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Computers and Structures 77 (2000) 485–493

Computers & Structures

www.elsevier.com/locate/compstruc

Nonlinear normal modes of planar frames discretised by the finite element method

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Received 19 March 1999; accepted 11 November 1999

Abstract

This article describes the implementation of a computer program to calculate nonlinear normal modes of structural systems. The procedure follows the invariant manifold approach, adapted to handle equations of motion of systems discretised by finite element techniques. In its current version, it generates individual modes of planar framed structures exhibiting geometrically nonlinear behaviour.

The program was tested in simple examples available in the literature, and showed very good results. Because finite element discretisations are not restrictive to the geometry of the structural system, it was also possible to generate, for the first time, nonlinear modes of framed structures. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Nonlinear modes; Normal modes; Invariant manifolds; Reduction techniques; Finite elements; Nonlinear oscillations

1. Introduction

The concept of nonlinear normal modes was first introduced by Rosenberg [1], in an attempt to extend, as much as possible, the particular properties of modal solutions to nonlinear systems. In the context of conservative mass-spring systems with many degrees of freedom, Rosenberg defined normal modes as synchronous motions in which there are fixed relations, possibly nonlinear, between the generalised coordinates. Following this definition, many authors developed the original idea to the point of discussing mode bifurcations in strongly nonlinear two-degree-of-freedom oscillators [2,3].

In 1991, Shaw and Pierre [4] proposed a redefinition of normal modes based on geometrical properties of the trajectory of a modal solution in a linear system's phase space. This so-called *invariant manifold approach* is equally suited to nonconservative problems, and was applied to some simple structural systems with elastic – and sometimes inertial – nonlinearities even in conditions of internal resonance, where modal coupling had to be consistently considered [5–8]. A derived methodology, restricted to conservative systems, was developed by King and Vakakis [9,10].

Parallel to these, many studies [11,12] regarding nonlinear oscillations in elastic bars were conducted, detecting by theory and experiment the essential features of this kind of motion, such as dependence of natural frequencies and mode shapes on the amplitudes, without explicitly connecting them to nonlinear modes.

Theoretical methods already used to construct nonlinear modes or to study large amplitude oscillations in structural systems have been restricted to very simple problems, mainly single bars, because their Galerkin type discretisation scheme makes use of shape functions – the linear mode shapes, in general – whose definition may be difficult in more complicated cases. In that sense, finite element strategies are clearly preferable, widening the range of applicability of such methods.

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0045-7949/00/$ - see front matter
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In this article, we describe a procedure to automatically calculate nonlinear normal modes of planar frames with geometric nonlinearities discretised by the finite element method. This procedure is based on the invariant manifold approach and follows very closely the steps suggested by Shaw and Pierre [13], adapted to handle a relatively large number of equations of motion more efficiently. The computer program thus implemented was tested in single bar problems. A very good agreement between its results and those available in the literature was observed. It was also used to generate, for the first time, nonlinear modes of some simple framed structures.

2. The invariant manifold approach

Normal modes are traditionally defined as particular vibratory motions of linear conservative systems, during which all points oscillate with the same frequency in such a way that a displacement pattern is preserved, except for the amplitude. The concept can be easily extended to nonconservative systems, provided the equations of motion are set in their first order form. Exploring geometric properties of these modal solutions, we arrive at a new definition of normal modes, which in its turn admits an almost natural second extension to nonlinear systems.

Consider a linear oscillatory system with n degrees of freedom, governed by the first order matrix equation

$$\dot{\mathbf{z}} = \mathbf{T}\mathbf{z}, \tag{1}$$

where $\mathbf{z} = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the vector formed by grouping the generalised coordinates x_j and velocities $y_j = \dot{x}_j,\ j = 1, \ldots, n$, and $\mathbf{T}$ is an operator supposed to have n distinct pairs of complex conjugate eigenvalues

$$\lambda_j = \alpha_j \pm \mathrm{i}\beta_j, \quad j = 1, \ldots, n \tag{2}$$

associated to the eigenvectors

$$\mathbf{z}^j = \mathbf{z}^j_{\mathbf{R}} \pm \mathrm{i}\mathbf{z}^j_{\mathbf{I}}, \quad j = 1, \ldots, n. \tag{3}$$

It can be shown [14] that $\mathbf{z}^j_{\mathbf{R}}, \mathbf{z}^j_{\mathbf{I}},\ j = 1, \ldots, n$ are linearly independent, and may constitute a new basis in the phase space, in which Eq. (1) takes the block-diagonal form

$$\begin{Bmatrix} \dot{\zeta}_{2j-1} \\ \dot{\zeta}_{2j} \end{Bmatrix} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \begin{Bmatrix} \zeta_{2j-1} \\ \zeta_{2j} \end{Bmatrix}, \quad j = 1, \ldots, n \tag{4}$$

having as solutions

$$\zeta_{2j-1}(t) = \mathrm{e}^{\alpha_j t}(a_j \cos\beta_j t + b_j \sin\beta_j t),$$
$$\zeta_{2j}(t) = \mathrm{e}^{\alpha_j t}(-a_j \sin\beta_j t + b_j \cos\beta_j t), \quad j = 1, \ldots, n, \tag{5}$$

where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{2n})$ is the new state variable and $a_j, b_j,\ j = 1, \ldots, n$ are real constants.

In the original variables, the general solution to Eq. (1) can be written as

$$\mathbf{z}(t) = \zeta_1(t)\mathbf{z}^1_{\mathbf{R}} + \zeta_2(t)\mathbf{z}^1_{\mathbf{I}} + \cdots + \zeta_{2n-1}(t)\mathbf{z}^n_{\mathbf{R}} + \zeta_{2n}(t)\mathbf{z}^n_{\mathbf{I}}, \tag{6}$$

a superposition of n different harmonics. Assuming a modal motion to contain a single harmonic, it can be understood as the solution associated with a particular set of initial conditions that leads to

$$\mathbf{z}(t) = \zeta_{2r-1}(t)\mathbf{z}^r_{\mathbf{R}} + \zeta_{2r}(t)\mathbf{z}^r_{\mathbf{I}}, \tag{7}$$

where r is the order of the corresponding mode. Eq. (7) clarifies very important features of modal solutions. Remembering, for example, that $\mathbf{z}^r_{\mathbf{R}}$ and $\mathbf{z}^r_{\mathbf{I}}$ are linearly independent, and observing the phase shift between $\zeta_{2r-1}(t)$ and $\zeta_{2r}(t)$, it is easy to justify the nonstationary character of modal motions of nonproportionally damped systems. Besides that, it shows that the trajectory in phase space of the rth modal solution is confined to a two-dimensional subspace of R^{2n} spanned by $\mathbf{z}^r_{\mathbf{R}}$ and $\mathbf{z}^r_{\mathbf{I}}$, the so-called *invariant manifold* associated with this mode.

The attachment of modal solutions to invariant manifolds suggested to Shaw and Pierre a redefinition of normal mode as a motion that takes place on a two-dimensional invariant manifold in the system's phase space. During such a motion, every generalised displacement or velocity can be written as a function of two of them, under certain nondegeneracy conditions. This new definition has the advantage of being suitable to nonconservative problems and, what is most important, *applies to weakly nonlinear oscillatory systems as well.* In this case, the invariant manifolds may be slightly curved; as a consequence, the functions relating generalised displacements and velocities during modal motions may be nonlinear.

The search for these functions is the key to evaluate nonlinear modes. Consider a nonlinear system governed by the first-order equations of motion

$$\dot{x}_i = y_i,$$
$$\dot{y}_i = f_i(x_1, \ldots, x_n, y_1, \ldots, y_n), \quad i = 1, \ldots, n, \tag{8}$$

where $f_i,\ i = 1, \ldots, n$ are analytical functions such that

$$f_i(0, \ldots, 0, 0, \ldots, 0) = 0, \quad i = 1, \ldots, n. \tag{9}$$

Suppose that, when linearised about the equilibrium position $\mathbf{z} = (\mathbf{x}, \mathbf{y}) = \mathbf{0}$, this system takes the form of Eq. (1), with distinct pairs of complex conjugate eigenvalues and eigenvectors represented by Eqs. (2) and (3).

We expect to find n two-dimensional invariant manifolds in the system's phase space, each of them associated to a particular normal mode and, consequently, to a set of functions relating all generalised coordinates and velocities to two of them. If we choose

x_k and y_k as independent variables and denote them by u and v, respectively, the modal relations we are looking for may be expressed as

$$x_i(t) = X_i(u(t), v(t)),$$
$$y_i(t) = Y_i(u(t), v(t)), \quad i = 1, \ldots, n, \tag{10}$$

where X_i, Y_i, $i = 1, \ldots, n$ are supposed to be analytical functions. It is easy to see that, in particular,

$$X_k(u, v) = u, \quad Y_k(u, v) = v. \tag{11}$$

The substitution of Eq. (10) into Eq. (8) leads to

$$\frac{\partial X_i}{\partial u} v + \frac{\partial X_i}{\partial v} f_k(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = Y_i.$$
$$\frac{\partial Y_i}{\partial u} v + \frac{\partial Y_i}{\partial v} f_k(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$
$$= f_i(X_1, \ldots, X_n, Y_1, \ldots, Y_n), \quad i = 1, \ldots, n, \tag{12}$$

a nonlinear system of partial differential equations having as unknowns the modal relations. Each solution to this system describes geometrically one of the invariant manifolds.

In most cases, it is impossible to find out the exact solutions of Eq. (12), and a power series approximation is needed. Expanding the equations of motion up to the third order, we arrive at

$$\begin{aligned} f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = {} & B_{ij} x_j + C_{ij} y_j + E_{ijm} x_j x_m \\ & + F_{ijm} x_j y_m + G_{ijm} y_j y_m \\ & + H_{ijmp} x_j x_m x_p \\ & + L_{ijmp} x_j x_m y_p \\ & + N_{ijmp} x_j y_m y_p \\ & + R_{ijmp} y_j y_m y_p, \end{aligned} \tag{13}$$

where $B_{ij}, C_{ij}, E_{ijm}, F_{ijm}, G_{ijm}, H_{ijmp}, L_{ijmp}, N_{ijmp}$ and R_{ijmp} are known coefficients and $i, j, m, p = 1, \ldots, n$. The approximate modal relations are written in the polynomial form

$$\begin{aligned} X_i(u, v) = {} & a_{1i} u + a_{2i} v + a_{3i} u^2 + a_{4i} uv + a_{5i} v^2 + a_{6i} u^3 \\ & + a_{7i} u^2 v + a_{8i} uv^2 + a_{9i} v^3, \\ Y_i(u, v) = {} & b_{1i} u + b_{2i} v + b_{3i} u^2 + b_{4i} uv + b_{5i} v^2 + b_{6i} u^3 \\ & + b_{7i} u^2 v + b_{8i} uv^2 + b_{9i} v^3, \quad i = 1, \ldots, n, \end{aligned} \tag{14}$$

where $a_{ji}, b_{ji}, j = 1, \ldots, 9, \ i = 1, \ldots, n$ are constants to be determined. After substituting Eqs. (14) and (13) into Eq. (12), and collecting terms of equal order in u and v in the resulting polynomial equations, a large system of nonlinear algebraic equations having the coefficients $a_{ji}, b_{ji}, \ j = 1, \ldots, 9, \ i = 1, \ldots, n$ as unknowns is constructed. There must be n different solutions to this system, corresponding to the n distinct invariant manifolds. It can be shown [13] that these equations can be ordered in such a manner that, instead of solving them all at once, we can solve a much smaller system of nonlinear algebraic equations having as unknowns just the coefficients of the linear terms in the modal relations; after that, two *linear* systems are constructed and solved, one for the coefficients of quadratic terms and the other for the coefficients of cubic terms.

Once known, a particular set of modal relations in the form of Eq. (14), the dynamics on the corresponding invariant manifold can be generated by substituting them in the kth pair of equations of motion in Eq. (8) – considering the expansion given by Eq. (13) – and solving the resulting modal oscillator, generally nonlinear, to obtain $u(t)$ and $v(t)$.

3. Equations of motion

In this article, the invariant manifold approach described in Section 2 was adopted to generate nonlinear modes of planar framed structures with elastic and inertial nonlinearities. The finite element formulation used to discretise the structural system is based on Bernoulli–Euler theory, under the additional hypothesis of invariance of axial force inside the element.

Details of this formulation can be found in Ref. [15]. For our purposes, it suffices to say that, after assembling the entire system, the equations of motion take the form

$$m_{ij} \ddot{x}_j + d_{ij} \dot{x}_j + k_{ij} x_j = 0, \quad i, j = 1, \ldots, n, \tag{15}$$

where

$$m_{ij} = M^0_{ij} + M^1_{ijk} x_k + M^2_{ijkl} x_k x_l,$$
$$d_{ij} = D^0_{ij} + D^1_{ijk} \dot{x}_k + D^2_{ijkl} \dot{x}_k x_l,$$
$$k_{ij} = K^0_{ij} + K^1_{ijk} x_k + K^2_{ijkl} x_k x_l \tag{16}$$

and M^0_{ij}, M^1_{ijk}, M^2_{ijkl}, D^0_{ij}, D^1_{ijk}, D^2_{ijkl}, K^0_{ij}, K^1_{ijk} e K^2_{ijkl}, $i, j, k, l = 1, \ldots, n$ are constants.

This system of n second-order equations must be transformed into a system of $2n$ first-order equations such as Eq. (8). In theory, this can be accomplished by simply solving Eq. (15) in terms of the accelerations and expanding the result in power series. However, the operation involves a symbolic inversion of the nonconstant inertia matrix, and in practice this is not feasible.

Again, we can find an approximate solution to this problem by using Taylor series. Substituting the desired expanded form – given by Eq. (13) – of Eq. (8) into the second-order equation (Eq. (15)) and equating coefficients of like powers of x_i and $y_i = \dot{x}_i$, we arrive at a system of linear algebraic equations having as unknowns the coefficients B_{ij}, C_{ij}, E_{ijm}, F_{ijm}, G_{ijm}, H_{ijmp}, L_{ijmp}, N_{ijmp} and R_{ijmp} $(i, j, m, p = 1, \ldots, n)$.

4. Invariant manifolds and eigenspaces

Section 2 showed that, in order to determine the expanded modal relations, it is necessary to solve a nonlinear system of algebraic equations for the coefficients of linear terms. In fact, this solution can be avoided by observing that these coefficients describe a two-dimensional planar surface in the phase space which is tangential to the corresponding curved invariant manifold at the equilibrium point; this planar surface is coincident with the invariant manifold of the linearised system, shown by Eq. (7) to be related to the eigenvectors of the mode of interest. Hence, the linear part of the modal relations can be alternatively generated from the solution to an eigenvalue problem.

Consider again the linearised system governed by Eq. (1), in which the operator has distinct pairs of complex conjugate eigenvalues. It was already mentioned that, in a basis constructed taking the real and imaginary parts of the eigenvalues, the equations take the block-diagonal form of Eq. (4). It is easy to find an additional coordinate transformation, for example

$$\mathbf{w} = \mathbf{S}\boldsymbol{\zeta},$$

$$\text{where } \mathbf{S} = \text{blockdiag}\left\{ \begin{bmatrix} 1/\beta_1 & 0 \\ \alpha_1/\beta_1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 1/\beta_n & 0 \\ \alpha_n/\beta_n & 1 \end{bmatrix} \right\} \tag{17}$$

that leads to

$$\begin{Bmatrix} \dot{w}_{2j-1} \\ \dot{w}_{2j} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_j^2 - \beta_j^2 & 2\alpha_j \end{bmatrix} \begin{Bmatrix} w_{2j-1} \\ w_{2j} \end{Bmatrix}, \quad j = 1, \ldots, n. \tag{18}$$

Each pair of Eq. (18) represents a harmonic oscillator with a damped frequency β_j and a damping ratio $-\alpha_j/(\alpha_j^2 + \beta_j^2)^{1/2}$, which are shown by Eq. (5) to be dynamic properties associated to the jth mode. As a consequence,

$$w_{2j-1}(t) = u_j(t), \qquad w_{2j}(t) = v_j(t), \tag{19}$$

where $u_j(t)$ and $v_j(t)$ represent, respectively, the modal displacement and the modal velocity of the jth mode.

Starting from the original state variables (x_i, y_i) and performing two coordinate transformations, we arrived at the modal variables (u_i, v_i), $i = 1, \ldots, n$. Supposing we are interested in a single mode, for example, the mth mode, the complete transformation takes the form

$$\begin{Bmatrix} z_1(t) \\ \vdots \\ z_{2n}(t) \end{Bmatrix} = \begin{bmatrix} z^m_{R,1} & z^m_{I,1} \\ \vdots & \vdots \\ z^m_{R,2n} & z^m_{I,2n} \end{bmatrix} \begin{bmatrix} \beta_m & 0 \\ -\alpha_m & 1 \end{bmatrix} \begin{Bmatrix} u_m(t) \\ v_m(t) \end{Bmatrix}, \tag{20}$$

which is precisely the linear part of the modal relations, we were searching. It must be observed that Eq. (20) depends on the way eigenvectors are normalised. The correct normalization produces a transformation that takes into account the correspondence between modal variables and actual displacements expressed by Eqs. (11).

5. Examples

Considering the alternative procedure of Section 4, the invariant manifold approach described in Section 2 was implemented in a computer program to evaluate nonlinear modes of systems governed by equations of motion generated as shown in Section 3. After reading a standard finite element input file, the program performs an eigenanalysis and asks the user for the definition of the mode of interest and the generalised coordinate to be considered as a modal variable. The output is generated as a table containing coefficients of the modal relations and the equation of the associated modal oscillator.

5.1. Simply supported beam

As a first example, a simply supported slender beam of dimensions $2 \times 20 \times 610$ mm, mass density $\rho = 2770$ kg/m^3 and Young's modulus $E = 7.33 \times 10^{10}$ N/m^2 was chosen, inspired by [11]. A finite element model was constructed, describing half the beam with 29 degrees of freedom, and the first mode was chosen for the analysis.

Fig. 1 shows the nonlinear mode shapes calculated at three different values of the modal displacement, as well as the corresponding linear ones ($w(x)$ stands for displacement at position x along the axis). It can be seen that, up to a cubic approximation, there is no difference between nonlinear and linear shapes, a result that is in complete agreement with those available in the literature [11,12,16].

As regards the dynamics on the invariant manifold, there is a nonlinear effect to be considered. Solving the modal oscillator equation

$$\ddot{u} + c_1u + c_2v + c_3u^2 + c_4uv + c_5v^2 + c_6u^3 + c_7u^2v + c_8uv^2 + c_9v^3 = 0, \tag{21}$$

whose coefficients are listed in Table 1, we note that the free-vibration frequency depends on the amplitude. As depicted in Fig. 2, there is an increase in the frequency ω at higher values of the amplitude $u_{\max}$, a hardening effect that should be attributed to tensile stresses introduced by the supports.

5.2. Clamped–clamped beam

After changing the boundary conditions, the same model was used to generate the first nonlinear mode of a clamped–clamped beam, a case for which experimental results are available [11].

Fig. 3 shows a comparison between the nonlinear mode shape obtained here, calculated at a particular

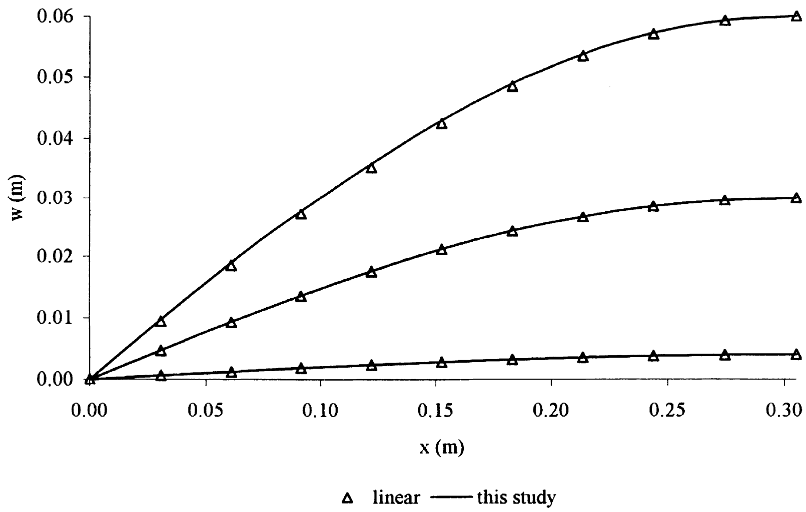

Fig. 1. First mode shape of a simply supported beam, calculated at three different amplitudes.

Table 1
Nonzero coefficients of the modal oscillator equations

	c_1	c_6	c_8
Simply supported	6.206×10^3	4.654×10^9	2.485
Clamped–clamped	3.189×10^4	5.368×10^9	3.448×10^4

amplitude, and experimental results. As a reference, the linear mode shape is also plotted. Both nonlinear results exhibit the same tendency: as the amplitude grows, the mode shape approaches that of a simply supported beam, increasing the curvature near the clamps.

In Fig. 4, the frequency–amplitude relation for this case is plotted, considering the coefficients listed in Table 1. Again, there is a hardening effect due to tension introduced by the clamps. As with the mode shape, a very good agreement between theoretical and experimental results is observed.

5.3. Framed structure

Fig. 5 shows a portal frame having three bars with the same cross-sectional and material properties as the preceding examples. In this case, evaluation of nonlinear modes by traditional techniques may be extremely

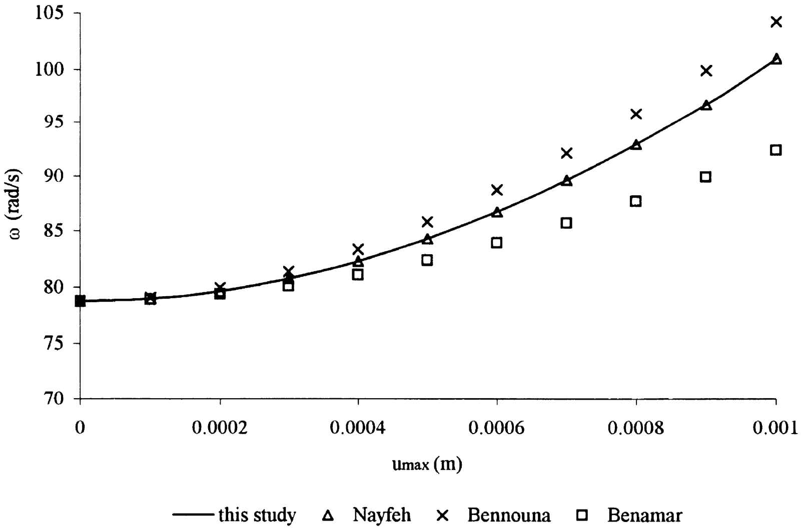

Fig. 2. Comparison between frequency–amplitude relations obtained by different models, for the first mode of a simply supported beam.

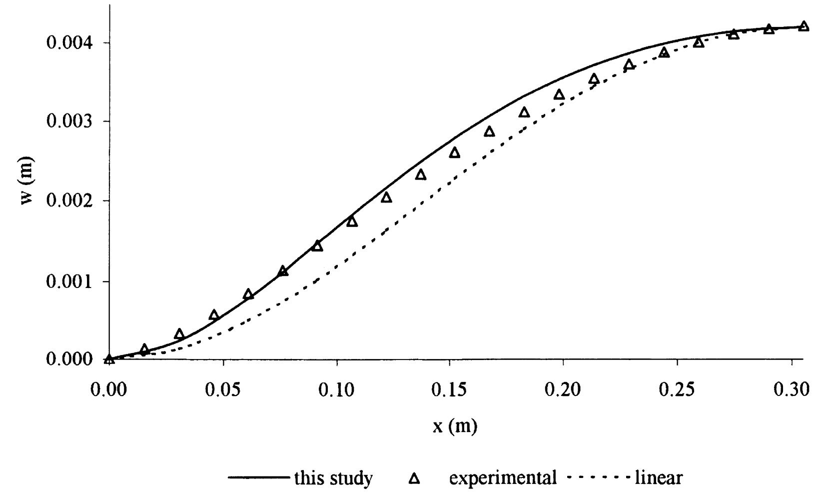

Fig. 3. First mode shape of a clamped–clamped beam.

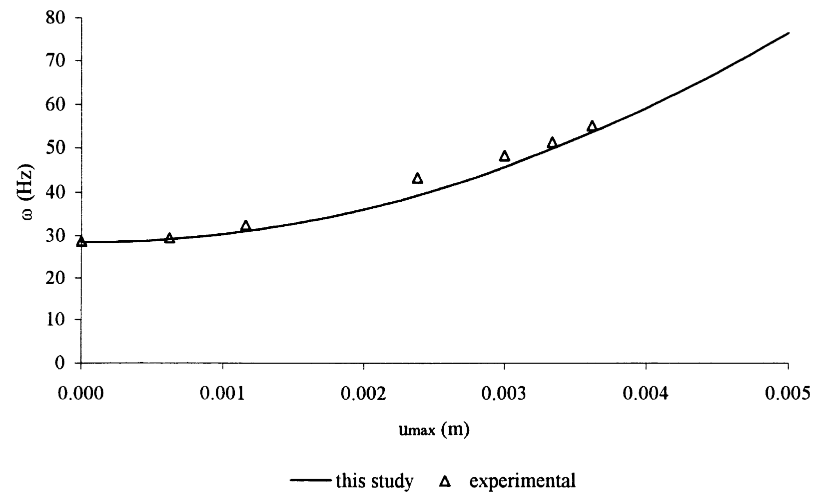

Fig. 4. Frequency–amplitude relation for the first mode of a clamped–clamped beam.

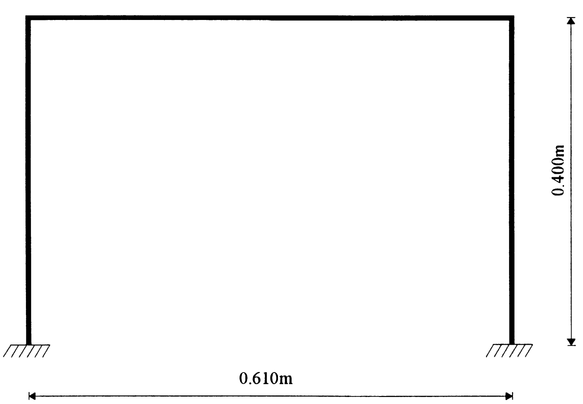

Fig. 5. Portal frame dimensions and boundary conditions.

difficult, because even a linear mode shape cannot be easily described as a function of position along the axis.

A finite element model was constructed, having four elements along each vertical bar and six elements along the horizontal bar, adding up to 39 degrees of freedom. This model was analysed by the computer program described in this article, and the first two nonlinear modes were determined.

Fig. 6 depicts the first mode shape, calculated at $u = 0.04$ and $v = \dot{u} = 0$, where the modal coordinate u was chosen to represent the horizontal displacement at mid-section. It can be seen that, at this level of amplitude, the linear and nonlinear mode shapes are very close, except for the horizontal bar, shown in detail in Fig. 7. By itself, the effect of longitudinal displacements in the vertical bars should produce a simple vertical

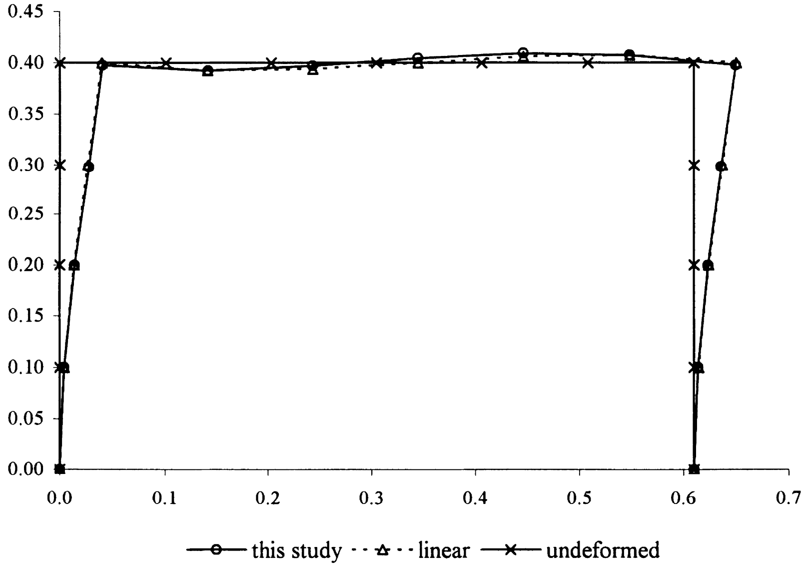

Fig. 6. First mode of the portal frame.

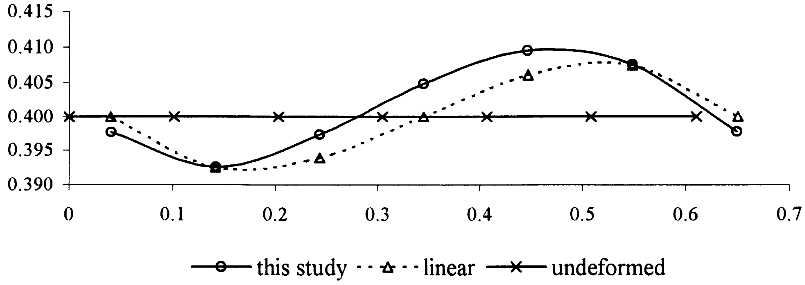

Fig. 7. Deformed horizontal bar in the first mode of the portal frame.

translation of the horizontal bar mode shape; however, Fig. 7 shows that the nonlinear mode shape seems to result from a superposition of this translated mode and a half-sine (symmetrical) displacement pattern.

Fig. 8 shows the second mode shape, calculated at $u = 0.06$ and $v = \dot{u} = 0$, where u stands for the vertical displacement at the beam mid-section. In this case, the differences between nonlinear and linear mode shapes can be attributed to longitudinal displacements of the horizontal bar.

In terms of frequency correction, there is a hardening effect in the first mode and softening in the second one. However, at this level of amplitude the difference between nonlinear and linear frequencies does not exceed 1%.

6. Conclusions and future directions

In this article, a procedure for the automatic generation of nonlinear modes of finite element systems was implemented, based on the invariant manifold approach and following very closely the solution steps proposed by Shaw and Pierre [13].

The procedure was tested in a number of cases, and showed very good results as compared to theoretical and experimental solutions available in the literature. In these tests, some distinguishing characteristics of nonlinear modal motions could be easily observed, as for example the dependence of deformed configuration and vibration frequency upon the amplitude. Besides, it was very useful for structural applications to confirm that, as shown in Fig. 7, the deformed configuration in a nonlinear mode cannot always be artificially constructed by simply superposing on the linear mode the field of longitudinal displacements, despite this being a good approximation in many cases. Simplifications like this have been successfully used in some analytical models, but care should be taken.

As regards performance, simulations in a Sun Enterprise 3000 Server revealed strong limitations to the use of the implemented procedure in systems with more

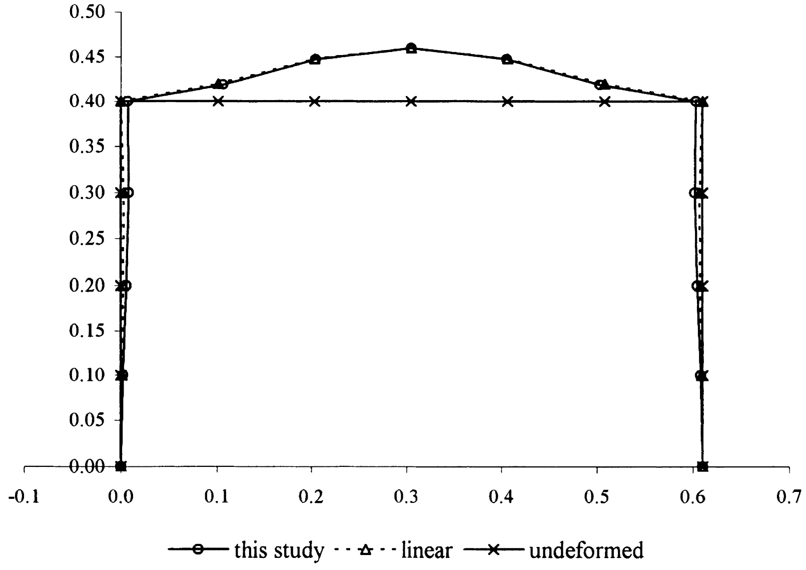

Fig. 8. Second mode of the portal frame.

than 50 degrees of freedom. Paradoxically, most of the computational time required is dedicated to converting the original system of second order differential equations into the first-order equations, which corresponds to the starting point of the invariant manifold approach. Additional studies should be devoted to optimise data storage and manipulation at this step, but a significant improvement in performance seems to be much more related to the use of lower order approximations of inertial forces at the element formulation level, a subject demanding careful attention in this research.

Other topics to be covered in future developments of this research include the adaptation of the procedure to construction of multimode manifolds, thus allowing an adequate treatment of systems exhibiting internal resonances, the implementation of different finite elements (provided an explicit formulation of the element is available) and the use of the invariant manifolds of the corresponding autonomous system to the reduction of a forced system. Some of these subjects have already been treated in the literature, but their application to finite element problems is entirely new.

Acknowledgements

This work received the financial support of FAPESP-Fundação de Amparo à Pesquisa do Estado de São Paulo.

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