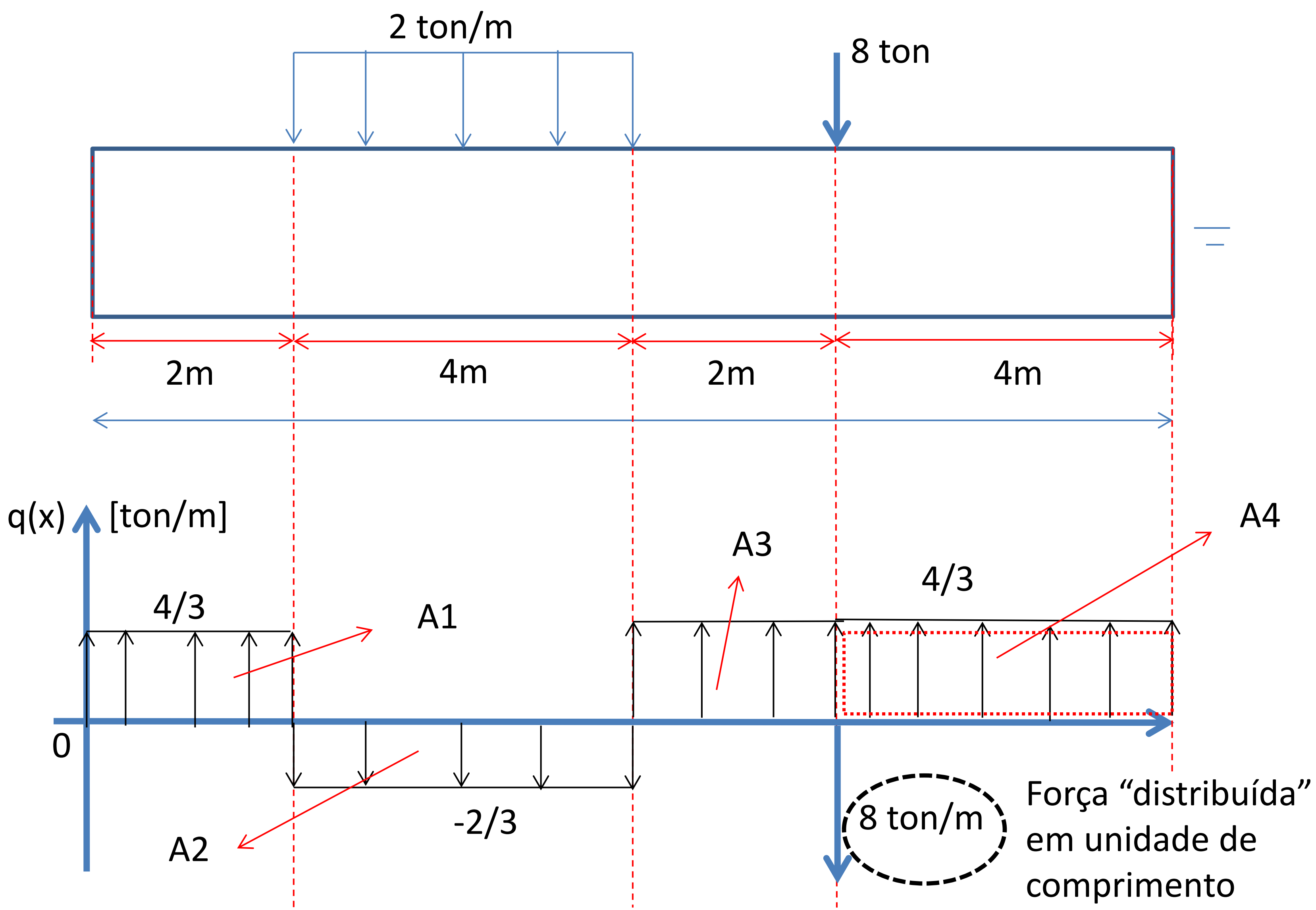
Exemplo 2



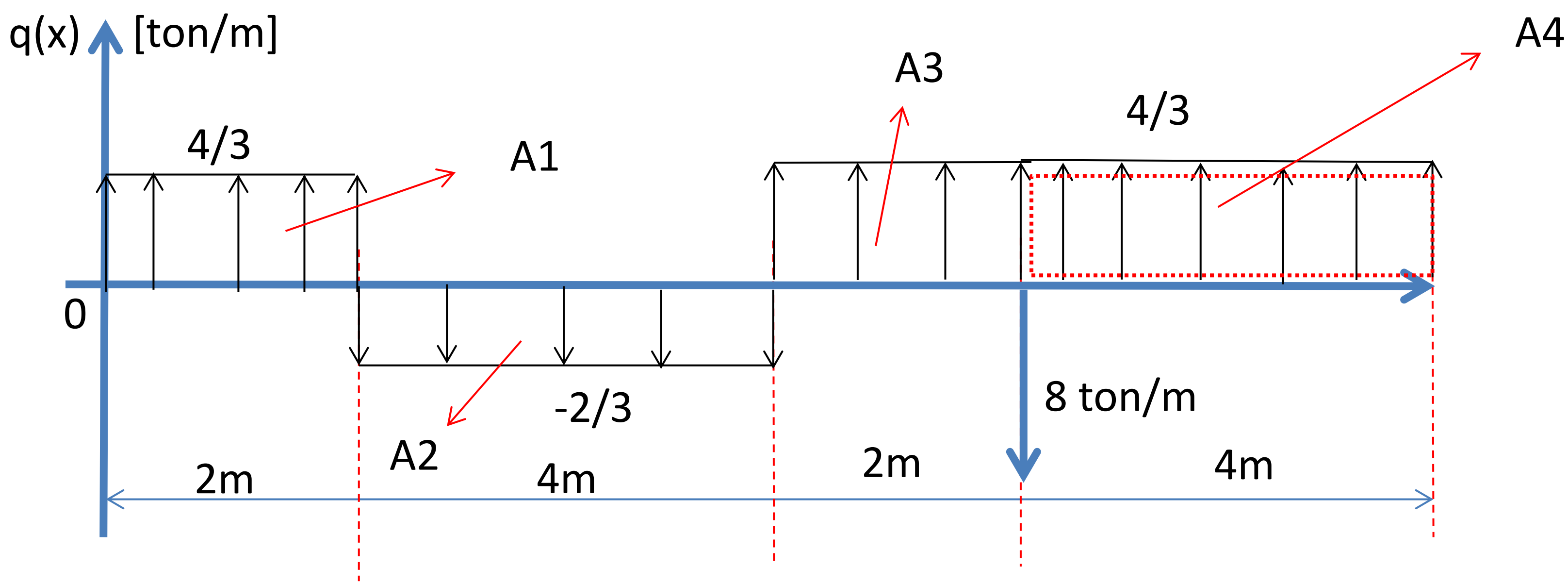
Força “distribuída” em unidade de comprimento

Exemplo 2



2 ton/m
8 ton
2m
4m
2m
4m
q(x)
[ton/m]
4/3
A1
A3
4/3
A4
0
-2/3
A2
8 ton/m
Força “distribuída” em unidade de comprimento

Exemplo 2

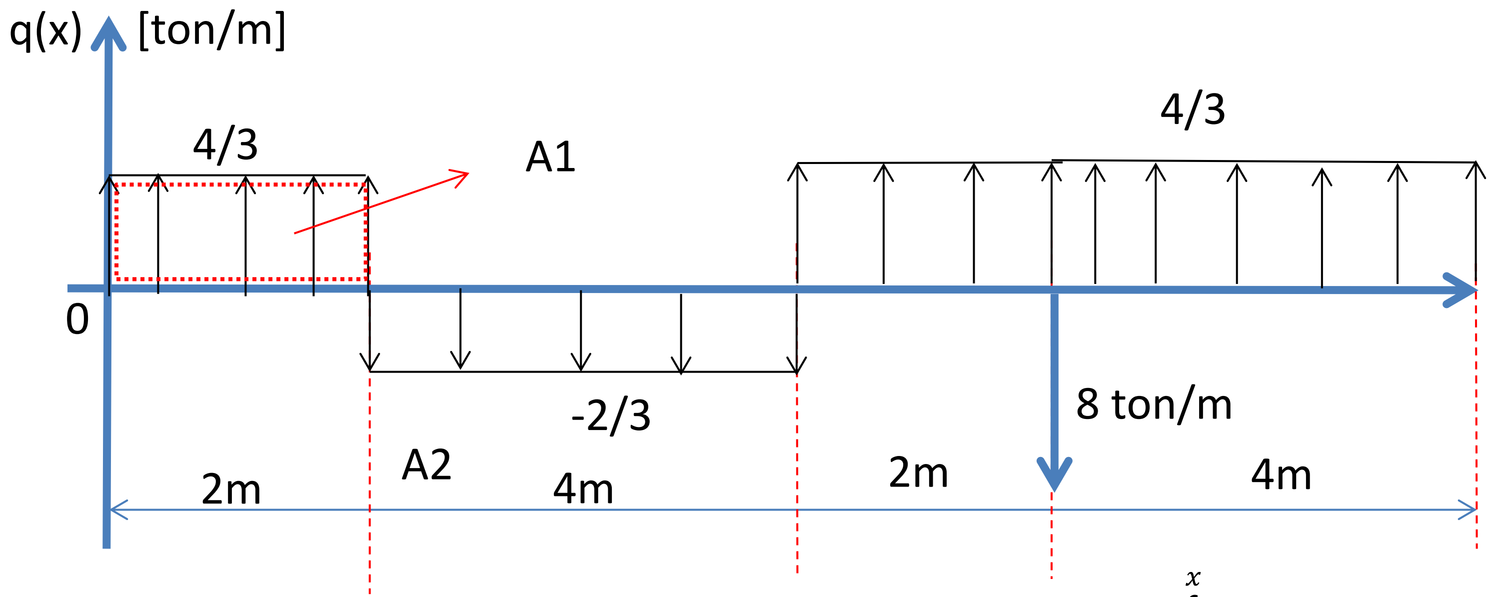


Força Cortante

$$V(x) = -\int_0^x q(x)\, dx + C_1$$

<u>Área sob a curva q(x)</u>

Exemplo 2



Força Cortante

$$V(x) = -\int_0^x q(x)\,dx + C_1$$

- x=0

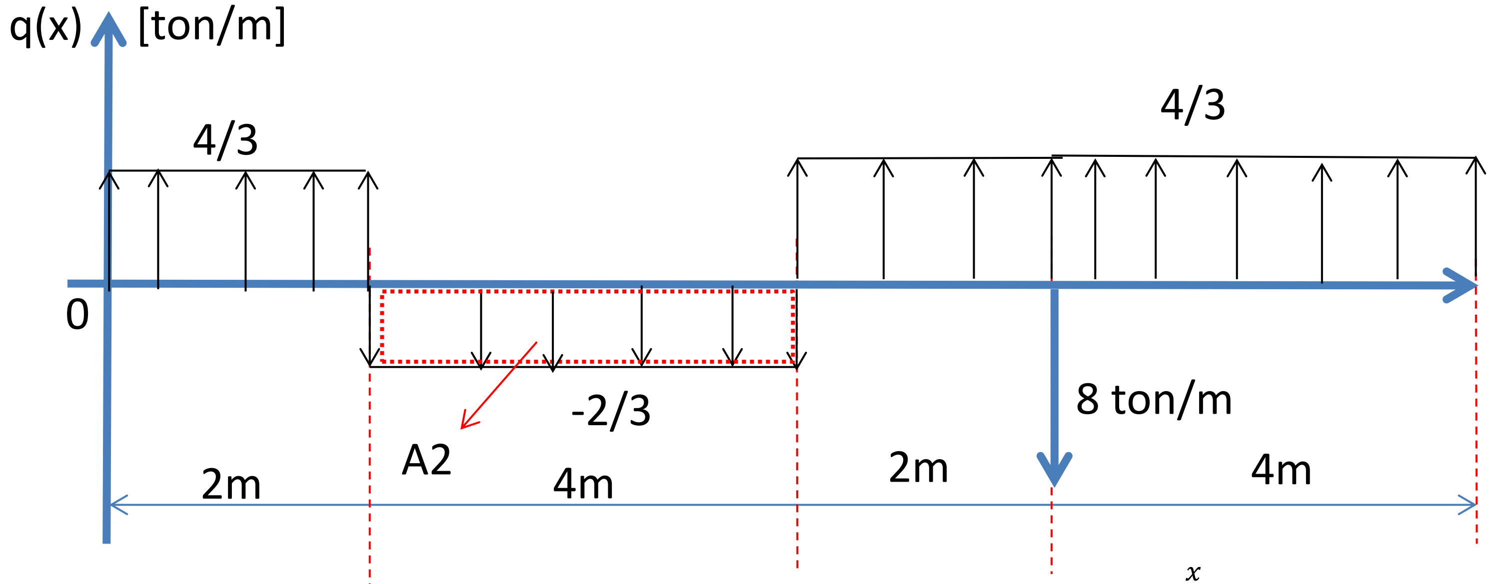
$$V(0) = 0$$ “Viga” Livre-Livre nas extremidades

- x=2

$$V(x) = -A1 + V(0)$$

$$V(2) = -\left[\frac{4}{3} \times 2\right] + 0 \rightarrow V(2) = -\frac{8}{3}$$ [ton]

Exemplo 2



Força Cortante

$$V(x) = -\int_0^x q(x)\,dx + C_1$$

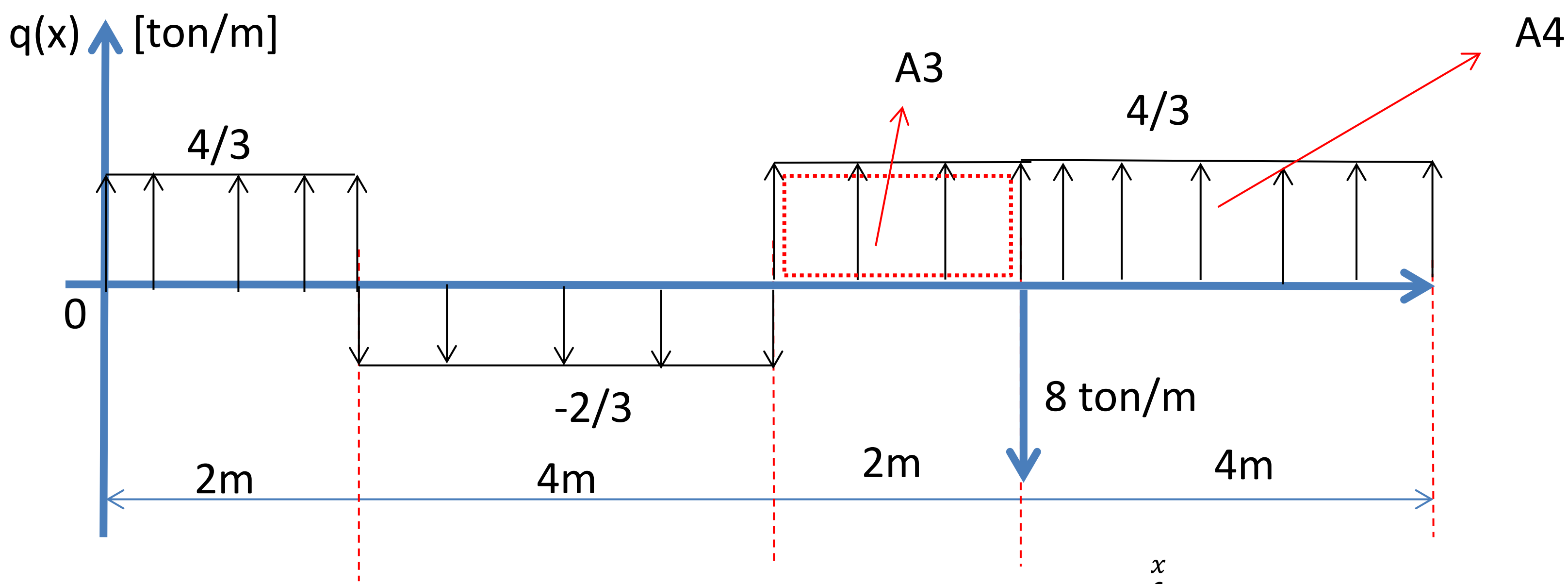
- x=6

$$V(6) = -A2 + V(2)$$

$$V(6) = -\left[-\frac{2}{3} \times 4\right] - \frac{8}{3}$$

$$V(6) = 0 \quad \text{[ton]}$$

Exemplo 2



Força Cortante

$$V(x) = -\int_0^x q(x)\,dx + C_1$$

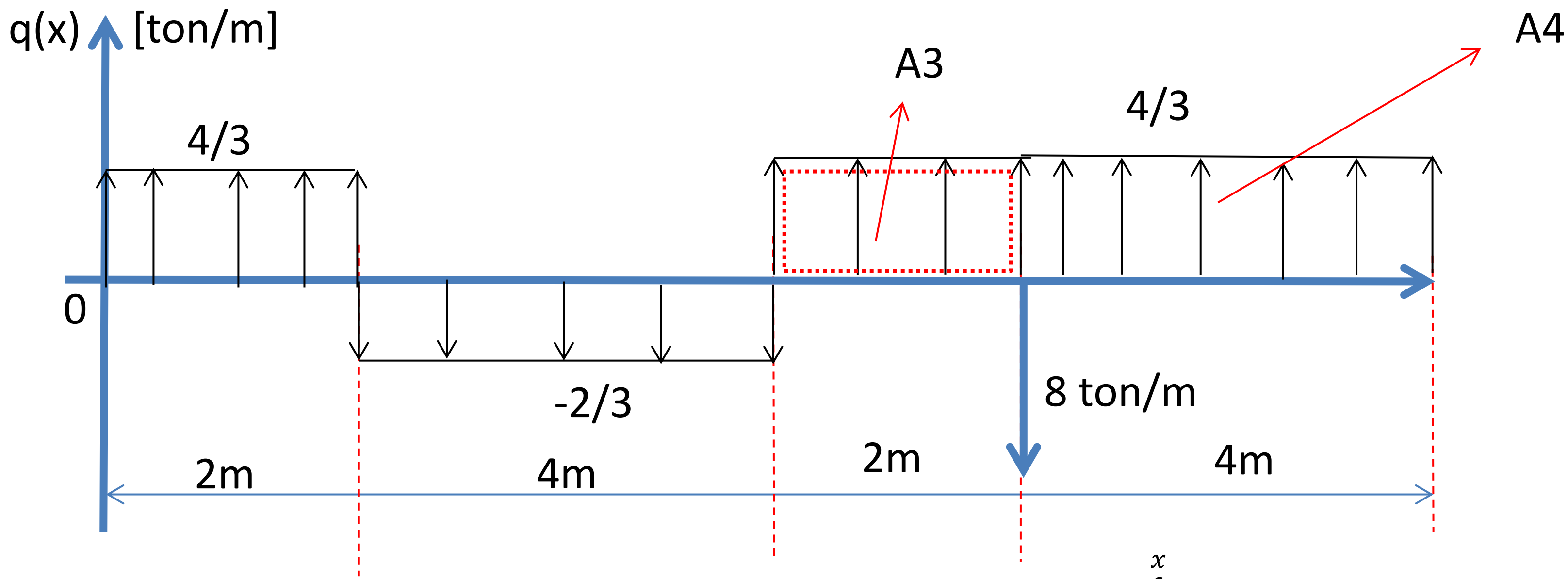
- $x = 8^-$

$$V(8^-) = -A3 + V(6)$$

$$V(8^-) = -\left[\frac{4}{3} \times 2\right] - 0$$

$$V(8^-) = -\frac{8}{3} \quad \text{[ton]}$$

Exemplo 2



Força Cortante

$$V(x) = -\int_0^x q(x)\,dx + C_1$$

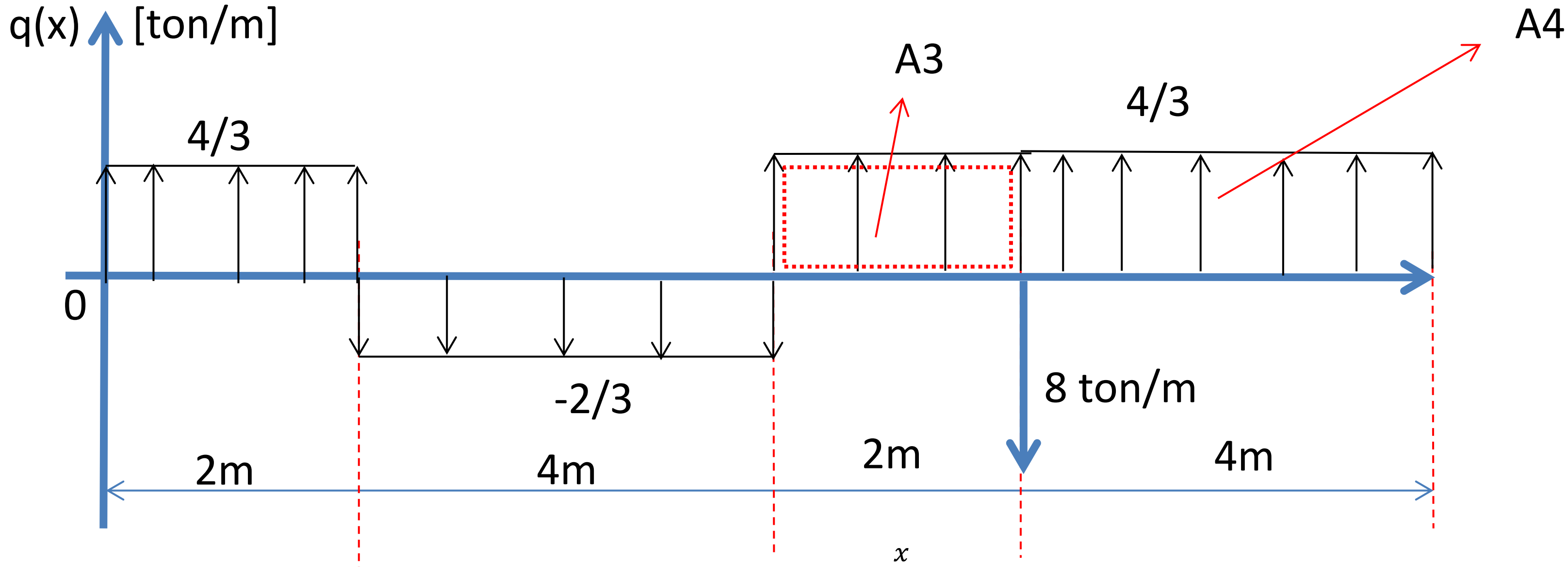
- $x = 8^-$

$$V(8^-) = -A3 + V(6)$$

$$V(8^-) = -\left[\frac{4}{3} \times 2\right] - 0$$

$$V(8^-) = -\frac{8}{3} \quad \text{[ton]}$$

Exemplo 2



Força Cortante

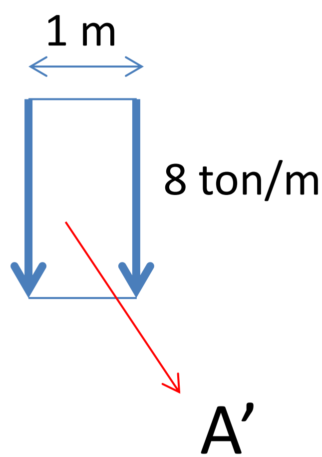
$$V(x) = -\int_0^x q(x)\,dx + C_1$$

- $x = 8$

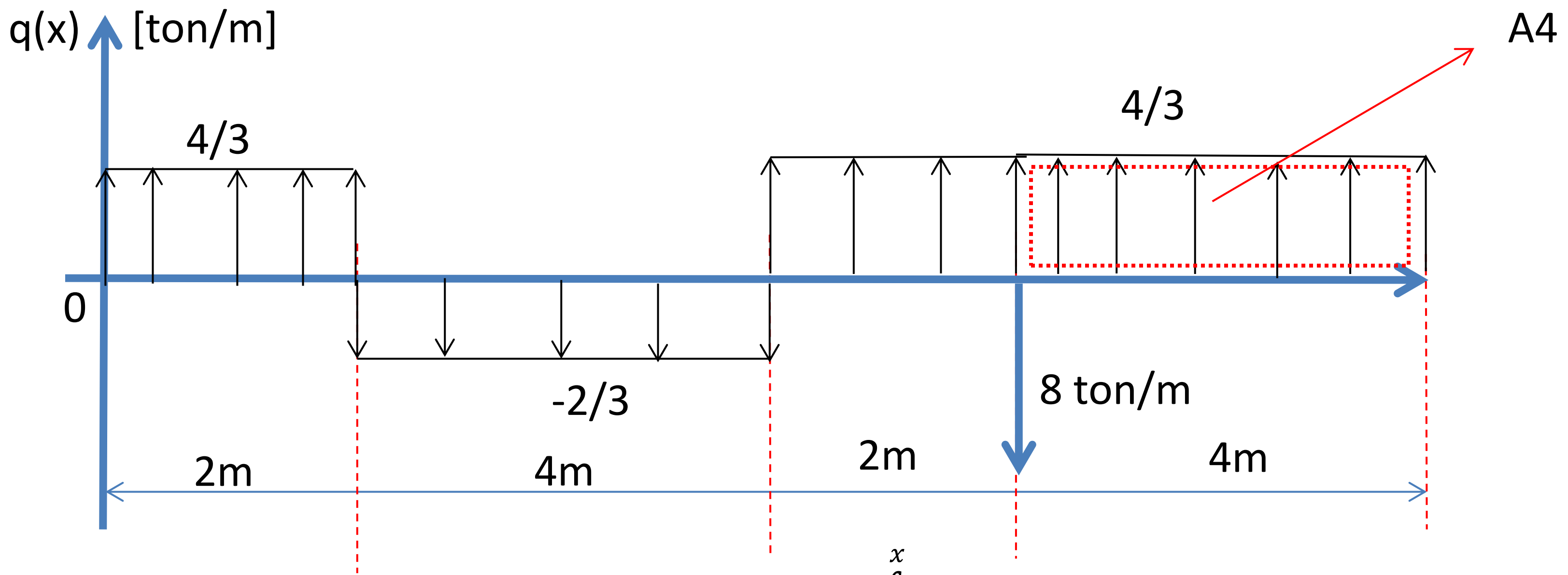
$$V(8) = -A' + V(8^-)$$

$$V(8) = -[-8 \times 1] - \frac{8}{3}$$

$$V(8) = \frac{16}{3} \quad \text{[ton]}$$



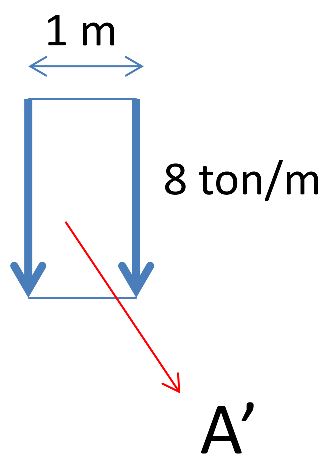
Exemplo 2



$$V(x) = -\int_0^x q(x)\,dx + C_1$$

Força Cortante

- x =12

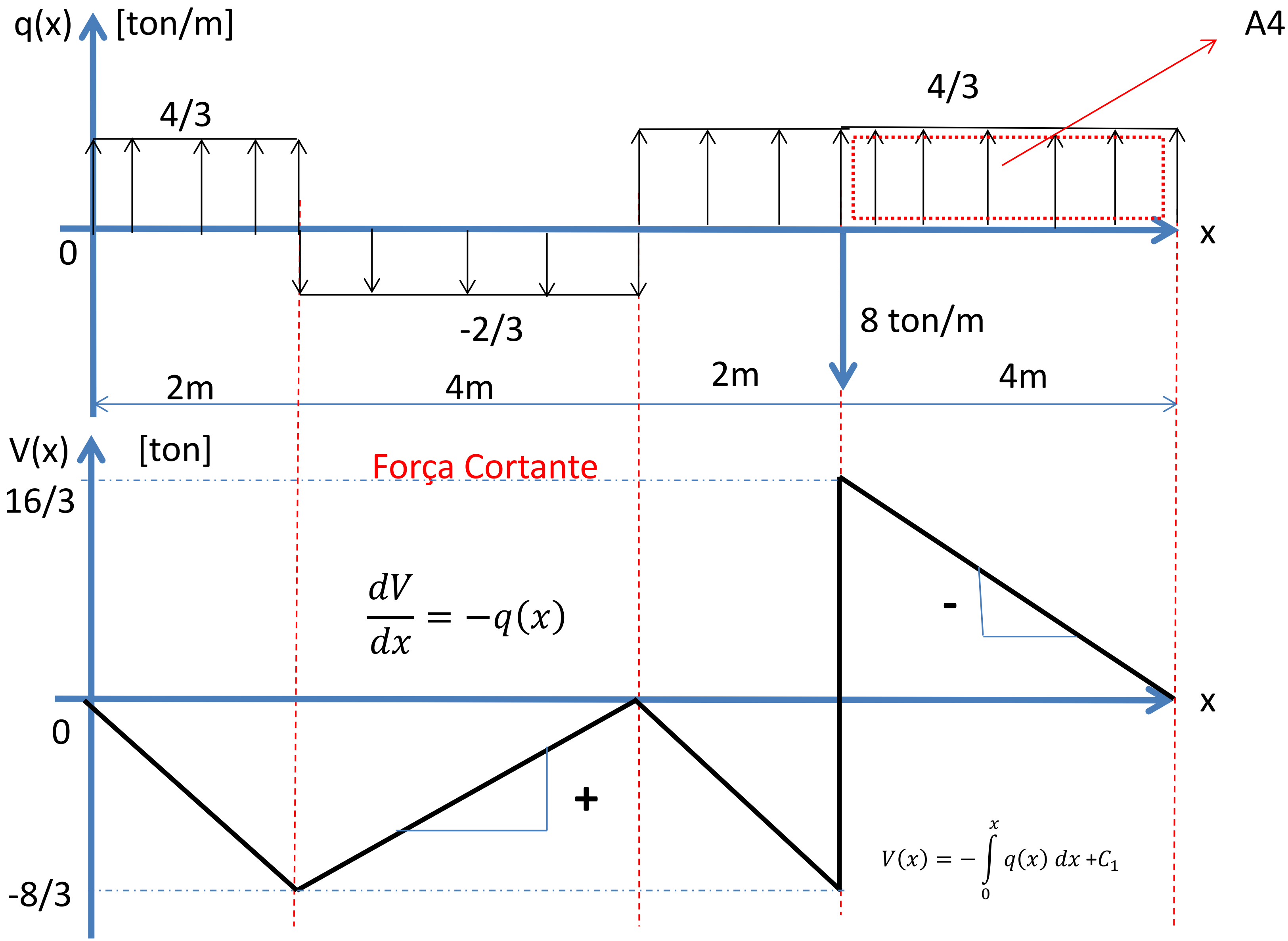


$$V(12) = -A4 + V(8)$$

$$V(12) = -\left[\frac{4}{3} \times 4\right] + \frac{16}{3}$$

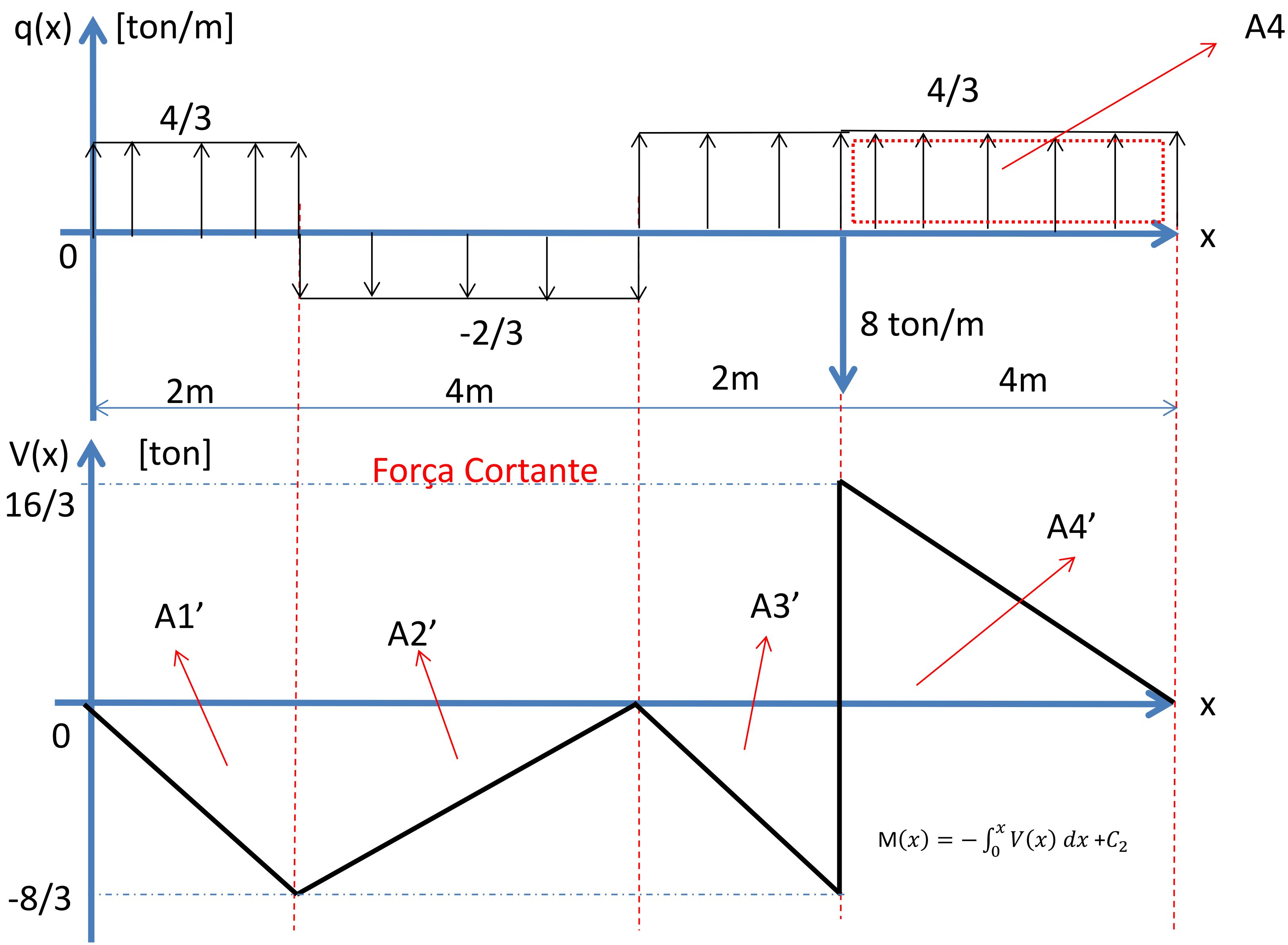
$$V(12) = 0 \quad \text{[ton]}$$

Exemplo 2



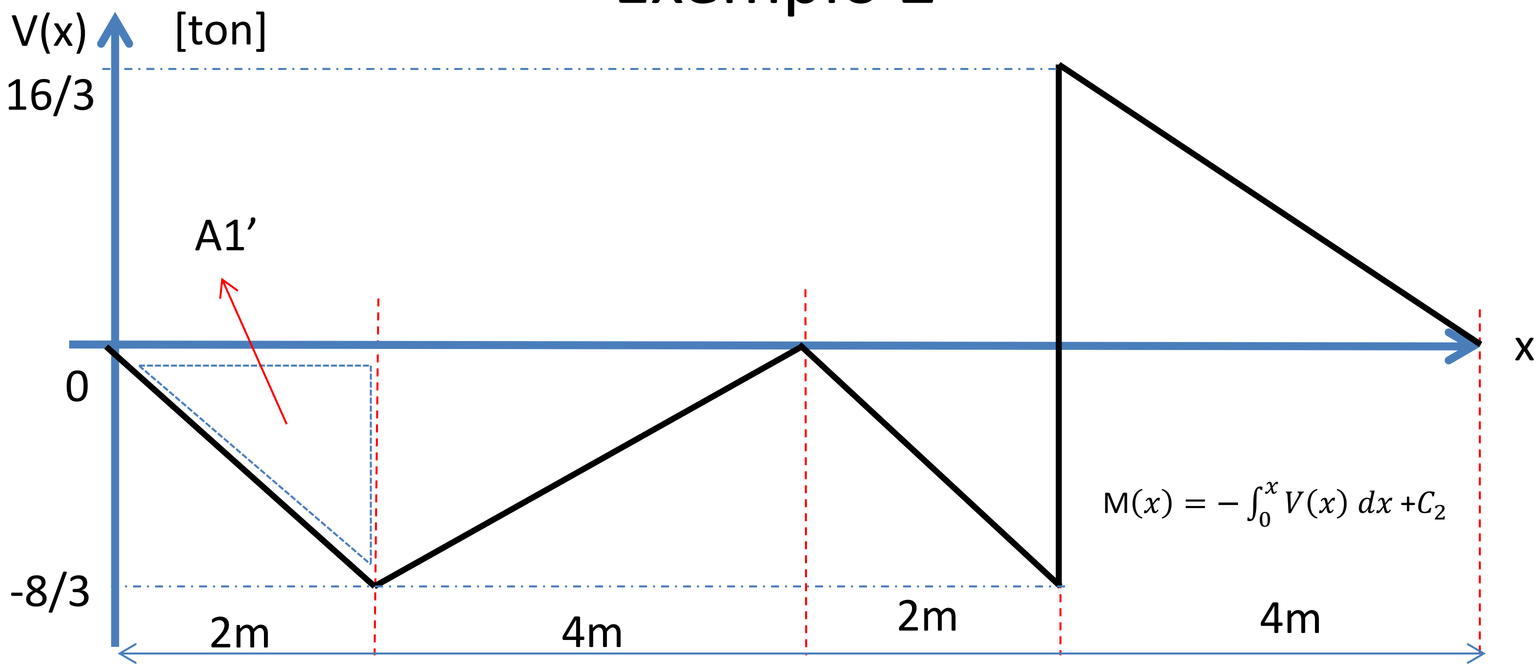
q(x)
[ton/m]
A4
4/3
4/3
0
x
-2/3
8 ton/m
2m
4m
2m
4m
V(x)
[ton]
Força Cortante
16/3
$\frac{dV}{dx} = -q(x)$
-
+
0
x
-8/3
$V(x) = -\int_0^x q(x)\,dx + C_1$

Exemplo 2



q(x) [ton/m]
4/3
4/3
A4
0
x
-2/3
8 ton/m
2m
4m
2m
4m
V(x) [ton]
Força Cortante
16/3
A4’
A1’
A2’
A3’
0
x
-8/3
$M(x) = -\int_0^x V(x)\,dx + C_2$

Exemplo 2



$$M(x) = -\int_0^x V(x)\,dx + C_2$$

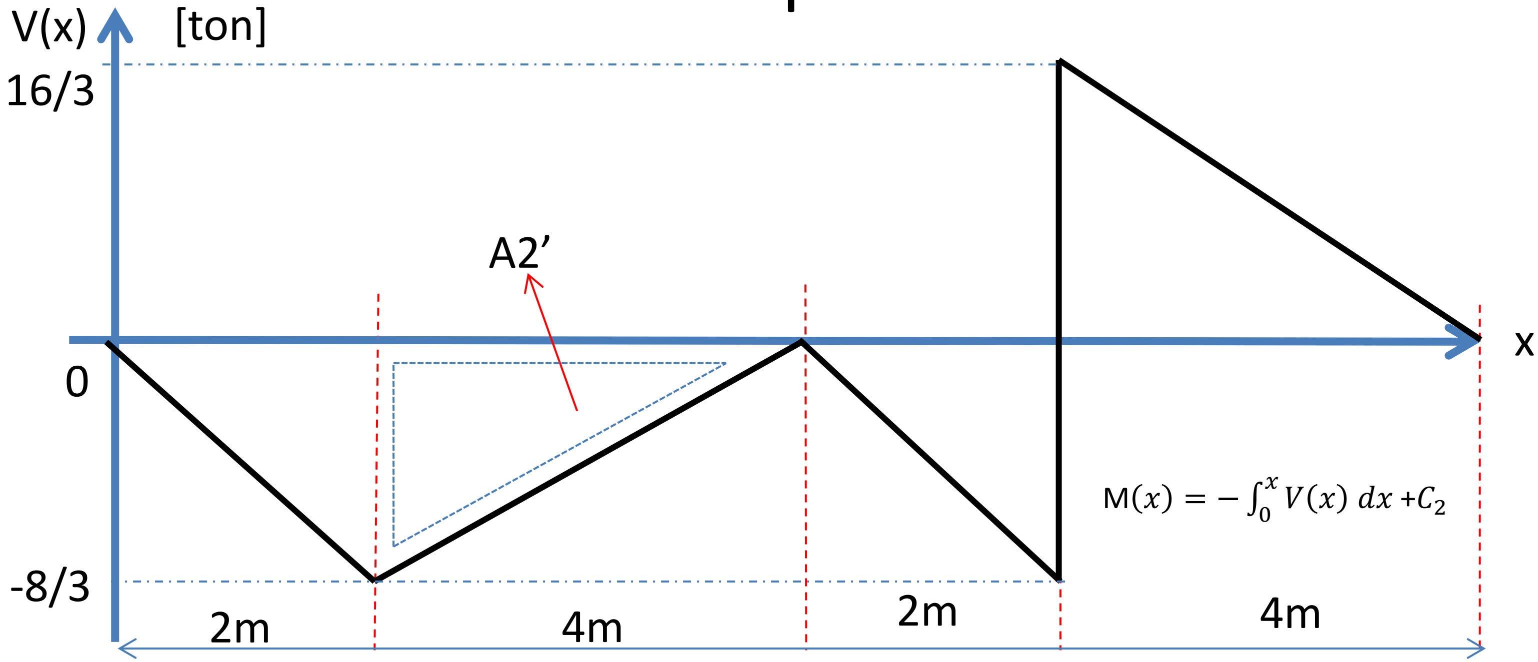
Momento Fletor

- x=0 → $M(0) = 0$ Why?
- x=2 → $M(2) = -A1' + M(0)$

$$M(2) = -\left[\frac{-\frac{8}{3}\times 2}{2}\right] + 0$$

$$M(2) = \frac{8}{3}$$ [ton -m]

Exemplo 2



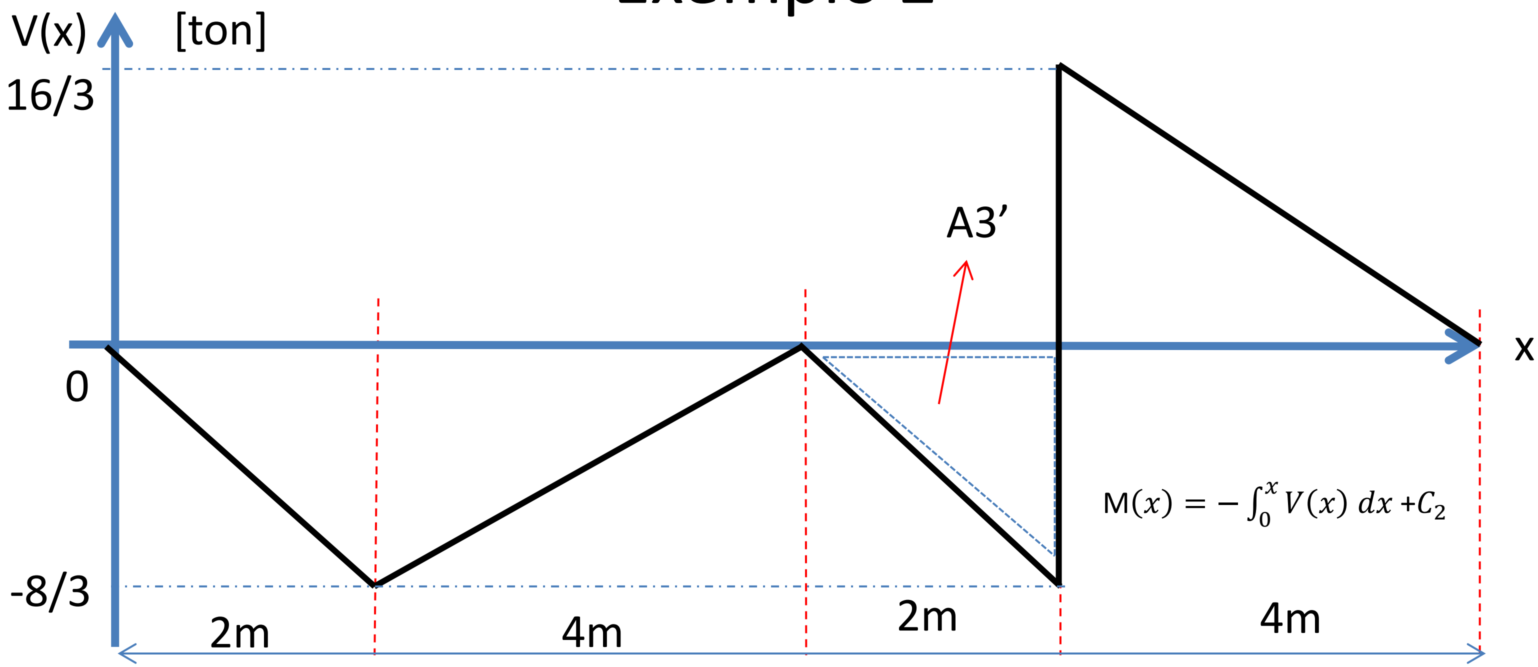
Momento Fletor

- x=6 → $M(6) = -A2' + M(2)$

$$M(6) = -\left[\frac{-\frac{8}{3}\times 4}{2}\right] + \frac{8}{3}$$

$$M(6) = \frac{24}{3} = 8 \quad \text{[ton -m]}$$

Exemplo 2



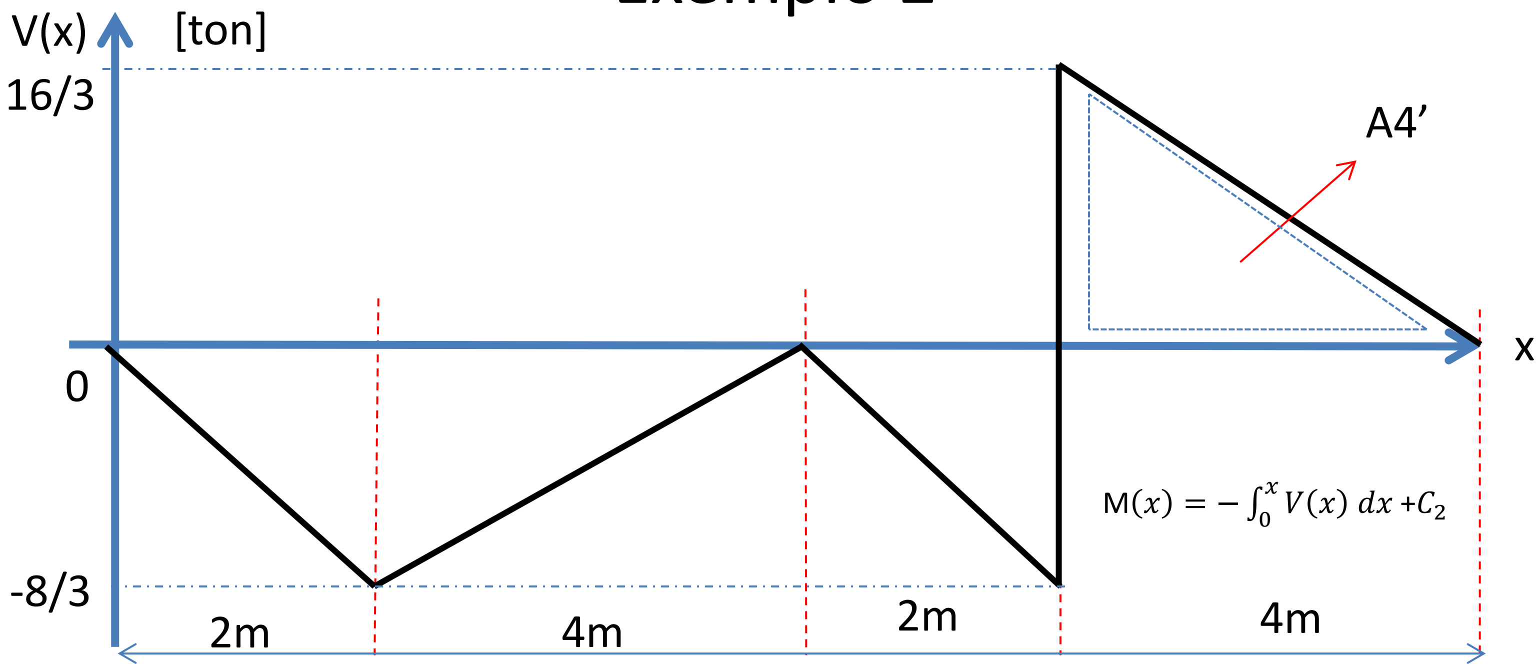
Momento Fletor

- x=8 → $M(8) = -A3' + M(6)$

$$M(8) = -\left[\frac{-\frac{8}{3}\times 2}{2}\right] + 8$$

$$M(8) = \frac{32}{3} \quad \text{[ton -m]}$$

Exemplo 2



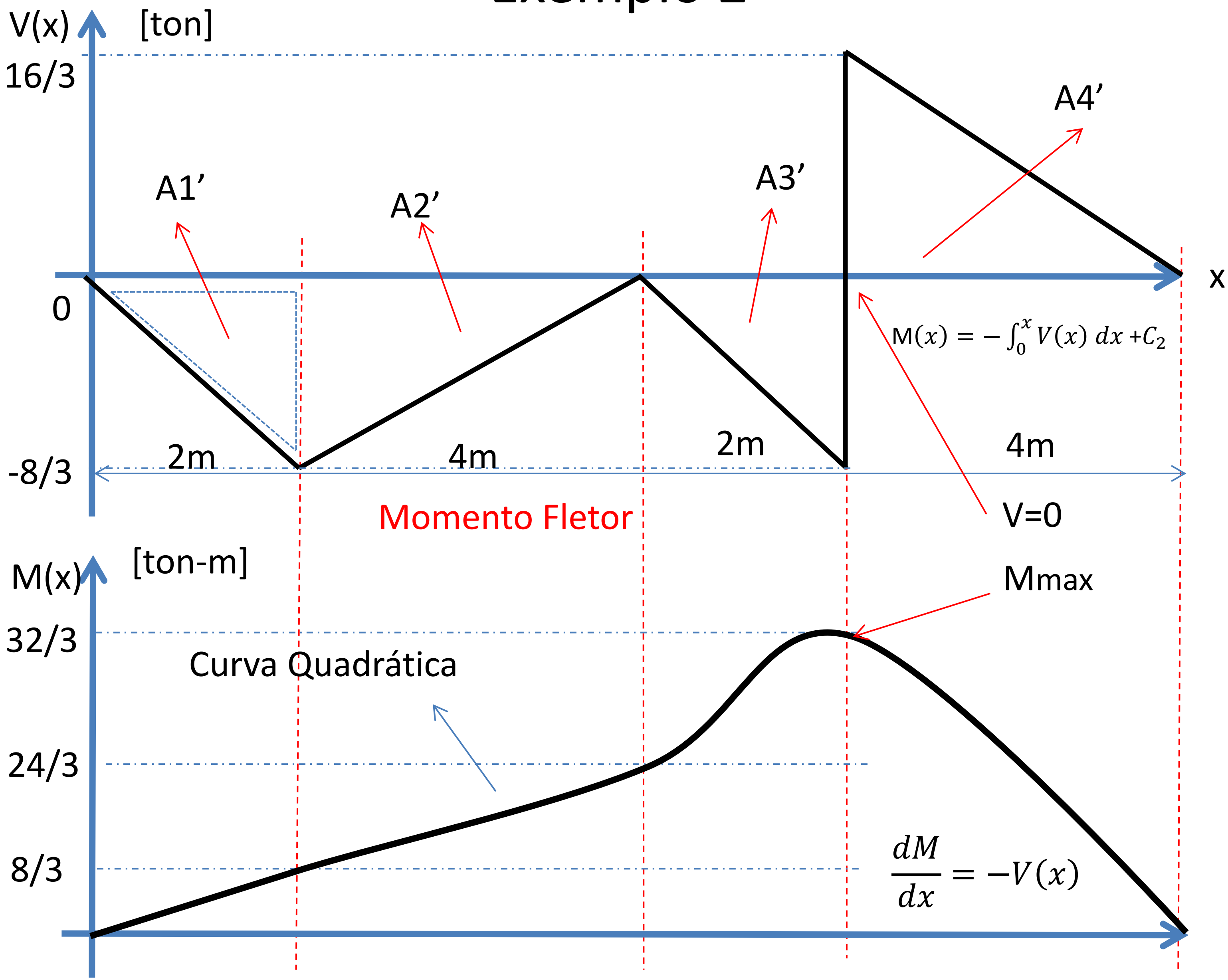
Momento Fletor

- x=12 → $M(12) = -A4' + M(8)$

$$M(12) = -\left[\frac{\frac{16}{3}\times 4}{2}\right] + \frac{32}{3} = -\frac{32}{3} + \frac{32}{3}$$

$M(12) = 0$ [ton -m] As expected!

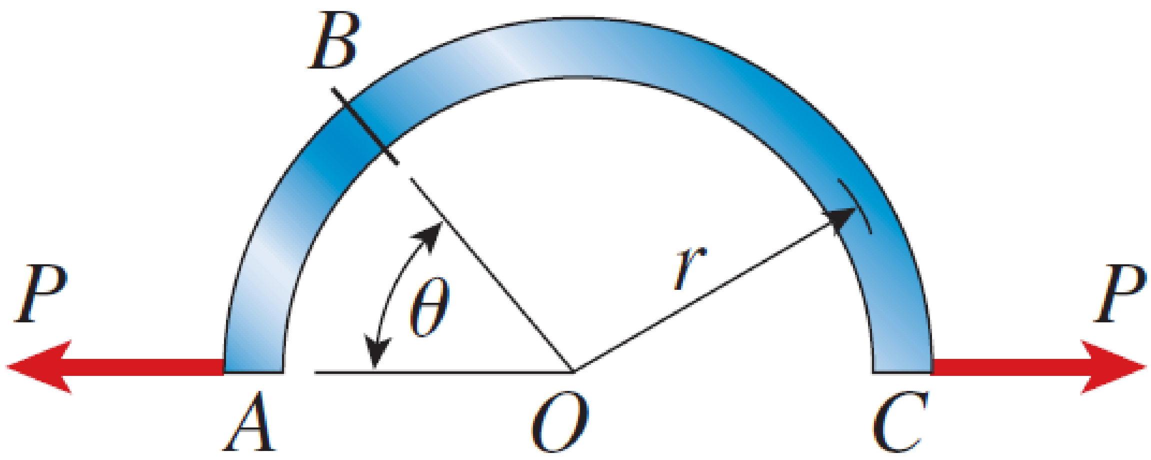
Exemplo 2



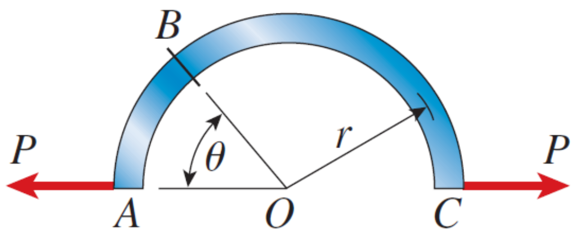
V(x)
[ton]
16/3
A1’
A2’
A3’
A4’
x
0
$M(x) = -\int_0^x V(x)\,dx + C_2$
2m
4m
2m
4m
-8/3
Momento Fletor
V=0
M(x)
[ton-m]
Mmax
32/3
Curva Quadrática
24/3
8/3
$\frac{dM}{dx} = -V(x)$

Exemplo 3

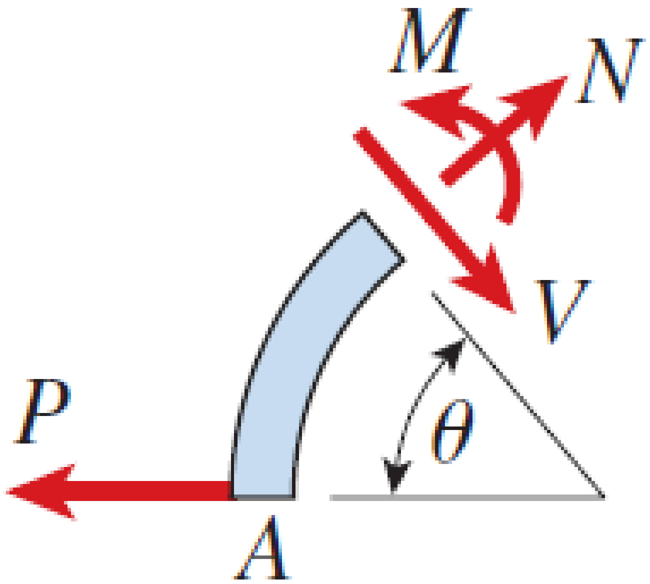
- **Para a barra semicircular ABC, Determine os diagramas de força de força axial P(x), força cortante V(x) e momento fletor M(x) como função do ângulo θ.**



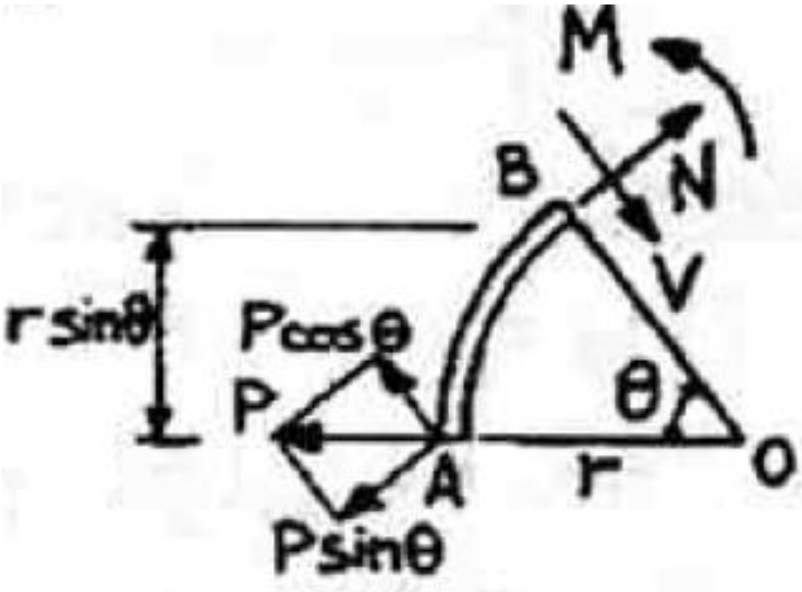
Exemplo 3



D.C.L



Exemplo 3



Direção circunferencial

$$\sum F_\theta = 0 \quad \nearrow +$$

$$N - P\sin\theta = 0 \rightarrow \boxed{N = P\sin\theta}$$

Direção Radial

$$\sum F_r = 0 \quad \searrow +$$

$$\text{V} - P\cos\theta = 0 \rightarrow \boxed{V = P\cos\theta}$$

Perpendicular ao plano x-y

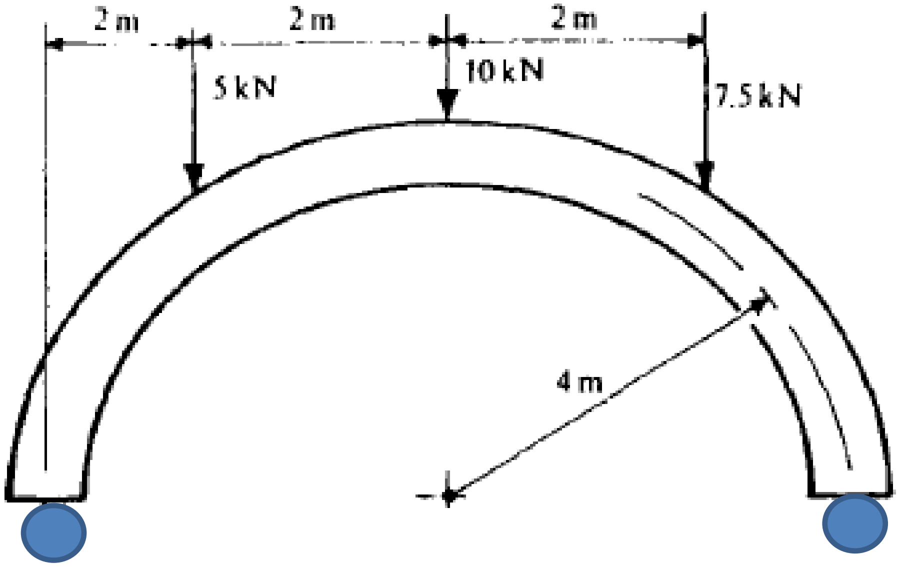
$$\sum (M_z)_o = 0 \quad \curvearrowleft +$$

$$\text{M} - N \times r = 0$$

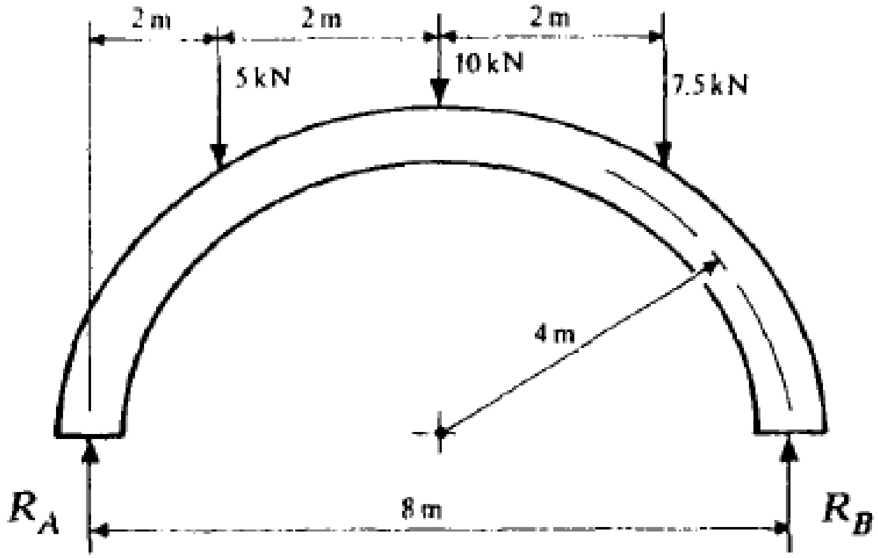
$$\boxed{\text{M} = P\sin\theta \times r}$$

Exemplo 4

- **Para a barra semicircular mostrada, determine o máximo momento fletor devido ao carregamento aplicado.**



Exemplo 4



Reações nos Suportes:

$$\sum (M_z)_A = 0 \; + \circlearrowleft$$

$$8R_B - (5 \times 2) - (10 \times 4) - (7.5 \times 6) = 0$$

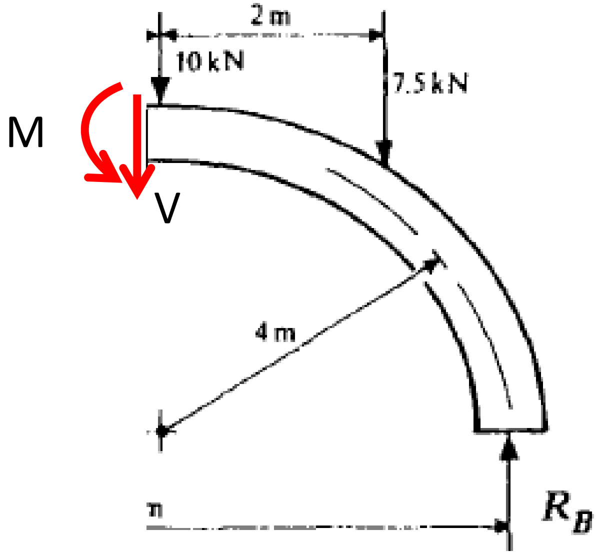
$$R_B = 11.87 \text{ kN}$$

$$\sum F_\theta = 0 \; \uparrow +$$

$$R_B + R_A = 5 + 10 + 7.5$$

$$R_A = 10.62 \text{ kN}$$

Exemplo 4



Momento máximo (no centro)

$$\sum (M_z)_{centro} = 0 \quad +$$

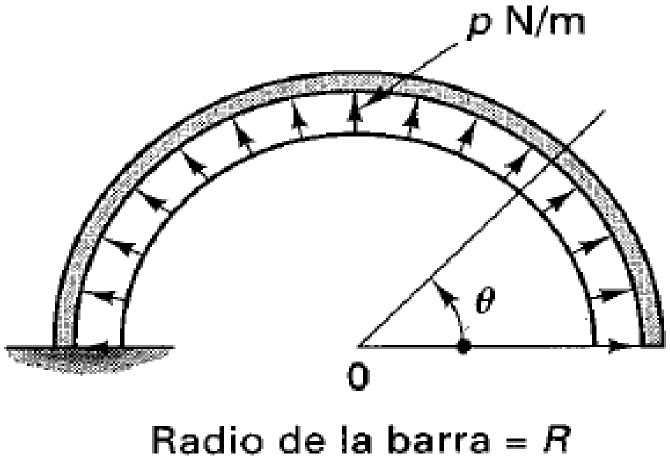
$$M + 4R_B - (7.5 \times 2) = 0$$

$$M + 4 \times 11.87 + (7.5 \times 2) = 0$$

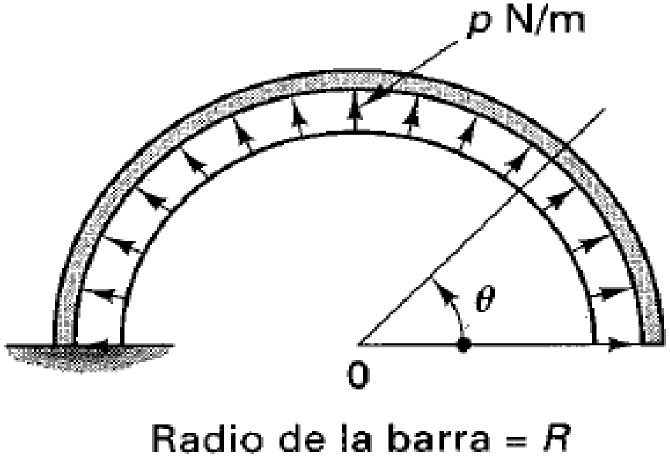
M=-32.5 kN

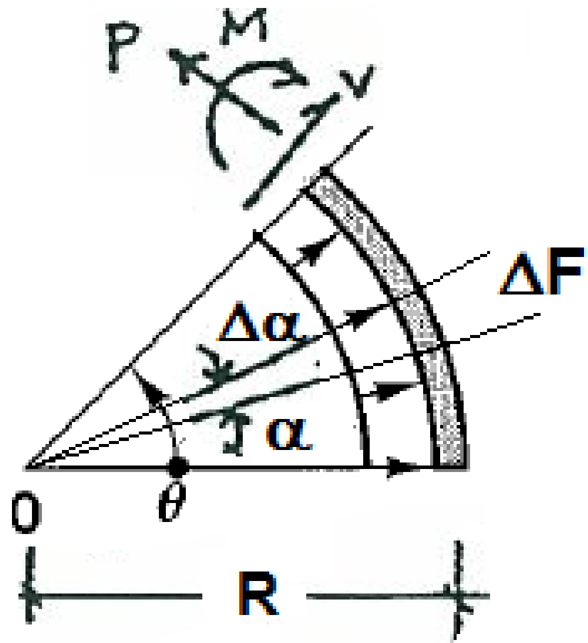
Exemplo 5

- **Para a barra semicircular, Determine os diagramas de força de força axial P(x), força cortante V(x) e momento fletor M(x) como função do ângulo θ.**



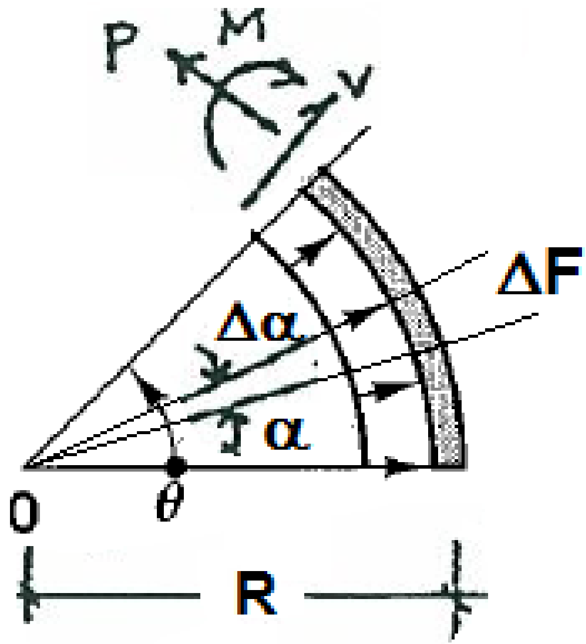
Exemplo 5





- Um corte perpendicular ao eixo da barra é feito em um ângulo θ $(0 \leq \theta \leq 180°)$.
- Na seção de corte aparecem três esforços internos para restaurar o equilíbrio do segmento da barra.

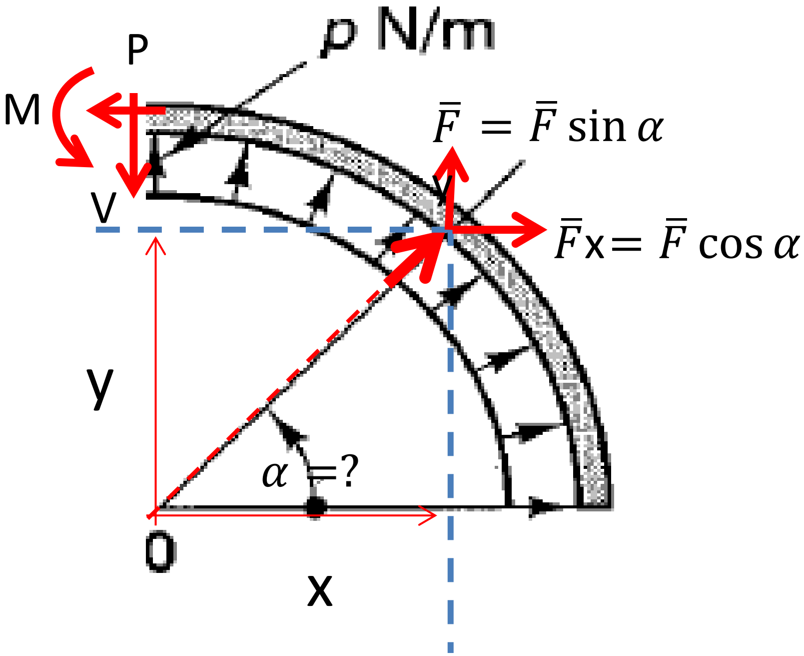
Exemplo 5



$$\Delta F = p\Delta s$$
$$\Delta F = pR\Delta\alpha$$
$$\bar{F} = \int_{0}^{\theta} pR\,d\alpha$$

Exemplo 5

θ=90°



$$\bar{F} = \int_0^{\pi/2} pR\,d\alpha \rightarrow \bar{F} = \frac{pR\pi}{2}$$

$$x = R\cos\alpha \qquad y = R\sin\alpha$$

$$\sum F_x = 0 \quad \overset{+}{\rightarrow}$$

$$\bar{F}x - P = 0 \rightarrow \quad P = \frac{pR\pi}{2}\cos\alpha$$

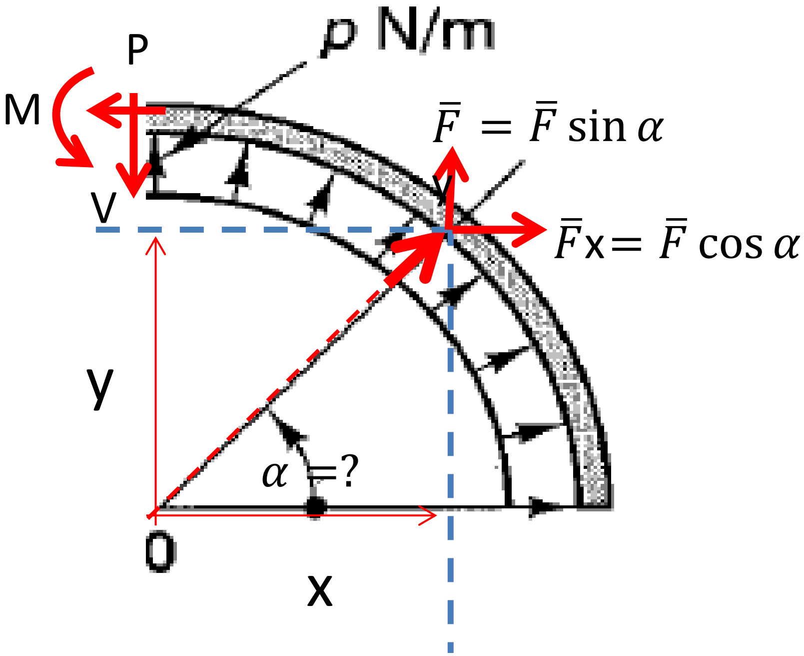
$$P = \frac{pR\pi}{2}\cos\alpha$$

$$P = k_P(\theta)pR$$

$$k_P = \cos\alpha \times \int_0^{\theta} d\alpha$$

Exemplo 5

θ=90°



$$\bar{F} = \int_0^{\pi/2} pRd\alpha \rightarrow \bar{F} = \frac{pR\pi}{2}$$

$$x = R\cos\alpha \qquad y = R\sin\alpha$$

$$\sum F_y = 0 \uparrow +$$

$$\bar{F}y - V = 0 \rightarrow \quad V = \frac{pR\pi}{2}\sin\alpha$$

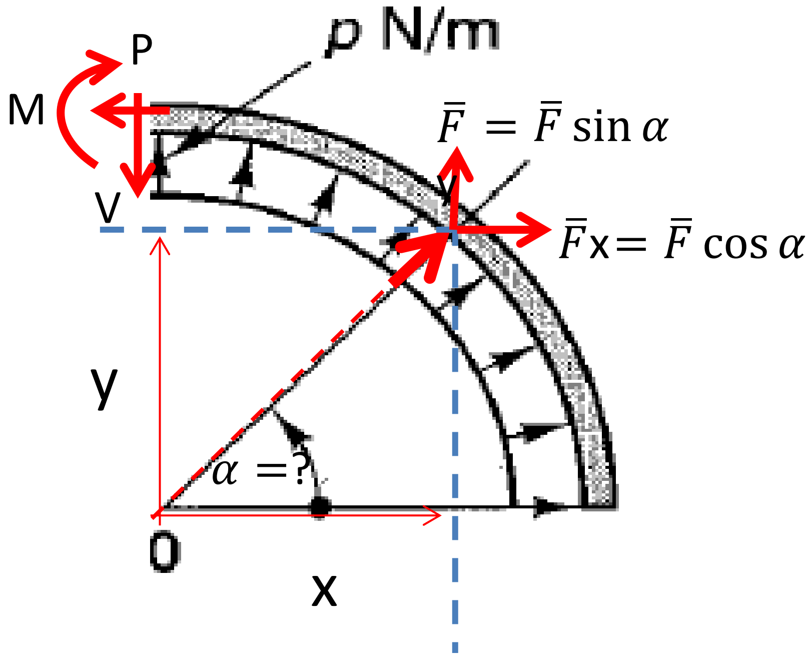
$$V = \frac{pR\pi}{2}\sin\alpha$$

$$V = k_V(\theta)pR$$

$$k_V = \sin\alpha \times \int_0^{\theta} d\,\alpha$$

Exemplo 5

θ=90°



$$\bar{F} = \int_0^{\pi/2} pRd\alpha \rightarrow \bar{F} = \frac{pR\pi}{2}$$

$x = R\cos\alpha$ y= $R\sin\alpha$

$$\sum (M_z)_0 = 0 \quad +$$

M$-P \times R = 0$

M= $P \times R = \frac{pR^2\pi}{2}\sin\alpha$

M= $\frac{pR^2\pi}{2}\sin\alpha$

M= $k_M(\theta)pR^2$

$$k_M = \sin\alpha \times \int_0^{\theta} d\,\alpha$$