Dinamica Non Lineare di Strutture e Sistemi Meccanici

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Lezione 5

Non-linear Multimodes

Historical notes

- Shaw & Pierre (1994-1999): singularity within invariant manifold approach
- Mazzilli & Baracho Neto (2002-2005): analytical methods; multiple time scales; application to finite-element models of 2D reticulated structures
- Gonçalves (2010-2015): hybrid method (Galerkin + numerical integration); application to rigid bars (Augusti’s problem) and shells

Non-linear Multimodes

Invariant manifold definition

Non-linear multimode with M modes is a motion that takes place on a 2M invariant manifold within the system phase space. Such a manifold contains an equilibrium point and, there, is tangent to the corresponding eigenplanes of the linearized system.

Non-linear Multimodes

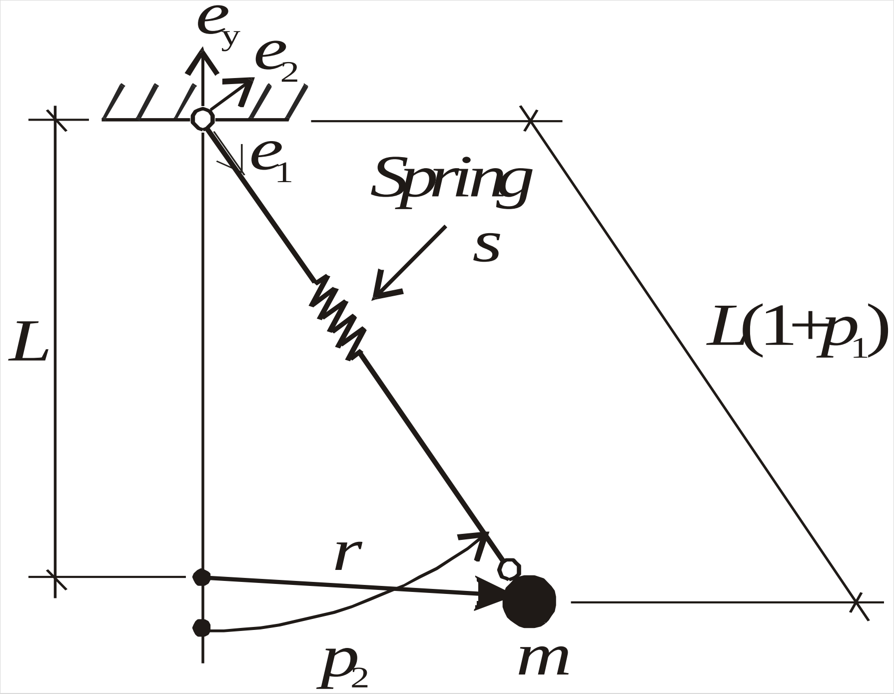
Multiple time scales approach

- Asymptotic analytical solutions for the second-order equations of motion are searched, using the multiple time scales approach (Nayfeh & Mook, 1979) , considering internal resonance between the coupled modes
- The time responses of all generalized coordinates are obtained explicitly.
- The topological structure of the embedded invariant manifold, if needed, has to be ‘extracted’ (post-processed)

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum



Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- A 2:1 internal resonance is assumed to happen between the radial and the angular modes
- Analytical studies show that there is a strong interaction between the radial and the angular modes for the great majority of initial conditions (Mazzilli, 1982)
- The energy imparted to one of the modes is continuously transferred to the other one along the time response
- Roughly speaking, when the maximum amplitude is attained for one mode, the other one has almost null amplitude
- Yet, there are particular starting conditions for which both modes have simple harmonic motions in steady states, without noticeable energy exchange

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- Equation of order ε $$[M]\{D_0^2 p_1\} + [K]\{p_1\} = \{0\}$$

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [K] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

- Solutions are sought in the form $$p_{s1} = A_{s1} \exp(\Psi T_0)$$
- Solvability condition: linearised system eigenproblem $$\left|\Psi^2 [M] + [K]\right| = 0$$

$$\Psi_I = i\beta_I \qquad \Psi_{II} = i\beta_{II}$$

$$\beta_I = \omega_2 = 4.525 \qquad \beta_{II} = \omega_1 = 9.050$$

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- Solution

$$p_{s1} = A_{s1}^{I} e^{\Psi_I T_0} + A_{s1}^{II} e^{\Psi_{II} T_0} + c.c.$$

$$A_{s1}^{I} = \phi_s^{I} A_1^{I} \qquad A_{s1}^{II} = \phi_s^{II} A_1^{II}$$

- Linear modes

$$\left\{\phi^{I}\right\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$ and $$\left\{\phi^{II}\right\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- Equation of order ε^2

$$[M]\{D_0^2 p_2\}+[K]\{p_2\} = \{x_1\}D_1 A_1^I e^{\Psi_I T_0} + \{x_2\}D_1 A_1^{II} e^{\Psi_{II} T_0} +$$
$$\{x_3\}\overline{A}_1^I A_1^{II} e^{(\Psi_{II}+\overline{\Psi}_I)T_0} + \{x_4\}(A_1^I)^2 e^{2\Psi_I T_0} + \{x_5\}A_1^I A_1^{II} e^{(\Psi_I+\Psi_{II})T_0} +$$
$$\{x_6\}(A_1^{II})^2 e^{2\Psi_{II} T_0} + \{x_7\}A_1^I \overline{A}_1^I + \{x_8\}A_1^{II} \overline{A}_1^{II}$$

- where $\{x_1\}\ldots\{x_8\}$ are complex vectors

- Solutions are sought in the form

$$p_{s2} = A_{s2}^I e^{\Psi_I T_0} + A_{s2}^{II} e^{\Psi_{II} T_0} + C_{s2} e^{(\Psi_I+\Psi_{II})T_0} + E_{s2} e^{2\Psi_{II} T_0} + F_{s2}^I + F_{s2}^{II} + c.c.$$

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- Substitute for p_{s2} in the equation of order ε^2
- Identify coefficients of each exponential and constant functions on both sides of the equality
- Arrive at systems of linear algebraic equations for

$\{C_{s2}\}$ $\{E_{s2}\}$ $\{F^{I}_{s2}\}$ $\{F^{II}_{s2}\}$

- And at systems of ordinary differential equations for

$\{A^{I}_{s2}\}$ $\{A^{II}_{s2}\}$

- These latter ones correspond to the imposition of the solvability conditions upon terms of

$\exp(\Psi_I T_0)$ $\exp(\Psi_{II} T_0)$

Non-linear Multimodes

Multiple time scales approach

Example: elastic pendulum

- Write the time response for the non-linear multi-mode as:

$$p_s = \frac{(\varepsilon a_I)}{2}\left(\gamma_s^I + i\delta_s^I\right)e^{i\varphi_I} + \frac{(\varepsilon a_{II})}{2}\left(\gamma_s^{II} + i\delta_s^{II}\right)e^{i\varphi_{II}} +$$
$$\frac{(\varepsilon a_I)(\varepsilon a_{II})}{4}\left(\sigma_s + i\tau_s\right)e^{i(\varphi_I+\varphi_{II})} + \frac{(\varepsilon a_{II})^2}{4}\left(\vartheta_s + i\xi_s\right)e^{i2\varphi_{II}} +$$
$$\frac{(\varepsilon a_I)^2}{4}\left(\zeta_s^I + i\nu_s^I\right) + \frac{(\varepsilon a_{II})^2}{4}\left(\zeta_s^{II} + i\nu_s^{II}\right) + c.c.$$

- with

$$A_1^I = \frac{1}{2}a_I \exp(i\theta_I) \qquad A_1^I = \frac{1}{2}a_I \exp(i\theta_I)$$

$$\varphi_I = \beta_I t + \theta_I \qquad \varphi_{II} = \beta_{II} t + \theta_{II}$$

Non-linear Multimodes

Multiple time scales approach

Example: first numerical simulation for the elastic pendulum

- Case of strong energy exchange between the two modes, resulting in a "beating" or modulation effect in the time responses, as seen in Figures 2 and 3

- In order to perform this study, the following initial conditions were chosen

$$\varepsilon a_I(0) = 0.01, \varepsilon a_{II}(0) = 0.1 \qquad \theta_I(0) = 0 , \theta_{II}(0) = 0$$

or

$$p_1(0) = 0.1000 , \dot{p}_1(0) = 0 \qquad p_2(0) = 0.0097, \dot{p}_2(0) = 0$$

Non-linear Multimodes

Multiple time scales approach

Example: first numerical simulation for the elastic pendulum

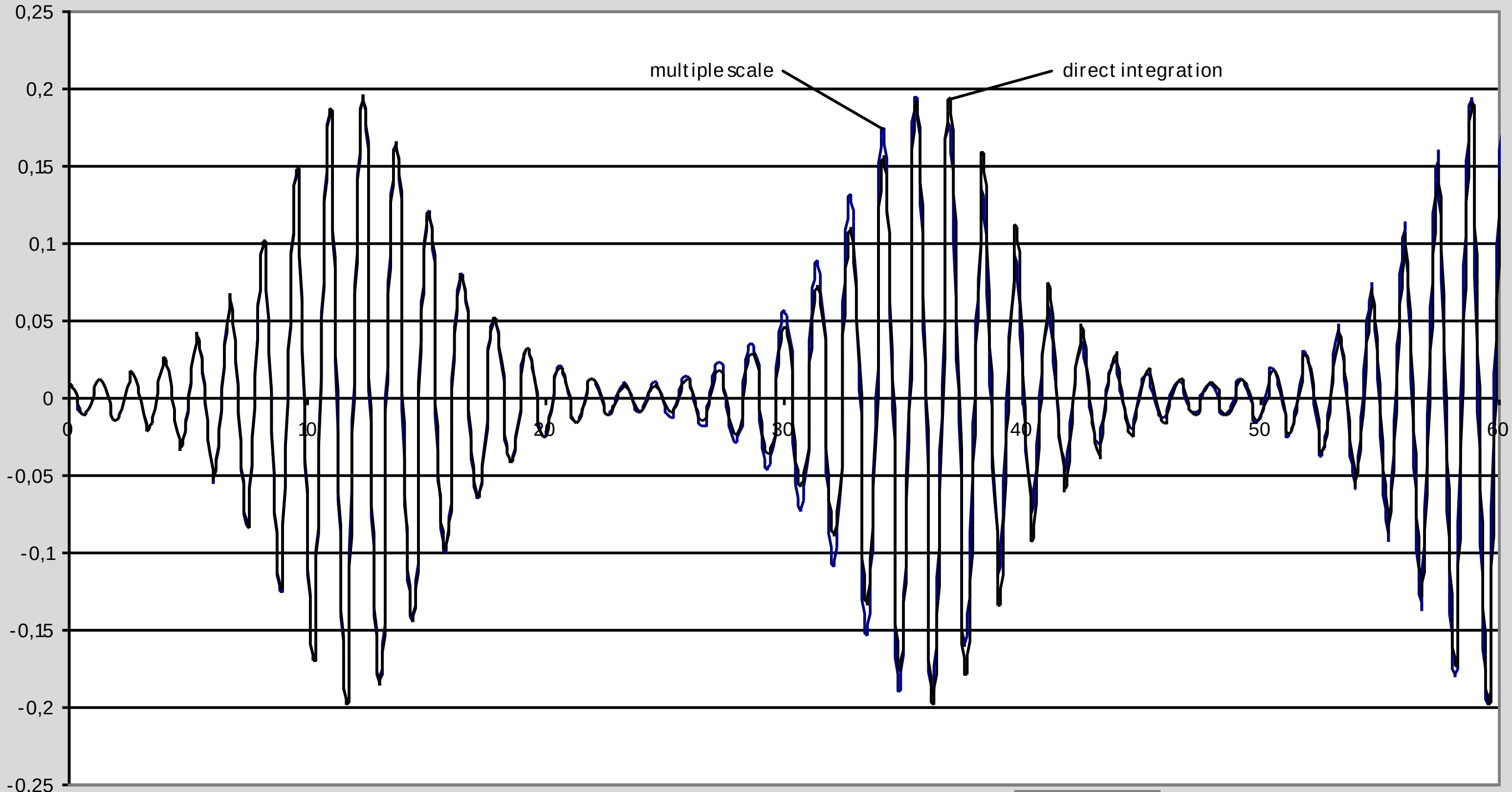


Figure 2: time response for $p_2(t)$

Non-linear Multimodes

Multiple time scales approach

Example: first numerical simulation for the elastic pendulum

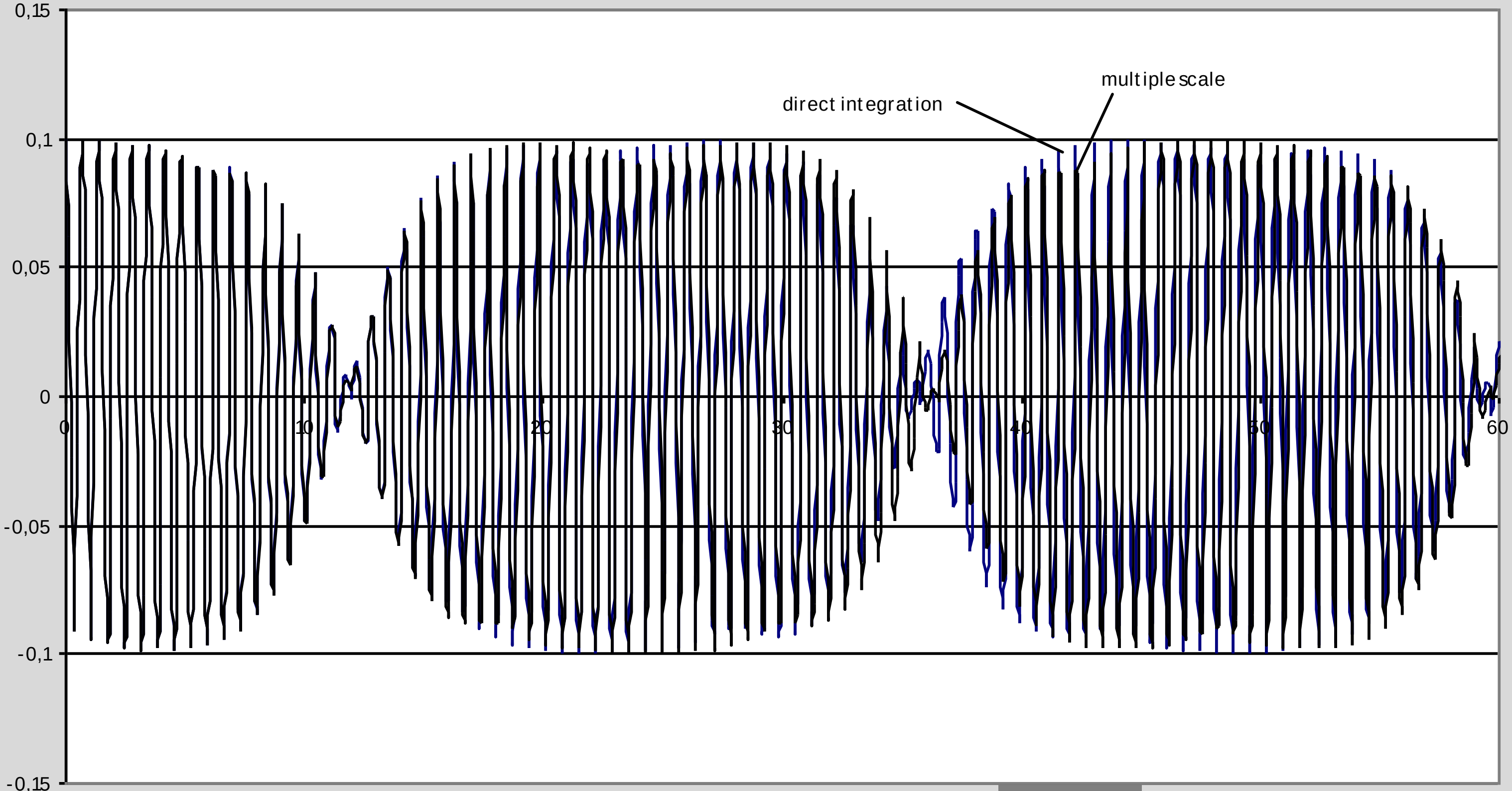


Figure 3: time response for $p_1(t)$

Non-linear Multimodes

Multiple time scales approach

Example: second numerical simulation for the elastic pendulum

- Case of a quasi-steady state, as result of a particular choice of initial conditions, as seen in Figure 4 and 5
- In order to perform this study, the following initial conditions were chosen

$$\varepsilon a_I(0) = 0.2829,\ \varepsilon a_{II}(0) = 0.1 \qquad \theta_I(0) = 0\ ,\ \theta_{II}(0) = 0$$

or

$$p_1(0) = 0.1050\ ,\ \dot{p}_1(0) = 0 \qquad p_2(0) = 0.2740,\ \dot{p}_2(0) = 0$$

Non-linear Multimodes

Multiple time scales approach

Example: second numerical simulation for the elastic pendulum

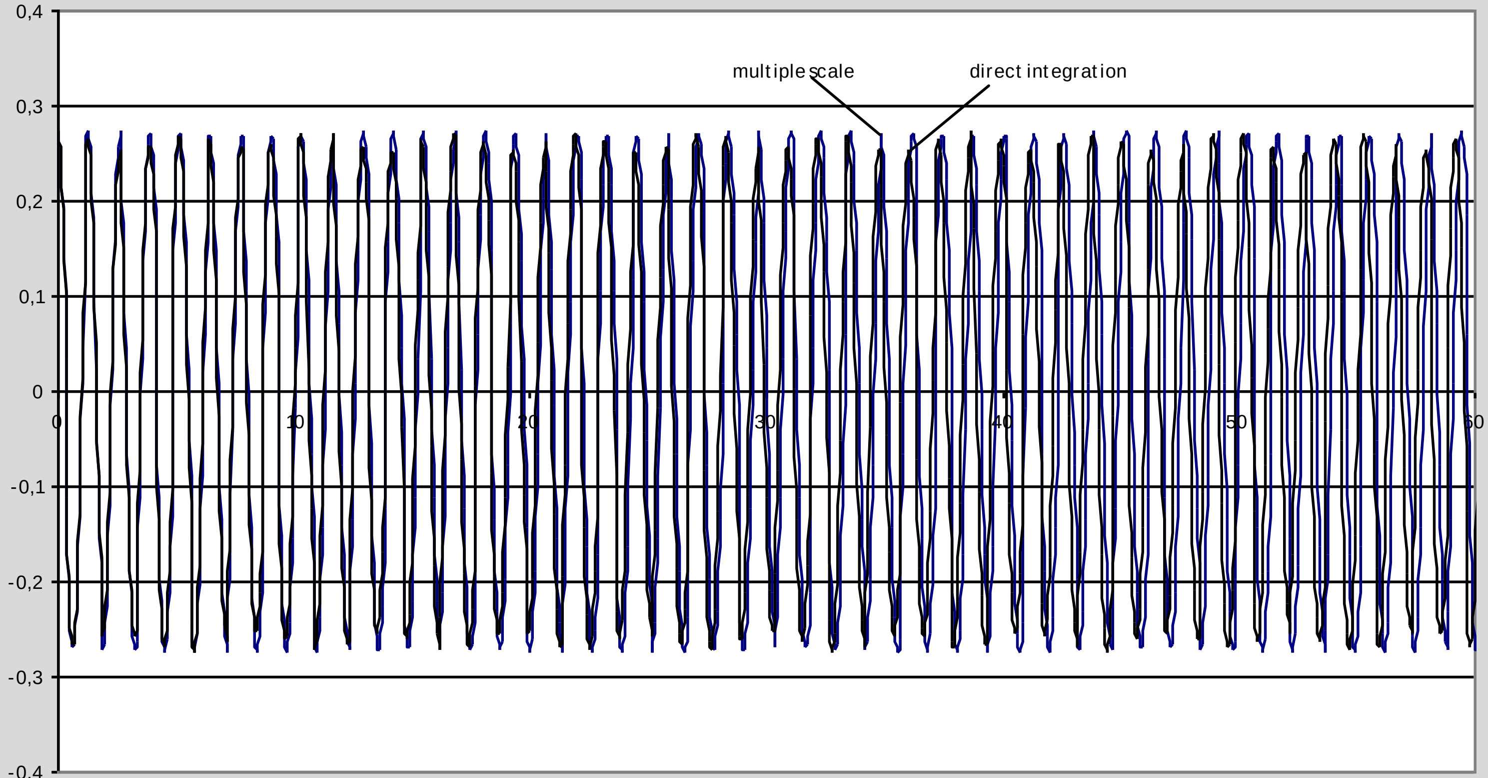


Figure 4: time response for $p_2(t)$

Non-linear Multimodes

Multiple time scales approach

Example: second numerical simulation for the elastic pendulum

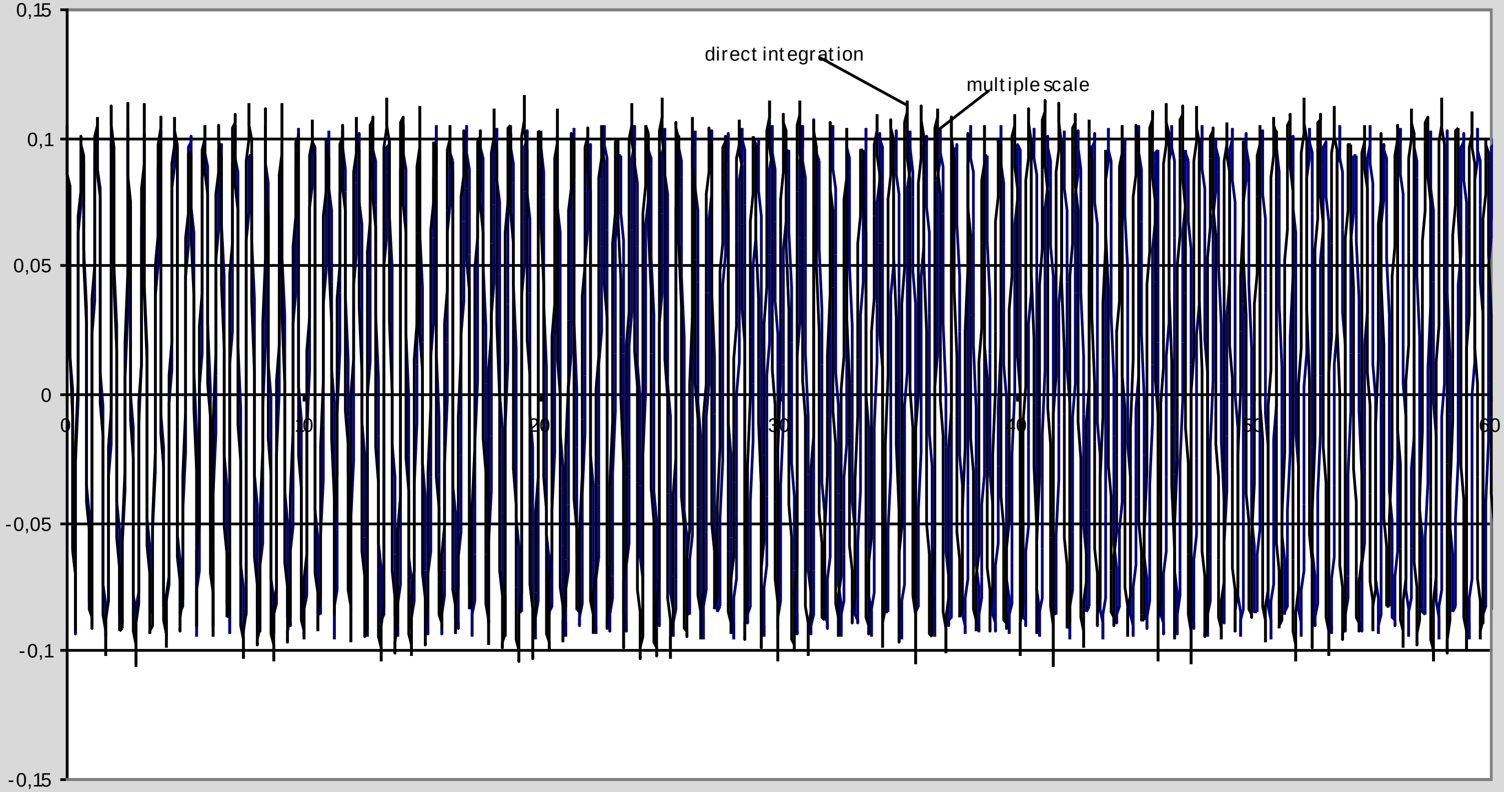
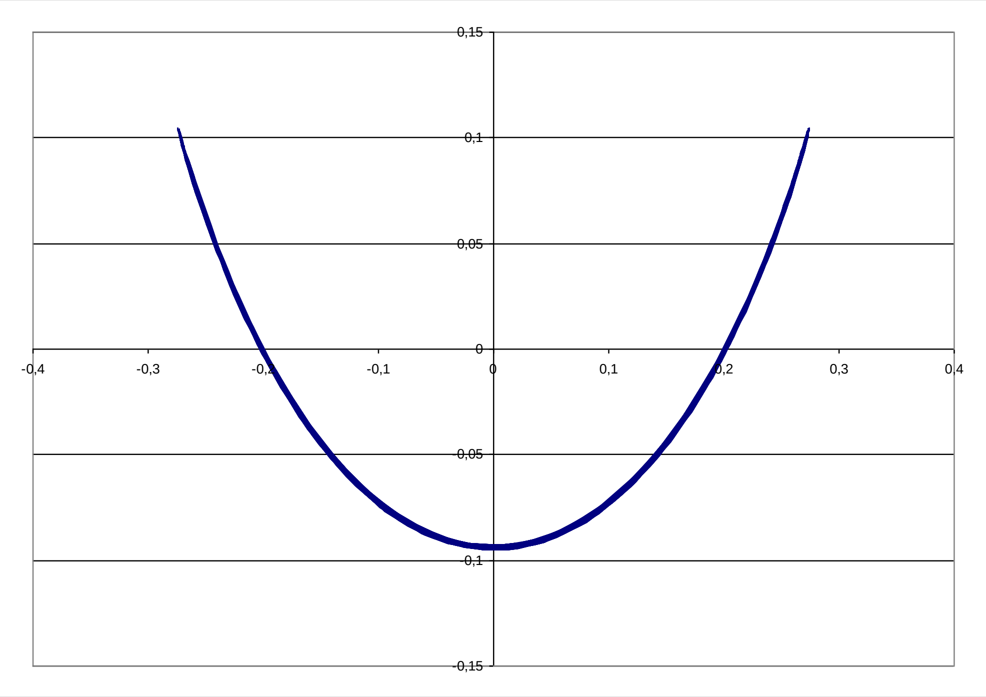


Figure 5: time response for $p_1(t)$

Non-linear Multimodes

Multiple time scales approach

Example: second numerical simulation for the elastic pendulum; invariant manifold projection

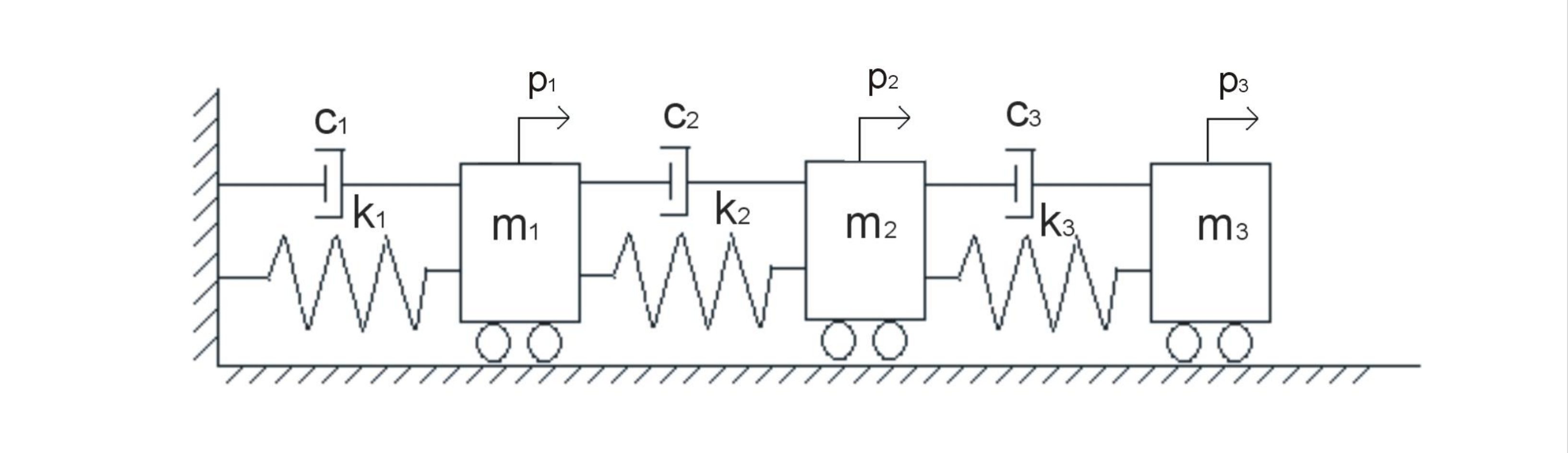


$$p_1 = p_1(p_2)$$

Non-linear Multimodes

Multiple time scales approach

Example: 3 DOF non-linear oscillator



$$m_1 = \rho_1 m \quad , \quad m_2 = \rho_2 m \quad , \quad m_3 = \rho_3 m$$

$$c_1 = \mu_1 c \quad , \quad c_2 = \mu_2 c \quad , \quad c_3 = \mu_3 c$$

$$k_1 = \eta_1 k \quad , \quad k_2 = \eta_2 k \quad , \quad k_3 = \eta_3 k$$

Non-linear Multimodes

Multiple time scales approach

- Assumed non-linear spring forces are $k_i \Delta_i - \lambda_i \Delta_i^2$ where Δ_i stands for the spring deformation
- Hence, for positive λ_i the spring has a softening non-linear behaviour, whereas for negative λ_i the spring has a hardening non-linear behaviour
- As for the linear viscous dampers, the dissipative forces are $c_i \dot{\Delta}_i$ where $\dot{\Delta}_i$ stands for the spring deformation velocity.
- An adequate choice of the system parameters can be made to assure that a 2:1 internal resonance happens between the third and the second linear modes

Non-linear Multimodes

Multiple time scales approach

$$[M]\{\ddot{p}\}+[C]\{\dot{p}\}+[K]\{p\}=\{NL\}$$

$$[M]=m\begin{bmatrix}\rho_1 & 0 & 0\\ 0 & \rho_2 & 0\\ 0 & 0 & \rho_3\end{bmatrix}$$

$$[C]=c\begin{bmatrix}\mu_1+\mu_2 & -\mu_2 & 0\\ -\mu_2 & \mu_2+\mu_3 & -\mu_3\\ 0 & -\mu_3 & \mu_3\end{bmatrix}$$

$$[K]=k\begin{bmatrix}\eta_1+\eta_2 & -\eta_2 & 0\\ -\eta_2 & \eta_2+\eta_3 & -\eta_3\\ 0 & -\eta_3 & \eta_3\end{bmatrix}$$

$$\{NL\}=\begin{Bmatrix}(\lambda_1-\lambda_2)p_1^2+2\lambda_2 p_1 p_2-\lambda_2 p_2^2\\ \lambda_2 p_1^2-2\lambda_2 p_1 p_2+(\lambda_2-\lambda_3)p_2^2+2\lambda_3 p_2 p_3-\lambda_3 p_3^2\\ \lambda_3 p_2^2-2\lambda_3 p_2 p_3+\lambda_3 p_3^2\end{Bmatrix}$$

Non-linear Multimodes

Multiple time scales approach

- The third undamped linear frequency is approximately twice the second one when the following choice of parameters is made:

$\rho_1 = 2$ $\rho_2 = \rho_3 = 1$ $\eta_1 = 1.54$ $\eta_2 = 3$ $\eta_3 = 1$

- In fact, the three undamped linear frequencies, in this case, are:

$\beta_1 = 0.5449$ $\beta_2 = 1.1809$ $\beta_3 = 2.3619 \approx 2\beta_2$

- For the non-linear multi-mode evaluation, the following additional parameters are assumed:

$m = 1$ $k = 1$ $\lambda_1 = \lambda_2 = \lambda_3 = 0.25$

- Two simulations are considered:

i. damped system with $c = 0.1$ $\mu_1 = \mu_2 = \mu_3 = 1$

ii. undamped system: non-linear effects for a longer period of time

Non-linear Multimodes

Multiple time scales approach

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ii. undamped system: non-linear effects for a longer period of time

Non-linear Multimodes

Multiple time scales approach

- Procedure based upon the method of multiple scales
- Assume weak non-linearities
- Express the generalised coordinates in terms of power series of a non-dimensional perturbation parameter $0<\varepsilon<<1$ & introduce the time scales, as usual…

$$p_s = \varepsilon p_{s1} + \varepsilon^2 p_{s2} + \ldots \quad ; \quad T_n = \varepsilon^n t$$

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \quad ; \quad D_i^n = \frac{d^n}{dT_i^n}$$

- Substitute in the equations of motion & eliminate secular terms. Take into account that the vibration modes of the structure are internally resonant (in this case they present a commensurable relationship between two of their frequencies of the 2:1 type)

Non-linear Multimodes

Multiple time scales approach

- Equation of order ε $[M]\{D_0^2 p_1\}+[C]\{D_0 p_1\}+[K]\{p_1\}=\{0\}$
- Solutions are sought in the form $p_{s1}=A_{s1}\exp(\Psi T_0)$
- Solvability condition: linearised damped eigenproblem $\left|\Psi^2[M]+\Psi[C]+[K]\right|=0$
- Eigenvalues $\Psi_I=\alpha_I+i\beta_I$ $\Psi_{II}=\alpha_{II}+i\beta_{II}$
- Internal resonance $\Psi_{II}=2\Psi_I+\Delta\Psi$ & $\Delta\Psi$ small complex number
- Solution $p_{s1}=A_{s1}^I e^{\Psi_I T_0}+A_{s1}^{II}e^{\Psi_{II}T_0}+c.c.$ $A_{s1}^I=\phi_s^I A_1^I$ $A_{s1}^{II}=\phi_s^{II}A_1^{II}$
- where $\{\phi^I\}$ and $\{\phi^{II}\}$ stand respectively for the eigenvectors of the second (*I*) and the third (*II*) damped modes

Non-linear Multimodes

Multiple time scales approach

- Equation of order ε^2

$$[M]\{D_0^2 p_2\}+[C]\{D_0 p_2\}+[K]\{p_2\}=\{x_1\}D_1 A_1^I e^{\Psi_I T_0}+\{x_2\}D_1 A_1^{II} e^{\Psi_{II} T_0}+$$
$$\{x_3\}\overline{A}_1^I A_1^{II} e^{(\Psi_{II}+\overline{\Psi}_I)T_0}+\{x_4\}(A_1^I)^2 e^{2\Psi_I T_0}+\{x_5\}A_1^I A_1^{II} e^{(\Psi_I+\Psi_{II})T_0}+$$
$$\{x_6\}(A_1^{II})^2 e^{2\Psi_{II} T_0}+\{x_7\}A_1^I \overline{A}_1^I e^{2\alpha_I T_0}+\{x_8\}A_1^{II}\overline{A}_1^{II} e^{2\alpha_{II} T_0}+c.c.$$

- where $\{x_1\}\ldots\{x_8\}$ are complex vectors

- Solutions are sought in the form

$$p_{s2}=A_{s2}^I e^{\Psi_I T_0}+A_{s2}^{II} e^{\Psi_{II} T_0}+C_{s2} e^{(\Psi_I+\Psi_{II})T_0}+E_{s2} e^{2\Psi_{II} T_0}+F_{s2}^I e^{2\alpha_I T_0}+F_{s2}^{II} e^{2\alpha_{II} T_0}+c.c.$$

Non-linear Multimodes

Multiple time scales approach

- Substitute for p_{s2} in the equation of order ε^2
- Identify coefficients of each exponential and constant functions on both sides of the equality
- Arrive at systems of linear algebraic equations for

  $\{C_{s2}\}$ $\{E_{s2}\}$ $\{F^{I}_{s2}\}$ $\{F^{II}_{s2}\}$

- And at systems of ordinary differential equations for

  $\{A^{I}_{s2}\}$ $\{A^{II}_{s2}\}$

- These latter ones correspond to the imposition of the solvability conditions upon terms of

  $\exp(\Psi_I T_0)$ $\exp(\Psi_{II} T_0)$

Non-linear Multimodes

Multiple time scales approach

- After solving the systems of algebraic and differential equations mentioned before, write time response for non-linear multi-mode as:

$$p_s = \frac{(\varepsilon a_I)}{2}\left(\gamma_s^I + i\delta_s^I\right)e^{i\varphi_I} + \frac{(\varepsilon a_{II})}{2}\left(\gamma_s^{II} + i\delta_s^{II}\right)e^{i\varphi_{II}} +$$

$$\frac{(\varepsilon a_I)(\varepsilon a_{II})}{4}\left(\sigma_s + i\tau_s\right)e^{i(\varphi_I + \omega\varphi_{II})} + \frac{(\varepsilon a_{II})^2}{4}\left(\vartheta_s + i\xi_s\right)e^{i2\varphi_{II}} +$$

$$\frac{(\varepsilon a_I)^2}{4}\left(\zeta_s^I + i\nu_s^I\right) + \frac{(\varepsilon a_{II})^2}{4}\left(\zeta_s^{II} + i\nu_s^{II}\right) + c.c.$$

- with

$$A_1^I = \frac{1}{2} a_I \exp(i\theta_I)$$

$$A_1^I = \frac{1}{2} a_I \exp(i\theta_I)$$

$$\varphi_I = \beta_I t + \theta_I$$

$$\varphi_{II} = \beta_{II} t + \theta_{II}$$

Non-linear Multimodes

Multiple time scales approach

$$p_s = F_{s1}U_1 + F_{s2}U_2 + F_{s3}V_1 + F_{s4}V_2 + F_{s5}(U_1)^2 + F_{s6}(U_2)^2 + F_{s7}U_1U_2 + F_{s8}U_1V_1 + F_{s9}U_2V_2 + F_{s10}U_1V_2 + F_{s11}U_2V_1 + F_{s12}(V_1)^2 + F_{s13}(V_2)^2 + F_{s14}V_1V_2$$

$$\dot{p}_s = G_{s1}U_1 + G_{s2}U_2 + G_{s3}V_1 + G_{s4}V_2 + G_{s5}(U_1)^2 + G_{s6}(U_2)^2 + G_{s7}U_1U_2 + G_{s8}U_1V_1 + G_{s9}U_2V_2 + G_{s10}U_1V_2 + G_{s11}U_2V_1 + G_{s12}(V_1)^2 + G_{s13}(V_2)^2 + G_{s14}V_1V_2$$

where $U_1 = p_v \quad ; \quad V_1 = \dot{p}_v \quad ; \quad U_2 = p_w \quad ; \quad V_2 = \dot{p}_w$

are, respectively, two non-identically-zero generalised coordinates and their corresponding velocities

Non-linear Multimodes

Multiple time scales approach

- Parameters $c = 0.1$ $\mu_1 = \mu_2 = \mu_3 = 1$
- In order to perform this study, the following initial conditions were chosen

$$\varepsilon a_I(0) = 0.10,\ \varepsilon a_{II}(0) = 0.01 \qquad \theta_I(0) = 0\,,\ \theta_{II}(0) = 0$$

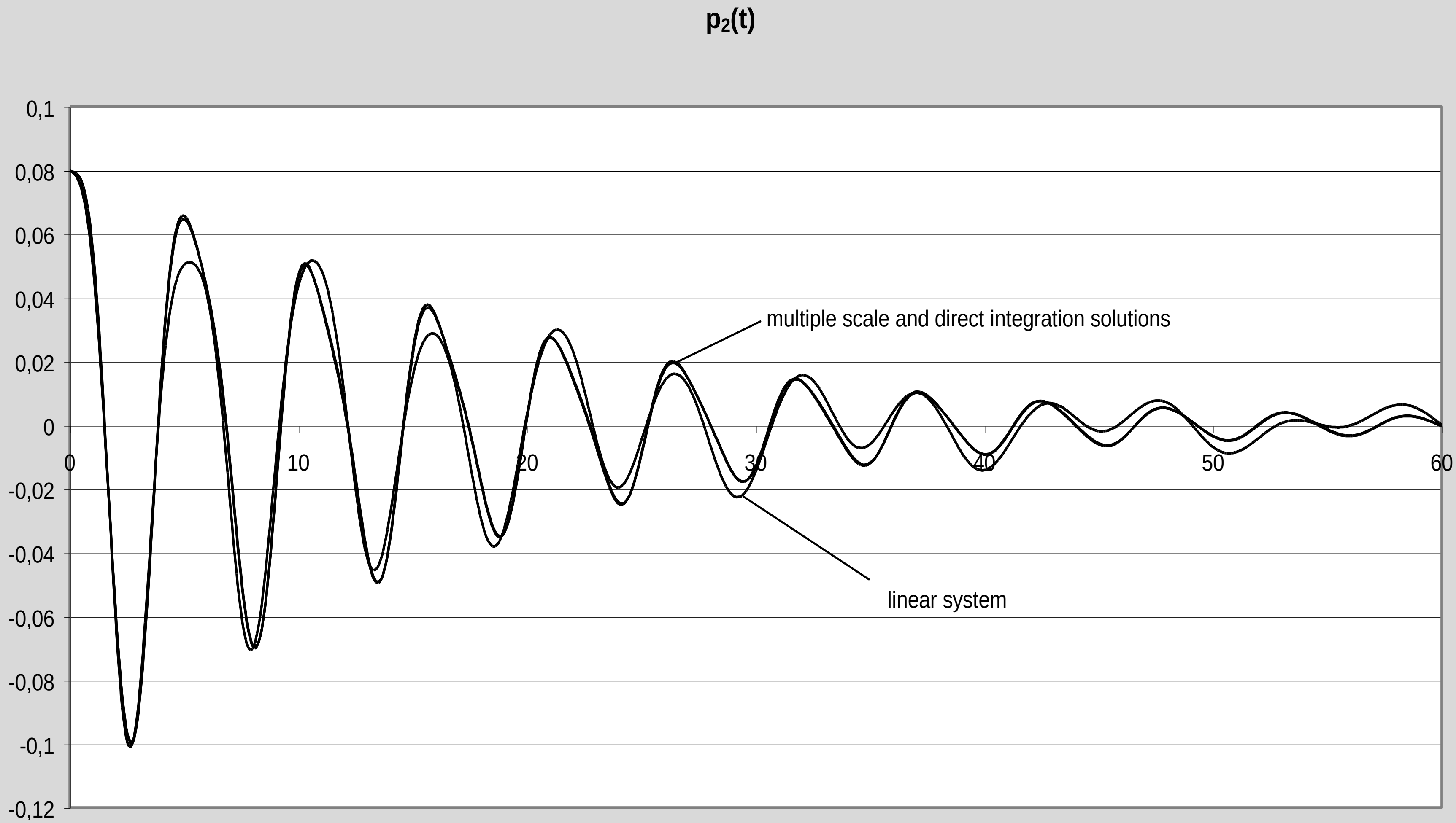
or

$$p_1(0) = 0.1838\,,\ \dot{p}_1(0) = 0 \qquad p_2(0) = 0.0802,\ \dot{p}_2(0) = 0$$

$$p_3(0) = -0.2301,\ \dot{p}_3(0) = 0$$

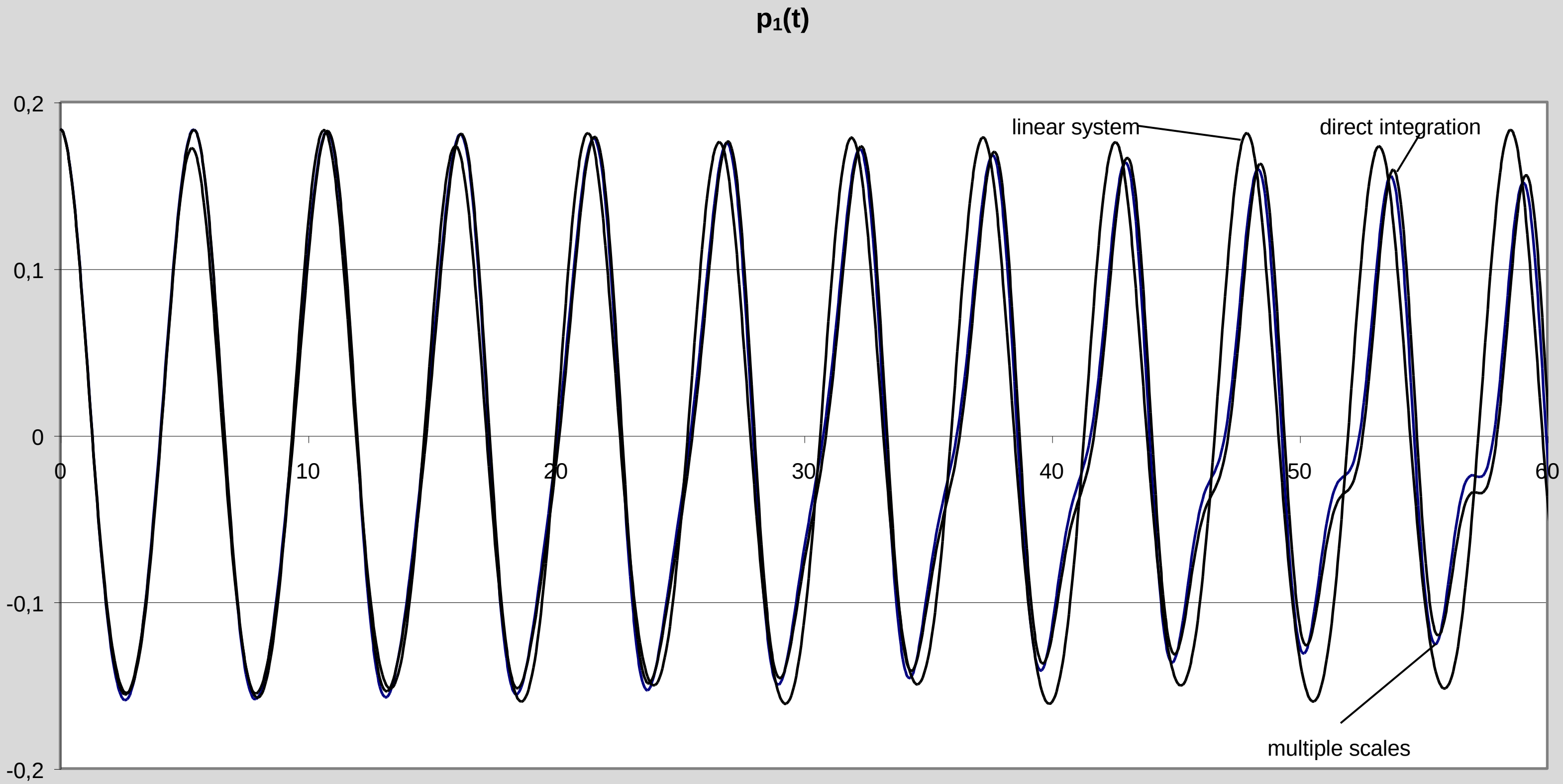
Non-linear Multimodes

Multiple time scales approach



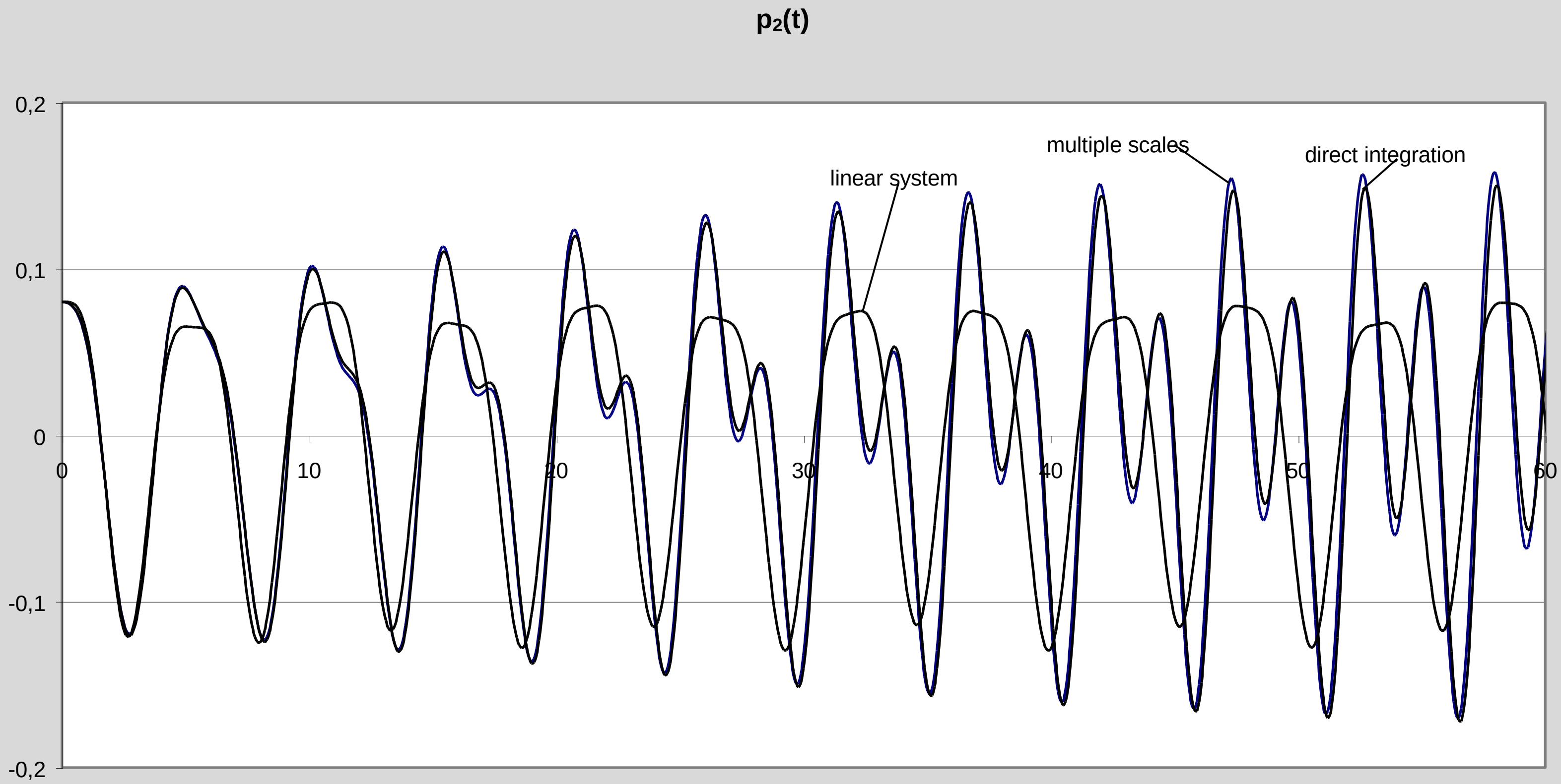
Non-linear Multimodes

Multiple time scales approach



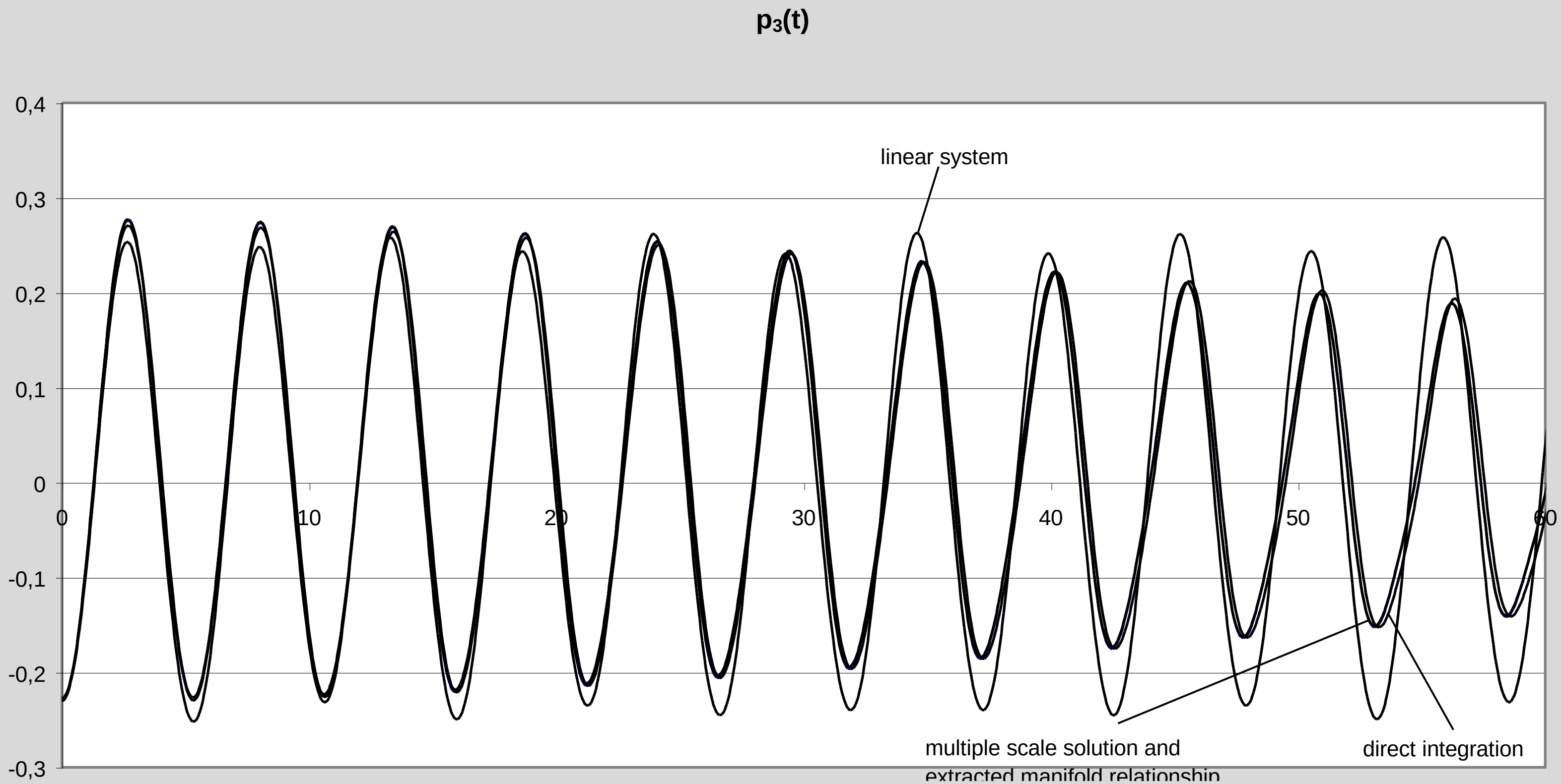
Non-linear Multimodes

Multiple time scales approach



Non-linear Multimodes

Multiple time scales approach



Non-linear Multimodes

Multiple time scales approach