THE PROBABILITY INTEGRAL OF THE RANGE IN SAMPLES OF n OBSERVATIONS FROM A NORMAL POPULATION

I. FOREWORD AND TABLES

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1. *Scope of the main table*

Denote by $x_1, x_2, \ldots, x_n$ a random sample of n observations, arranged in ascending order of magnitude, drawn from a normal or Gaussian population having for probability law

$$p(x) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \exp\left[-\tfrac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right] \tag{1}$$

where μ and σ are respectively the mean and standard deviation of the population. The range, sometimes described as the spread, in the sample is $x_n - x_1$ and we shall write the ratio of the range to the population standard deviation as

$$w = \frac{x_n - x_1}{\sigma} \tag{2}$$

No simple expression exists for the probability law $f_n(w)$ of w, but Table 1 below gives for specific values W of w computed values of the probability *integral*

$$P_n(W) = \int_0^W f_n(w)\,dw \tag{3}$$

This expression is the chance that the range in a sample of n observations is less than a given multiple of the population standard deviation. The table has been calculated for samples with n lying between 2 and 20 and for intervals of 0·05 of W. The values of the integral are given to 4-decimal place accuracy; linear interpolation is adequate except in the neighbourhood of the two quartiles in each column. The method of calculation is described below by Dr H. O. Hartley in a separate section.

2. *Auxiliary table for special uses* (Table 2)

When dealing with samples containing only a small number of observations the range or spread may often be usefully employed as a measure of dispersion in place either of the standard (root mean square) deviation or the mean deviation. For example, an estimate of the population standard deviation may be obtained by multiplying the range in a single sample or the mean range in a number of samples by the factors a_n shown in Table 2. The accuracy of this form of estimation of σ compared with that of other methods has been discussed by Davies & Pearson (1934).

In other circumstances it may be useful to plot in serial order on a control chart the values of range obtained from successive samples, e.g. when dealing with the control of quality in mass production. For this purpose it is necessary to know certain standard probability levels for w which will serve as control limits*. Twelve of these levels expressed as percentage points and obtained by interpolation in the main Table 1, are shown in Table 2. They replace approximate limits published a few years ago (Pearson, 1932). It will be found, however, that except in the case $n = 3$†, the discrepancies between the two tables are all small. As an example, the table shows that if samples of 7 observations are randomly drawn from a population with a standard deviation σ, then in the long run only 5 % of these should have a range greater than $4{\cdot}17\sigma$, while 95 % should satisfy the inequality

$$1{\cdot}25\sigma \leqslant x_n - x_1 \leqslant 4{\cdot}49\sigma$$

Probability levels for samples with $n > 12$ have not been included as the use of range for control purposes in larger samples is of doubtful value.

* For a discussion of the use of range in problems of industrial quality control see Reports issued by the British Standards Institution (Pearson, 1935, pp. 89–90; Dudding & Jennett, 1942).

† Correct values for the case $n = 3$ were given by McKay & Pearson (1933).

Table 1. *Probability integral of the range W in normal samples of size n*

W \ n	2	3	4	5	6	7	8	9	10
0·00	0·0000	0·0000							
0·05	·0282	·0007	0·0000						
0·10	·0564	·0028	·0001						
0·15	·0845	·0062	·0004	0·0000					
0·20	·1125	·0110	·0010	·0001					
0·25	0·1403	0·0171	0·0020	0·0002					
0·30	·1680	·0245	·0034	·0004	0·0000				
0·35	·1955	·0332	·0053	·0008	·0001				
0·40	·2227	·0431	·0079	·0014	·0002	0·0000			
0·45	·2497	·0543	·0111	·0022	·0004	·0001			
0·50	0·2763	0·0666	0·0152	0·0033	0·0007	0·0002	0·0000		
0·55	·3027	·0800	·0200	·0048	·0011	·0003	·0001		
0·60	·3286	·0944	·0257	·0068	·0017	·0004	·0001	0·0000	
0·65	·3542	·1099	·0323	·0092	·0025	·0007	·0002	·0001	
0·70	·3794	·1263	·0398	·0121	·0036	·0011	·0003	·0001	0·0000
0·75	0·4041	0·1436	0·0483	0·0157	0·0050	0·0016	0·0005	0·0002	0·0001
0·80	·4284	·1616	·0578	·0200	·0068	·0023	·0008	·0002	·0001
0·85	·4522	·1805	·0682	·0250	·0090	·0032	·0011	·0004	·0001
0·90	·4755	·2000	·0797	·0309	·0117	·0044	·0016	·0006	·0002
0·95	·4983	·2201	·0922	·0375	·0150	·0059	·0023	·0009	·0003
1·00	0·5205	0·2407	0·1057	0·0450	0·0188	0·0078	0·0032	0·0013	0·0005
1·05	·5422	·2618	·1201	·0535	·0234	·0101	·0043	·0018	·0008
1·10	·5633	·2833	·1355	·0629	·0287	·0129	·0057	·0025	·0011
1·15	·5839	·3051	·1517	·0733	·0348	·0163	·0075	·0035	·0016
1·20	·6039	·3272	·1688	·0847	·0417	·0203	·0098	·0047	·0022
1·25	0·6232	0·3495	0·1868	0·0970	0·0495	0·0250	0·0125	0·0062	0·0030
1·30	·6420	·3719	·2054	·1104	·0583	·0304	·0157	·0080	·0041
1·35	·6602	·3943	·2248	·1247	·0680	·0366	·0195	·0103	·0054
1·40	·6778	·4168	·2448	·1400	·0787	·0437	·0240	·0131	·0071
1·45	·6948	·4392	·2654	·1562	·0904	·0517	·0292	·0164	·0092
1·50	0·7112	0·4614	0·2865	0·1733	0·1031	0·0606	0·0353	0·0204	0·0117
1·55	·7269	·4835	·3080	·1913	·1168	·0705	·0422	·0250	·0148
1·60	·7421	·5053	·3299	·2101	·1316	·0814	·0499	·0304	·0184
1·65	·7567	·5269	·3521	·2296	·1473	·0934	·0587	·0366	·0227
1·70	·7707	·5481	·3745	·2498	·1639	·1064	·0684	·0437	·0277
1·75	0·7841	0·5690	0·3971	0·2706	0·1815	0·1204	0·0792	0·0517	0·0336
1·80	·7969	·5894	·4197	·2920	·2000	·1355	·0910	·0607	·0403
1·85	·8092	·6094	·4423	·3138	·2193	·1516	·1039	·0707	·0479
1·90	·8209	·6290	·4649	·3361	·2394	·1686	·1178	·0818	·0565
1·95	·8321	·6480	·4874	·3587	·2602	·1867	·1329	·0940	·0661
2·00	0·8427	0·6665	0·5096	0·3816	0·2816	0·2056	0·1489	0·1072	0·0768
2·05	·8528	·6845	·5317	·4046	·3035	·2254	·1661	·1216	·0886
2·10	·8624	·7019	·5534	·4277	·3260	·2460	·1842	·1371	·1015
2·15	·8716	·7187	·5748	·4508	·3489	·2673	·2033	·1536	·1156
2·20	·8802	·7349	·5957	·4739	·3720	·2893	·2232	·1712	·1307
2·25	0·8884	0·7505	0·6163	0·4969	0·3955	0·3118	0·2440	0·1899	0·1470
2·30	·8961	·7655	·6363	·5196	·4190	·3348	·2656	·2095	·1645
2·35	·9034	·7799	·6558	·5421	·4427	·3582	·2878	·2300	·1830
2·40	·9103	·7937	·6748	·5643	·4663	·3820	·3107	·2514	·2025
2·45	·9168	·8069	·6932	·5861	·4899	·4059	·3341	·2735	·2230
2·50	0·9229	0·8195	0·7110	0·6075	0·5132	0·4300	0·3579	0·2964	0·2443

Table 1 (cont.). *Probability integral of the range W in normal samples of size n*

W \ n	11	12	13	14	15	16	17	18	19	20
0·85	0·0000									
0·90	·0001									
0·95	·0001	0·0000								
1·00	0·0002	0·0001	0·0000							
1·05	·0003	·0001	·0001							
1·10	·0005	·0002	·0001	0·0000						
1·15	·0007	·0003	·0001	·0001						
1·20	·0010	·0005	·0002	·0001	0·0000					
1·25	0·0015	0·0007	0·0004	0·0002	0·0001	0·0000				
1·30	·0021	·0010	·0005	·0003	·0001	·0001	0·0000			
1·35	·0028	·0015	·0008	·0004	·0002	·0001	·0001	0·0000		
1·40	·0038	·0021	·0011	·0006	·0003	·0002	·0001	·0001		
1·45	·0051	·0028	·0016	·0009	·0005	·0003	·0001	·0001	0·0000	
1·50	0·0067	0·0038	0·0022	0·0012	0·0007	0·0004	0·0002	0·0001	0·0001	0·0000
1·55	·0087	·0051	·0030	·0017	·0010	·0006	·0003	·0002	·0001	·0001
1·60	·0111	·0067	·0040	·0024	·0014	·0008	·0005	·0003	·0002	·0001
1·65	·0140	·0086	·0053	·0032	·0020	·0012	·0007	·0004	·0003	·0002
1·70	·0175	·0111	·0069	·0043	·0027	·0017	·0011	·0007	·0004	·0003
1·75	0·0217	0·0140	0·0090	0·0058	0·0037	0·0024	0·0015	0·0010	0·0006	0·0004
1·80	·0266	·0175	·0115	·0075	·0049	·0032	·0021	·0013	·0009	·0006
1·85	·0323	·0217	·0145	·0097	·0065	·0043	·0028	·0019	·0012	·0008
1·90	·0388	·0266	·0182	·0124	·0084	·0057	·0039	·0026	·0018	·0012
1·95	·0463	·0323	·0225	·0156	·0108	·0075	·0052	·0035	·0024	·0017
2·00	0·0548	0·0389	0·0276	0·0195	0·0137	0·0097	0·0068	0·0048	0·0033	0·0023
2·05	·0643	·0465	·0335	·0241	·0173	·0124	·0088	·0063	·0045	·0032
2·10	·0749	·0550	·0403	·0295	·0215	·0156	·0114	·0082	·0060	·0043
2·15	·0866	·0646	·0481	·0357	·0264	·0196	·0144	·0106	·0078	·0057
2·20	·0994	·0753	·0569	·0429	·0323	·0242	·0181	·0136	·0102	·0076
2·25	0·1134	0·0872	0·0669	0·0511	0·0390	0·0297	0·0226	0·0172	0·0130	0·0099
2·30	·1286	·1003	·0779	·0605	·0468	·0361	·0279	·0215	·0165	·0127
2·35	·1450	·1145	·0902	·0709	·0556	·0435	·0340	·0265	·0207	·0161
2·40	·1625	·1300	·1037	·0825	·0655	·0519	·0411	·0325	·0256	·0202
2·45	·1811	·1466	·1183	·0953	·0766	·0615	·0493	·0394	·0315	·0251
2·50	0·2007	0·1644	0·1342	0·1094	0·0890	0·0722	0·0586	0·0474	0·0383	0·0309

Table 1 (cont.). *Probability integral of the range W in normal samples of size n*

W \ n	2	3	4	5	6	7	8	9	10
2·50	0·9229	0·8195	0·7110	0·6075	0·5132	0·4300	0·3579	0·2964	0·2443
2·55	·9286	·8315	·7282	·6283	·5364	·4541	·3820	·3198	·2665
2·60	·9340	·8429	·7448	·6487	·5592	·4782	·4064	·3437	·2894
2·65	·9390	·8537	·7607	·6685	·5816	·5022	·4309	·3680	·3130
2·70	·9438	·8640	·7759	·6877	·6036	·5259	·4555	·3927	·3372
2·75	0·9482	0·8737	0·7905	0·7063	0·6252	0·5494	0·4801	0·4175	0·3617
2·80	·9523	·8828	·8045	·7242	·6461	·5725	·5044	·4425	·3867
2·85	·9561	·8915	·8177	·7415	·6665	·5952	·5286	·4675	·4119
2·90	·9597	·8996	·8304	·7580	·6863	·6174	·5525	·4923	·4372
2·95	·9630	·9073	·8424	·7739	·7055	·6390	·5760	·5171	·4625
3·00	0·9661	0·9145	0·8537	0·7891	0·7239	0·6601	0·5991	0·5415	0·4878
3·05	·9690	·9212	·8645	·8036	·7416	·6806	·6216	·5656	·5129
3·10	·9716	·9275	·8746	·8174	·7587	·7003	·6436	·5892	·5378
3·15	·9741	·9334	·8842	·8305	·7750	·7194	·6649	·6124	·5623
3·20	·9763	·9388	·8931	·8429	·7905	·7377	·6856	·6350	·5864
3·25	0·9784	0·9439	0·9016	0·8546	0·8053	0·7553	0·7055	0·6569	0·6099
3·30	·9804	·9487	·9095	·8657	·8194	·7721	·7248	·6782	·6329
3·35	·9822	·9531	·9168	·8761	·8327	·7881	·7432	·6988	·6553
3·40	·9838	·9572	·9237	·8859	·8454	·8034	·7609	·7186	·6769
3·45	·9853	·9609	·9302	·8951	·8573	·8179	·7778	·7376	·6978
3·50	0·9867	0·9644	0·9361	0·9037	0·8685	0·8316	0·7939	0·7558	0·7180
3·55	·9879	·9677	·9417	·9117	·8790	·8446	·8091	·7732	·7373
3·60	·9891	·9706	·9468	·9192	·8889	·8568	·8236	·7898	·7558
3·65	·9901	·9734	·9516	·9261	·8981	·8683	·8372	·8055	·7735
3·70	·9911	·9759	·9559	·9326	·9067	·8790	·8501	·8204	·7902
3·75	0·9920	0·9782	0·9600	0·9386	0·9148	0·8891	0·8622	0·8345	0·8062
3·80	·9928	·9803	·9637	·9441	·9222	·8985	·8736	·8477	·8212
3·85	·9935	·9822	·9672	·9493	·9291	·9073	·8842	·8602	·8355
3·90	·9942	·9839	·9703	·9540	·9355	·9155	·8941	·8718	·8488
3·95	·9948	·9856	·9732	·9583	·9415	·9230	·9034	·8827	·8614
4·00	0·9953	0·9870	0·9758	0·9623	0·9469	0·9300	0·9120	0·8929	0·8731
4·05	·9958	·9883	·9782	·9660	·9519	·9365	·9199	·9024	·8841
4·10	·9963	·9895	·9804	·9693	·9566	·9425	·9273	·9112	·8943
4·15	·9967	·9906	·9824	·9724	·9608	·9480	·9341	·9193	·9038
4·20	·9970	·9916	·9842	·9752	·9647	·9530	·9404	·9269	·9126
4·25	0·9974	0·9925	0·9859	0·9777	0·9682	0·9576	0·9461	0·9338	0·9208
4·30	·9976	·9933	·9874	·9800	·9715	·9619	·9514	·9402	·9283
4·35	·9979	·9941	·9887	·9821	·9744	·9657	·9562	·9460	·9352
4·40	·9981	·9947	·9899	·9840	·9771	·9692	·9607	·9514	·9416
4·45	·9984	·9953	·9910	·9857	·9795	·9724	·9647	·9563	·9474
4·50	0·9985	0·9958	0·9920	0·9873	0·9817	0·9754	0·9684	0·9608	0·9527
4·55	·9987	·9963	·9929	·9887	·9837	·9780	·9717	·9649	·9575
4·60	·9989	·9967	·9937	·9899	·9855	·9804	·9747	·9686	·9620
4·65	·9990	·9971	·9944	·9911	·9871	·9825	·9775	·9719	·9660
4·70	·9991	·9974	·9951	·9921	·9885	·9845	·9799	·9750	·9696
4·75	0·9992	0·9977	0·9956	0·9930	0·9898	0·9862	0·9822	0·9777	0·9729
4·80	·9993	·9980	·9962	·9938	·9910	·9878	·9842	·9802	·9759
4·85	·9994	·9983	·9966	·9945	·9920	·9892	·9860	·9824	·9786
4·90	·9995	·9985	·9970	·9952	·9930	·9904	·9876	·9844	·9810
4·95	·9995	·9987	·9974	·9958	·9938	·9916	·9890	·9862	·9832
5·00	0·9996	0·9988	0·9977	0·9963	0·9946	0·9926	0·9903	0·9878	0·9851

Table 1 (cont.). *Probability integral of the range W in normal samples of size n*

W \ n	11	12	13	14	15	16	17	18	19	20
2·50	0·2007	0·1644	0·1342	0·1094	0·0890	0·0722	0·0586	0·0474	0·0383	0·0309
2·55	·2213	·1833	·1514	·1247	·1026	·0842	·0690	·0565	·0462	·0377
2·60	·2429	·2033	·1697	·1413	·1174	·0974	·0807	·0668	·0552	·0455
2·65	·2653	·2243	·1891	·1591	·1336	·1120	·0937	·0783	·0654	·0545
2·70	·2885	·2462	·2096	·1780	·1509	·1278	·1080	·0911	·0768	·0647
2·75	0·3124	0·2690	0·2311	0·1981	0·1696	0·1449	0·1236	0·1053	0·0896	0·0761
2·80	·3368	·2926	·2536	·2194	·1894	·1632	·1405	·1208	·1037	·0889
2·85	·3617	·3169	·2770	·2416	·2103	·1829	·1587	·1376	·1192	·1031
2·90	·3870	·3417	·3011	·2647	·2324	·2036	·1782	·1558	·1360	·1186
2·95	·4126	·3670	·3258	·2887	·2554	·2255	·1989	·1752	·1542	·1355
3·00	0·4382	0·3927	0·3512	0·3134	0·2792	0·2484	0·2207	0·1959	0·1737	0·1538
3·05	·4639	·4186	·3769	·3387	·3039	·2723	·2436	·2178	·1944	·1734
3·10	·4895	·4446	·4029	·3645	·3292	·2970	·2675	·2407	·2164	·1943
3·15	·5150	·4706	·4292	·3907	·3551	·3224	·2923	·2647	·2394	·2164
3·20	·5401	·4965	·4555	·4171	·3814	·3483	·3177	·2895	·2635	·2396
3·25	0·5649	0·5222	0·4817	0·4437	0·4081	0·3748	0·3438	0·3151	0·2885	0·2638
3·30	·5893	·5475	·5078	·4703	·4348	·4016	·3704	·3413	·3142	·2890
3·35	·6131	·5725	·5337	·4967	·4617	·4286	·3974	·3681	·3407	·3150
3·40	·6363	·5970	·5592	·5230	·4885	·4557	·4246	·3953	·3677	·3417
3·45	·6589	·6209	·5842	·5489	·5151	·4827	·4519	·4227	·3950	·3689
3·50	0·6807	0·6442	0·6087	0·5744	0·5413	0·5096	0·4792	0·4502	0·4226	0·3964
3·55	·7017	·6668	·6326	·5994	·5672	·5362	·5063	·4777	·4504	·4242
3·60	·7220	·6886	·6558	·6237	·5926	·5624	·5332	·5051	·4781	·4522
3·65	·7414	·7096	·6782	·6474	·6173	·5881	·5596	·5321	·5056	·4801
3·70	·7600	·7298	·6998	·6704	·6414	·6132	·5856	·5588	·5329	·5078
3·75	0·7776	0·7491	0·7206	0·6925	0·6648	0·6376	0·6110	0·5850	0·5598	0·5352
3·80	·7944	·7675	·7406	·7138	·6873	·6613	·6357	·6106	·5861	·5622
3·85	·8103	·7850	·7596	·7342	·7090	·6841	·6596	·6355	·6118	·5887
3·90	·8254	·8016	·7777	·7537	·7298	·7061	·6827	·6596	·6369	·6145
3·95	·8395	·8173	·7948	·7723	·7497	·7273	·7050	·6829	·6611	·6397
4·00	0·8528	0·8321	0·8111	0·7899	0·7686	0·7474	0·7263	0·7053	0·6845	0·6640
4·05	·8653	·8460	·8264	·8065	·7866	·7666	·7466	·7268	·7070	·6874
4·10	·8769	·8590	·8408	·8223	·8036	·7848	·7660	·7472	·7285	·7099
4·15	·8878	·8712	·8543	·8371	·8196	·8021	·7844	·7667	·7491	·7315
4·20	·8978	·8826	·8669	·8509	·8347	·8183	·8018	·7852	·7686	·7520
4·25	0·9072	0·8931	0·8787	0·8639	0·8488	0·8336	0·8182	0·8027	0·7871	0·7715
4·30	·9159	·9029	·8896	·8760	·8620	·8479	·8336	·8191	·8046	·7899
4·35	·9238	·9120	·8998	·8872	·8744	·8613	·8480	·8345	·8210	·8073
4·40	·9312	·9204	·9092	·8976	·8858	·8737	·8614	·8490	·8364	·8237
4·45	·9379	·9281	·9178	·9073	·8964	·8853	·8740	·8625	·8508	·8391
4·50	0·9441	0·9352	0·9258	0·9162	0·9062	0·8960	0·8856	0·8750	0·8643	0·8534
4·55	·9498	·9417	·9332	·9244	·9153	·9060	·8964	·8867	·8768	·8667
4·60	·9550	·9476	·9399	·9319	·9236	·9151	·9064	·8975	·8884	·8791
4·65	·9597	·9530	·9460	·9388	·9313	·9235	·9155	·9074	·8991	·8906
4·70	·9640	·9579	·9516	·9451	·9383	·9312	·9240	·9165	·9090	·9012
4·75	0·9678	0·9624	0·9567	0·9508	0·9446	0·9383	0·9317	0·9249	0·9180	0·9110
4·80	·9713	·9665	·9614	·9560	·9505	·9447	·9387	·9326	·9264	·9199
4·85	·9745	·9702	·9656	·9608	·9558	·9505	·9452	·9396	·9340	·9281
4·90	·9774	·9735	·9694	·9650	·9605	·9559	·9510	·9460	·9409	·9356
4·95	·9799	·9765	·9728	·9689	·9649	·9607	·9563	·9518	·9472	·9424
5·00	0·9822	0·9791	0·9759	0·9724	0·9688	0·9650	0·9611	0·9571	0·9529	0·9486

Table 1 (cont.). *Probability integral of the range W in normal samples of size n*

W \ n	2	3	4	5	6	7	8	9	10
5·00	0·9996	0·9988	0·9977	0·9963	0·9946	0·9926	0·9903	0·9878	0·9851
5·05	·9996	·9990	·9980	·9967	·9952	·9935	·9915	·9893	·9869
5·10	·9997	·9991	·9982	·9971	·9958	·9942	·9925	·9906	·9884
5·15	·9997	·9992	·9985	·9975	·9963	·9950	·9934	·9917	·9898
5·20	·9998	·9993	·9986	·9978	·9968	·9956	·9942	·9927	·9911
5·25	0·9998	0·9994	0·9988	0·9981	0·9972	0·9961	0·9949	0·9936	0·9922
5·30	·9998	·9995	·9990	·9983	·9975	·9966	·9956	·9944	·9931
5·35	·9998	·9995	·9991	·9985	·9979	·9971	·9961	·9951	·9940
5·40	·9999	·9996	·9992	·9987	·9981	·9974	·9966	·9957	·9948
5·45	·9999	·9997	·9993	·9989	·9984	·9978	·9971	·9963	·9954
5·50	0·9999	0·9997	0·9994	0·9991	0·9986	0·9981	0·9975	0·9968	0·9960
5·55	·9999	·9997	·9995	·9992	·9988	·9983	·9978	·9972	·9965
5·60	·9999	·9998	·9996	·9993	·9989	·9985	·9981	·9976	·9970
5·65	·9999	·9998	·9996	·9994	·9991	·9987	·9983	·9979	·9974
5·70	0·9999	·9998	·9997	·9995	·9992	·9989	·9986	·9982	·9977
5·75	1·0000	0·9999	0·9997	0·9995	0·9993	0·9991	0·9988	0·9984	0·9981
5·80		·9999	·9998	·9996	·9994	·9992	·9989	·9986	·9983
5·85		·9999	·9998	·9997	·9995	·9993	·9991	·9988	·9986
5·90		·9999	·9998	·9997	·9996	·9994	·9992	·9990	·9988
5·95		·9999	·9998	·9998	·9996	·9995	·9993	·9991	·9989
6·00		0·9999	0·9999	0·9998	0·9997	0·9996	0·9994	0·9993	0·9991
6·05		·9999	·9999	·9998	·9997	·9996	·9995	·9994	·9992
6·10		0·9999	·9999	·9998	·9998	·9997	·9996	·9995	·9993
6·15		1·0000	·9999	·9999	·9998	·9997	·9996	·9995	·9994
6·20			·9999	·9999	·9998	·9998	·9997	·9996	·9995
6·25			0·9999	0·9999	0·9999	0·9998	0·9997	0·9997	0·9996
6·30			0·9999	·9999	·9999	·9998	·9998	·9997	·9996
6·35			1·0000	·9999	·9999	·9999	·9998	·9998	·9997
6·40				0·9999	·9999	·9999	·9998	·9998	·9997
6·45				1·0000	·9999	·9999	·9999	·9998	·9998
6·50					0·9999	0·9999	0·9999	0·9999	0·9998
6·55					0·9999	·9999	·9999	·9999	·9998
6·60					1·0000	·9999	·9999	·9999	·9999
6·65						0·9999	·9999	·9999	·9999
6·70						1·0000	0·9999	·9999	·9999
6·75							1·0000	0·9999	0·9999
6·80								0·9999	·9999
6·85								1·0000	·9999
6·90									0·9999
6·95									1·0000
7·00									
7·05									
7·10									
7·15									
7·20									
7·25									

Table 1 (cont.). *Probability integral of the range W in normal samples of size n*

W \ n	11	12	13	14	15	16	17	18	19	20
5·00	0·9822	0·9791	0·9759	0·9724	0·9688	0·9650	0·9611	0·9571	0·9529	0·9486
5·05	·9843	·9815	·9786	·9756	·9723	·9690	·9655	·9618	·9581	·9543
5·10	·9861	·9837	·9811	·9784	·9755	·9725	·9694	·9661	·9628	·9593
5·15	·9878	·9856	·9833	·9809	·9783	·9757	·9729	·9700	·9670	·9639
5·20	·9893	·9874	·9853	·9832	·9809	·9785	·9760	·9735	·9708	·9681
5·25	0·9906	0·9889	0·9871	0·9852	0·9832	0·9811	0·9789	0·9766	0·9742	0·9718
5·30	·9917	·9903	·9887	·9870	·9852	·9833	·9814	·9794	·9773	·9751
5·35	·9928	·9915	·9901	·9886	·9870	·9854	·9836	·9819	·9800	·9781
5·40	·9937	·9925	·9913	·9900	·9886	·9872	·9856	·9841	·9824	·9807
5·45	·9945	·9935	·9924	·9912	·9900	·9888	·9874	·9860	·9846	·9831
5·50	0·9952	0·9943	0·9934	0·9924	0·9913	0·9902	0·9890	0·9878	0·9865	0·9852
5·55	·9958	·9951	·9942	·9933	·9924	·9914	·9904	·9893	·9882	·9870
5·60	·9964	·9957	·9950	·9942	·9934	·9925	·9916	·9907	·9897	·9887
5·65	·9969	·9963	·9956	·9950	·9943	·9935	·9927	·9919	·9910	·9901
5·70	·9973	·9968	·9962	·9956	·9950	·9944	·9937	·9929	·9922	·9914
5·75	0·9976	0·9972	0·9967	0·9962	0·9957	0·9951	0·9945	0·9939	0·9932	0·9925
5·80	·9980	·9976	·9972	·9967	·9963	·9958	·9952	·9947	·9941	·9935
5·85	·9982	·9979	·9976	·9972	·9968	·9963	·9959	·9954	·9949	·9944
5·90	·9985	·9982	·9979	·9976	·9972	·9968	·9964	·9960	·9956	·9952
5·95	·9987	·9985	·9982	·9979	·9976	·9973	·9969	·9966	·9962	·9958
6·00	0·9989	0·9987	0·9984	0·9982	0·9979	0·9977	0·9974	0·9971	0·9967	0·9964
6·05	·9990	·9989	·9987	·9984	·9982	·9980	·9977	·9975	·9972	·9969
6·10	·9992	·9990	·9989	·9987	·9985	·9983	·9981	·9978	·9976	·9973
6·15	·9993	·9992	·9990	·9989	·9987	·9985	·9983	·9981	·9979	·9977
6·20	·9994	·9993	·9992	·9990	·9989	·9987	·9986	·9984	·9982	·9980
6·25	0·9995	0·9994	0·9993	0·9992	0·9991	0·9989	0·9988	0·9986	0·9985	0·9983
6·30	·9996	·9995	·9994	·9993	·9992	·9991	·9990	·9988	·9987	·9986
6·35	·9996	·9996	·9995	·9994	·9993	·9992	·9991	·9990	·9989	·9988
6·40	·9997	·9996	·9996	·9995	·9994	·9993	·9992	·9992	·9991	·9990
6·45	·9997	·9997	·9996	·9996	·9995	·9994	·9994	·9993	·9992	·9991
6·50	0·9998	0·9997	0·9997	0·9996	0·9996	0·9995	0·9995	0·9994	0·9993	0·9993
6·55	·9998	·9998	·9997	·9997	·9996	·9996	·9995	·9995	·9994	·9994
6·60	·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995
6·65	·9999	·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995
6·70	·9999	·9999	·9998	·9998	·9998	·9998	·9997	·9997	·9997	·9996
6·75	0·9999	0·9999	0·9999	0·9999	0·9998	0·9998	0·9998	0·9997	0·9997	0·9997
6·80	·9999	·9999	·9999	·9999	·9998	·9998	·9998	·9998	·9998	·9997
6·85	·9999	·9999	·9999	·9999	·9999	·9999	·9998	·9998	·9998	·9998
6·90	0·9999	·9999	·9999	·9999	·9999	·9999	·9999	·9998	·9998	·9998
6·95	1·0000	0·9999	·9999	·9999	·9999	·9999	·9999	·9999	·9999	·9998
7·00		1·0000	0·9999	0·9999	0·9999	0·9999	0·9999	0·9999	0·9999	0·9999
7·05			0·9999	0·9999	·9999	·9999	·9999	·9999	·9999	·9999
7·10			1·0000	1·0000	0·9999	0·9999	·9999	·9999	·9999	·9999
7·15					1·0000	1·0000	0·9999	0·9999	·9999	·9999
7·20							1·0000	1·0000	0·9999	0·9999
7·25									1·0000	1·0000

3. *The origin of the present tables*

Tables giving the expected or mean value and the standard deviation of range in random samples from the normal population of equation (1) were calculated by L. H. C. Tippett (1925) in the Department of Applied Statistics, University College, London. Since the probability distribution $f_n(w)$ is itself far from normal in form, it was evident that its mean and standard deviation alone would not provide all the information generally needed in practice. Tippett included in his paper some values of the constants β_1 and β_2 of the distribution and his work was extended by the present writer (Pearson, 1926, 1932) who

Table 2

Size of sample n	Factor a_n	Lower percentage points						Upper percentage points					
		0·1	0·5	1·0	2·5	5·0	10·0	10·0	5·0	2·5	1·0	0·5	0·1
2	0·8862	0·00	0·01	0·02	0·04	0·09	0·18	2·33	2·77	3·17	3·64	3·97	4·65
3	0·5908	0·06	0·13	0·19	0·30	0·43	0·62	2·90	3·31	3·68	4·12	4·42	5·06
4	0·4857	0·20	0·34	0·43	0·59	0·76	0·98	3·24	3·63	3·98	4·40	4·69	5·31
5	0·4299	0·37	0·55	0·66	0·85	1·03	1·26	3·48	3·86	4·20	4·60	4·89	5·48
6	0·3946	0·54	0·75	0·87	1·06	1·25	1·49	3·66	4·03	4·36	4·76	5·03	5·62
7	0·3698	0·69	0·92	1·05	1·25	1·44	1·68	3·81	4·17	4·49	4·88	5·15	5·73
8	0·3512	0·83	1·08	1·20	1·41	1·60	1·83	3·93	4·29	4·61	4·99	5·26	5·82
9	0·3367	0·96	1·21	1·34	1·55	1·74	1·97	4·04	4·39	4·70	5·08	5·34	5·90
10	0·3249	1·08	1·33	1·47	1·67	1·86	2·09	4·13	4·47	4·79	5·16	5·42	5·97
11	0·3152	1·20	1·45	1·58	1·78	1·97	2·20	4·21	4·55	4·86	5·23	5·49	6·04
12	0·3069	1·30	1·55	1·68	1·88	2·07	2·30	4·29	4·62	4·92	5·29	5·54	6·09

Estimate of $\sigma = a_n \times$ range (or mean range) in a sample of n observations.

developed an approximate method of determining probability levels for w and provided some provisional tables of these. The need has, however, been felt for some time for a full and accurate table of the probability integral of the range to fit into place among other fundamental tables associated with the normal distribution. The completion of this objective has been made possible by a grant from the Department of Scientific and Industrial Research, whose assistance in the matter is acknowledged with warm appreciation. The actual method of computation was planned by Dr H. O. Hartley and the calculations were carried out under his supervision by Scientific Computing Service, Ltd. The scope of the main table was limited to $n \leqslant 20$. As n increases beyond this value there is an increasing risk that the table may be misleading in practice, since $f_n(w)$ becomes very sensitive to relatively slight departures from normality in the tails of the population distribution.

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II. NUMERICAL EVALUATION OF THE PROBABILITY INTEGRAL

By H. O. HARTLEY

The formula used for the tabulation of the probability integral $P_n(W)$ of the range in normal samples of size n is given in the paper printed on pp. 334–48 below, where it proved that

$$P_n(W) = \left(\int_{-\frac{1}{2}W}^{+\frac{1}{2}W} z(x)\,dx\right)^n + 2n\int_{\frac{1}{2}W}^{\infty} z(u)\left(\int_{u-W}^{u} z(x)\,dx\right)^{n-1} du, \tag{1}$$

where $$z(x) = (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}x^2}.$$

Certain properties of this formula and the facilities provided by certain modern calculating machines make this integral amenable to tabulation.

The main work consists in the evaluation of the integral

$$2n\int_{\frac{1}{2}W}^{\infty} z(u)\left(\int_{u-W}^{u} z(x)\,dx\right)^{n-1} du, \tag{2}$$

by quadrature for a two variable network of values of n and W. The range of integration is from $\frac{1}{2}W$ up to a point where the integrand

$$z(u)\left(\int_{u-W}^{u} z(x)\,dx\right)^{n-1} \tag{3}$$

vanishes to 7-decimal accuracy.* For each point of the network n, W, therefore, the integrand (3) was tabulated for a set of equidistant values of u covering the range of integration. The interval of integration was chosen as $\Delta u = 0{\cdot}2$ throughout. This was sufficient to obtain about 5-decimal accuracy in the integral (2).

The interval in W was taken as wide as possible but sufficiently fine to permit checking by differencing and the subsequent subtabulation of $P_n(W)$ to interval 0·05, which is the interval in the final table. An interval of $\Delta W = 0{\cdot}25$ was therefore chosen for the n, W network.

For small values of W it was necessary to tabulate the integrand for all integers n for which $P_n(W)$ is required in the final table. For larger values of n and W, however, it was sufficient to calculate the final integral (1) for odd n and to obtain intermediate values by interpolation. Below, then, is shown the two variable network for which the integral (2) was produced by quadrature:

$$\left.\begin{array}{lll} W = 0{\cdot}00\ (0{\cdot}25)\ 1{\cdot}25 & \text{and} & n = 3\ (1)\ 20. \\ W = 1{\cdot}50\ (0{\cdot}25)\ 2{\cdot}75 & & n = 3\ (1)\ \ 9\ (2)\ 23. \\ W = 3{\cdot}00\ (0{\cdot}25)\ 3{\cdot}25 & & n = 3\ (1)\ \ 5\ (2)\ 23. \\ W = 3{\cdot}50\ (0{\cdot}25)\ 8{\cdot}00 & & n = 3\ (2)\ 23. \end{array}\right\} \tag{4}$$

For $n = 2$ the final integral $P_2(W)$ is given directly by the normal integral and may be obtained by interpolation in Table II of *Tables for Statisticians and Biometricians*, Part I. Using the notation of that table (Sheppard's original notation) we have

$$P_2(W) = \alpha\left(\frac{W}{\sqrt{2}}\right).$$

Moreover, for purposes of interpolation, use was made of the formal relation

$$P_1(W) = 1 \quad \text{for} \quad W > 0.$$

For fixed u and W and for values of n in the arithmetic progression (4), the integrand (3) is a geometric progression with

$$z(u)\left(\int_{u-W}^{u} z(x)\,dx\right)^2$$

* The integrand was calculated to 7-decimal accuracy in order to obtain $P_n(W)$ to about 5-decimal accuracy.

as leading term and $\left(\int_{u-W}^{u} z(x)\,dx\right)$ or $\left(\int_{u-W}^{u} z(x)\,dx\right)^2$

as common ratio. This leading term as well as the common ratios were easily obtained from Table II of *Tables for Statisticians and Biometricians*, Part I and the terms of the progression were then automatically produced on a Mercedes calculating machine Model 38 M.S. and copied down in two-way tables with u as row heading, n as column heading and W as table heading. The values of the integrand were then checked by differencing column-wise and added to yield the main term of the integral (2). The correction terms which, according to Gregory's formula, convert the integrand-sum into the integral were calculated from the differences and checked by the application of Gauss' formula of integration. Finally, to obtain $P_n(W)$ the term

$$\left(\int_{-\frac{1}{2}W}^{+\frac{1}{2}W} z(x)\,dx\right)^n$$

was produced by continued multiplication on the Mercedes and added to the corresponding integral (2) to yield $P_n(W)$ for all points of the above network.

For odd values of n the integral $P_n(W)$ was then differenced W-wise on the National machine which, incidentally, produced column totals $\sum_W P_n(W)$ for these values of n. Two checks were applied at this stage. One consisted in inspecting the fourth order differences. As a second check, the mean range, $\overline{w}_n$, was calculated from the formula

$$\overline{w}_n = 8 - \int_0^8 P_n(w)\,dw,^*$$

and compared with the correct mean range given in Table XXII of *Tables for Statisticians and Biometricians*, Part II. Finally, the function $P_n(W)$ was subtabulated to interval 0·05 on the National machine by a method similar to that described in detail by L. J. Comrie (1936).

The values of $P_n(W)$ for even n were then obtained by interpolation with the help of two interpolation formulae of Lagrangian type:

$$\begin{aligned} 2048P_n(W) = &-5[P_{n-7}(W)+P_{n+7}(W)]+49[P_{n-5}(W)+P_{n+5}(W)] \\ &-245[P_{n-3}(W)+P_{n+3}(W)]+1225[P_{n-1}(W)+P_{n+1}(W)], \end{aligned} \tag{5}$$

$$\begin{aligned} 20P_n(W) = &\,P_{n-3}(W)+P_{n+3}(W)-6[P_{n-2}(W)+P_{n+2}(W)] \\ &+15[P_{n-1}(W)+P_{n+1}(W)]. \end{aligned} \tag{6}$$

Formula (5) yields the interpolate for even n from the given values of $P_n(W)$ at adjacent odd values of n. This formula was used throughout. In some cases, however, the resulting interpolate was accurate to about 3 places of decimals only. In such cases values of P_{n-3}, P_{n+3}, P_{n-1}, P_{n+1} accurate to 5 places of decimals and values of P_{n-2}, P_{n+2} accurate to (say) 3 places of decimals were substituted in formula (6). This yielded a 'corrected value' of $P_n(W)$. The process was then repeated for $n = n+2$ and so on until all values of $P_n(W)$ had 'settled down' for even values of n. It is easy to see that the process is convergent and that the maximum error in the interpolate is 2 units for the 5th decimal.

After completion of the interpolation n-wise, the interpolates $P_n(W)$ for even n were differenced W-wise, checked and subtabulated as for odd values of n.

* This is true provided $P_n(8)=1$ to 6-decimal place accuracy.

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