$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r})$, and in that case the total energy $T + V$ is a constant. The condition for the existence of such a function is that $\boldsymbol{F}$ should be a function of position such that $\boldsymbol{\nabla} \wedge \boldsymbol{F} = \boldsymbol{0}$.

The rate of change of angular momentum of a particle is equal to the moment of the force, $\dot{\boldsymbol{J}} = \boldsymbol{r} \wedge \boldsymbol{F}$. When the force is central, the angular momentum is conserved. Then the motion is confined to a plane, and the rate of sweeping out area in this plane is a constant.

The use of these conservation laws greatly simplifies the treatment of any problem involving central or conservative forces. When the force is both central and conservative, they provide all the information we need to determine the motion of the particle, as we shall see in the following chapter.

Lagrange's equations are of great importance in advanced treatments of classical mechanics (and also in quantum mechanics). We have seen that they can be used to write down equations of motion in any system of co-ordinates, as soon as we have found the expressions for the kinetic energy and potential energy. In later chapters, we shall see that the method can readily and usefully be extended to more complicated systems than a single particle.

Problems

1. Find which of the following forces are conservative, and for those that are find the corresponding potential energy function (a and b are constants, and $\boldsymbol{a}$ is a constant vector):
   (a) $F_x = ax + by^2, \quad F_y = az + 2bxy, \quad F_z = ay + bz^2;$
   (b) $F_x = ay, \quad F_y = az, \quad F_z = ax;$
   (c) $F_r = 2ar \sin\theta \sin\varphi, \quad F_\theta = ar \cos\theta \sin\varphi, \quad F_\varphi = ar \cos\varphi;$
   (d) $\boldsymbol{F} = \boldsymbol{a} \wedge \boldsymbol{r};$
   (e) $\boldsymbol{F} = r\boldsymbol{a};$
   (f) $\boldsymbol{F} = \boldsymbol{a}(\boldsymbol{a} \cdot \boldsymbol{r}).$
2. Given that the force is as in Problem 1(a), evaluate the work done in taking a particle from the origin to the point $(1, 1, 0)$: (i) by moving first along the x-axis and then parallel to the y-axis, and (ii) by going in a straight line. Verify that the result in each case is equal to minus the change in the potential energy function.
3. Repeat the calculations of Problem 2 for the force in 1(b).
4. Compute the work done in taking a particle around the circle $x^2 + y^2 =$

$a^2, z = 0$ if the force is (a) $\boldsymbol{F} = y\boldsymbol{i}$, and (b) $\boldsymbol{F} = x\boldsymbol{i}$. What do you conclude about these forces? (Use the parametrization $x = a\cos\varphi, y = a\sin\varphi, z = 0$.)

5. Evaluate the force corresponding to the potential energy function $V(\boldsymbol{r}) = cz/r^3$, where c is a constant. Write your answer in vector notation, and also in spherical polars, and verify that it satisfies $\boldsymbol{\nabla} \wedge \boldsymbol{F} = \boldsymbol{0}$.
6. A projectile is launched with velocity $100\,\mathrm{m\,s^{-1}}$ at $60°$ to the horizontal. Atmospheric drag is negligible. Find the maximum height attained and the range. What other angle of launch would give the same range? Find the time of flight in each of the two cases.
7. *Find the equation for the trajectory of a projectile launched with velocity v at an angle α to the horizontal, assuming negligible atmospheric resistance. Given that the ground slopes at an angle β, show that the range of the projectile (measured horizontally) is

$$x = \frac{2v^2}{g}\frac{\sin(\alpha-\beta)\cos\alpha}{\cos\beta}.$$

At what angle should the projectile be launched to achieve the maximum range?
8. *By expanding the logarithm in (3.17), find the approximate equation for the trajectory of a projectile subject to small atmospheric drag to first order in γ. (Note that this requires terms up to order γ^3 in the logarithm.) Show that to this order the range (on level ground) is

$$x = \frac{2uw}{g} - \frac{8\gamma uw^2}{3g^2},$$

and hence that to maximize the range for given launch speed v the angle of launch should be chosen to satisfy $\cos 2\alpha = \sqrt{2}\gamma v/3g$. (*Hint*: In the term containing γ, you may use the zeroth-order approximation for the angle.) For a projectile whose terminal speed if dropped from rest (see Chapter 2, Problem 13) would be $500\,\mathrm{m\,s^{-1}}$, estimate the optimal angle and the range if the launch speed is $100\,\mathrm{m\,s^{-1}}$.
9. *Show that in the limit of strong damping (large γ) the time of flight of a projectile (on level ground) is approximately $t \approx (w/g + 1/\gamma)(1 - \mathrm{e}^{-1-\gamma w/g})$. Show that to the same order of accuracy the range is $x \approx (u/\gamma)(1 - e^{-1-\gamma w/g})$. For a projectile launched at $800\,\mathrm{m\,s^{-1}}$ with $\gamma = 0.1\,\mathrm{s^{-1}}$, estimate the range for launch angles of $30°, 20°$ and $10°$.
10. A particle of mass m is attached to the end of a light string of length l. The other end of the string is passed through a small hole and is

slowly pulled through it. Gravity is negligible. The particle is originally spinning round the hole with angular velocity ω. Find the angular velocity when the string length has been reduced to $\frac{1}{2}l$. Find also the tension in the string when its length is r, and verify that the increase in kinetic energy is equal to the work done by the force pulling the string through the hole.

11. A particle of mass m is attached to the end of a light spring of equilibrium length a, whose other end is fixed, so that the spring is free to rotate in a horizontal plane. The tension in the spring is k times its extension. Initially the system is at rest and the particle is given an impulse that starts it moving at right angles to the spring with velocity v. Write down the equations of motion in polar co-ordinates. Given that the maximum radial distance attained is $2a$, use the energy and angular momentum conservation laws to determine the velocity at that point, and to find v in terms of the various parameters of the system. Find also the values of $\ddot{r}$ when $r = a$ and when $r = 2a$.
12. *A light rigid cylinder of radius $2a$ is able to rotate freely about its axis, which is horizontal. A particle of mass m is fixed to the cylinder at a distance a from the axis and is initially at rest at its lowest point. A light string is wound on the cylinder, and a steady tension F applied to it. Find the angular acceleration and angular velocity of the cylinder after it has turned through an angle θ. Show that there is a limiting tension F_0 such that if $F < F_0$ the motion is oscillatory, but if $F > F_0$ it continues to accelerate. Estimate the value of F_0 by numerical approximation.
13. In the system of Problem 12, instead of a fixed tension applied to the string, a weight of mass $m/2$ is hung on it. Use the energy conservation equation to find the angular velocity as a function of θ. Find also the angular acceleration and the tension in the string. (Compare your results with those in Problem 12.) Show that there is a point at which the tension falls to zero, and find the angle at which this occurs. What happens immediately beyond this point?
14. A wedge-shaped block of mass M rests on a smooth horizontal table. A small block of mass m is placed on its upper face, which is also smooth and inclined at an angle α to the horizontal. The system is released from rest. Write down the horizontal component of momentum, and the kinetic energy of the system, in terms of the velocity v of the wedge and the velocity u of the small block relative to it. Using conservation of momentum and the equation for the rate of change of kinetic energy,

find the accelerations of the blocks. Given that $M = 1\,\mathrm{kg}$ and $m = 250\,\mathrm{g}$, find the angle α that will maximize the acceleration of the wedge.

15. *A particle starts from rest and slides down a smooth curve under gravity. Find the shape of the curve that will minimize the time taken between two given points. [Take the origin as the starting point and the z axis downwards. Show that the time taken is

$$\int_0^{z_1} \left[\frac{1 + (\mathrm{d}x/\mathrm{d}z)^2}{2gz}\right]^{1/2} \mathrm{d}z,$$

and hence that for a minimum

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = \frac{c^2 z}{1 - c^2 z},$$

where c is an integration constant. To complete the integration, use the substitution $z = c^{-2}\sin^2\theta$. This famous curve is known as the *brachistochrone*. It is in fact an example of a *cycloid*, the locus of a point on the rim of a circle of radius $1/2c^2$ being rolled beneath the x-axis.

This problem was first posed by Johann Bernoulli on New Year's Day 1697 as an open challenge. Newton's brilliant solution method initiated the calculus of variations. Bernoulli had an equally brilliant idea, using an optical analogy with refraction of a light ray through a sequence of plates and Fermat's principle of least time.]

16. *The position on the surface of a cone of semi-vertical angle α is specified by the distance r from the vertex and the azimuth angle φ about the axis. Show that the shortest path (or *geodesic*) along the surface between two given points is specified by a function $r(\varphi)$ obeying the equation

$$r\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} - 2\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - r^2\sin^2\alpha = 0.$$

Show that the solution is $r = r_0 \sec[(\varphi - \varphi_0)\sin\alpha]$, where r_0 and φ_0 are constants. [The equation may be solved by the standard technique of introducing a new dependent variable $u = \mathrm{d}r/\mathrm{d}\varphi$, and writing

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2} = \frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}u}{\mathrm{d}r} = u\frac{\mathrm{d}u}{\mathrm{d}r}.$$

The substitution $u^2 = v$ then reduces the equation to a linear form that may be solved by using an integrating factor (see Chapter 2, Problem

13). Finally, to calculate $\varphi = \int (1/u)\mathrm{d}r$, one may use the substitution $r = 1/x$.]

17. *Find the geodesics on a sphere of unit radius. [*Hint*: Use θ as independent variable, and look for the path $\varphi = \varphi(\theta)$. To perform the integration, use the substitution $x = \cot\theta$.]
18. **Parabolic co-ordinates* (ξ, η) in a plane are defined by $\xi = r + x, \eta = r - x$. Find x and y in terms of ξ and η. Show that the kinetic energy of a particle of mass m is

$$T = \frac{m}{8}(\xi + \eta)\left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta}\right).$$

Hence find the equations of motion.

19. Write down the equations of motion in polar co-ordinates for a particle of unit mass moving in a plane under a force with potential energy function $V = -k\ln r + cr + gr\cos\theta$, where k, c and g are positive constants. Find the positions of equilibrium (a) if $c > g$, and (b) if $c < g$. By considering the equations of motion near these points, determine whether the equilibrium is stable (*i.e.*, will the particle, if given a small displacement, tend to return repeatedly?).
20. If q_1, q_2, q_3 are orthogonal curvilinear co-ordinates, and the element of length in the q_i direction is $h_i \mathrm{d}q_i$, write down (a) the kinetic energy T in terms of the generalized velocities $\dot{q}_i$, (b) the generalized momentum p_i and (c) the component $\boldsymbol{e}_i \cdot \boldsymbol{p}$ of the momentum vector $\boldsymbol{p}$ in the q_i direction. (Here $\boldsymbol{e}_i$ is a unit vector in the direction of increasing q_i.)
21. *By comparing the Euler–Lagrange equations with the corresponding components of the equation of motion $m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla}V$, show that the component of the acceleration vector in the q_i direction is

$$\boldsymbol{e}_i \cdot \ddot{\boldsymbol{r}} = \frac{1}{mh_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right].$$

Use this result to identify the components of the acceleration in cylindrical and spherical polars.

22. For the case of plane polar co-ordinates r, θ, write the unit vectors $\boldsymbol{e}_r$ $(= \hat{\boldsymbol{r}})$ and $\boldsymbol{e}_\theta$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$. Hence show that $\partial\boldsymbol{e}_r/\partial\theta = \boldsymbol{e}_\theta$ and $\partial\boldsymbol{e}_\theta/\partial\theta = -\boldsymbol{e}_r$. By starting with $\boldsymbol{r} = r\boldsymbol{e}_r$ and differentiating, rederive the expressions for the components of the velocity and acceleration vectors.
23. *Find the corresponding formulae for $\partial\boldsymbol{e}_i/\partial q_j$ for spherical polar co-ordinates, and hence verify the results obtained in Problem 21.

24. *The motion of a particle in a plane may be described in terms of *elliptic co-ordinates* λ, θ defined by

$$x = c\cosh\lambda\cos\theta, \qquad y = c\sinh\lambda\sin\theta, \qquad (\lambda \geq 0,\ 0 \leq \theta \leq 2\pi),$$

where c is a positive constant. Show that the kinetic energy function may be written

$$T = \tfrac{1}{2}mc^2(\cosh^2\lambda - \cos^2\theta)(\dot{\lambda}^2 + \dot{\theta}^2).$$

Hence write down the equations of motion.

25. *The method of *Lagrange multipliers* (see Appendix A, Problem 11) can be extended to the calculus of variations. To find the maxima and minima of an integral I, subject to the condition that another integral $J = 0$, we have to find the stationary points of the integral $I - \lambda J$ under variations of the function $y(x)$ *and* of the parameter λ. Apply this method to find the *catenary*, the shape in which a uniform heavy chain hangs between two fixed supports. [The required shape is the one that minimizes the total potential energy, subject to the condition that the total length is fixed. Show that this leads to a variational problem with

$$f(y, y') = (y - \lambda)\sqrt{1 + y'^2},$$

and hence to the equation

$$(y - \lambda)y'' = 1 + y'^2.$$

Solve this equation by introducing the new variable $u = y'$ and solving for $u(y)$.]

26. *A curve of given total length is drawn in a plane between the points $(\pm a, 0)$. Using the method of Problem 25, find the shape that will enclose the largest possible area between the curve and the x-axis.

4.8 Summary

For a particle moving under any central, conservative force, information about the radial motion may be obtained from the radial energy equation, which results from eliminating $\dot{\theta}$ between the conservation equations for energy and angular momentum. The values of E and J can be determined from the initial conditions, and this equation then tells us the radial velocity at any value of r.

When information about the angle θ is needed, we must find the equation of the orbit. For the inverse square law, the orbit is an ellipse or a hyperbola, according as $E < 0$ or $E > 0$. The semi-major axis is fixed by E, and the semi-*latus rectum* by J.

If we are concerned with finding the time taken to traverse part of the orbit, we can use the relation between the angular momentum and the rate of sweeping out area.

When a beam of particles strikes a target, the angular distribution of scattered particles may be found from the differential cross-section $\mathrm{d}\sigma/\mathrm{d}\Omega$. This may be calculated from a knowledge of the relation between the scattering angle and the impact parameter. The attenuation of the beam is related to the total cross-section σ, obtained by integrating $\mathrm{d}\sigma/\mathrm{d}\Omega$ over all solid angles.

Problems

1. The orbits of synchronous communications satellites have been chosen so that viewed from the Earth they appear to be stationary. Find the radius of the orbits.
2. Find the radii of synchronous orbits about Jupiter and about the Sun. [Their mean rotation periods are 10 hours and 27 days, respectively. The mass of Jupiter is 318 times that of the Earth. The semi-major axis of the Earth's orbit, or *astronomical unit* (AU) is 1.50×10^8 km.]
3. The semi-major axis of Jupiter's orbit is 5.20 AU. Find its orbital period in years, and its mean (time-averaged) orbital speed. (Mean orbital speed of Earth = $29.8\,\mathrm{km\,s^{-1}}$.)
4. The orbit of an asteroid extends from the Earth's to Jupiter's, just touching both. Find its orbital period. (Treat the planetary orbits as circular and coplanar.)

5. Find the maximum and minimum orbital speeds of the asteroid in Problem 4.
6. The Moon's mass and radius are $0.0123\,M_{\mathrm{E}}$ and $0.273\,R_{\mathrm{E}}$ (E = Earth). For Jupiter the corresponding figures are $318\,M_{\mathrm{E}}$ and $11.0\,R_{\mathrm{E}}$. Find in each case the gravitational acceleration at the surface, and the escape velocity.
7. Calculate the period of a satellite in an orbit just above the Earth's atmosphere (whose thickness may be neglected). Find also the periods for close orbits around the Moon and Jupiter.
8. The Sun has an orbital speed of about $220\,\mathrm{km\,s^{-1}}$ around the centre of the Galaxy, whose distance is 28 000 light years. Estimate the total mass of the Galaxy in solar masses.
9. A particle of mass m moves under the action of a harmonic oscillator force with potential energy $\frac{1}{2}kr^2$. Initially, it is moving in a circle of radius a. Find the orbital speed v. It is then given a blow of impulse mv in a direction making an angle α with its original velocity. Use the conservation laws to determine the minimum and maximum distances from the origin during the subsequent motion. Explain your results physically for the two limiting cases $\alpha = 0$ and $\alpha = \pi$.
10. Write down the effective potential energy function $U(r)$ for the system described in Chapter 3, Problem 11. Initially, the particle is moving in a circular orbit of radius $2a$. Find the orbital angular velocity ω in terms of the natural angular frequency ω_0 of the oscillator when not rotating. If the motion is lightly disturbed, the particle will execute small oscillations about the circular orbit. By considering the effective potential energy function $U(r)$ near its minimum, find the angular frequency ω' of small oscillations. Hence describe the disturbed orbit qualitatively.
11. Show that the comet discussed at the end of §4.4 crosses the Earth's orbit at opposite ends of a diameter. Find the time it spends inside the Earth's orbit. (To evaluate the area required, write the equation of the orbit in Cartesian co-ordinates. See Appendix B.)
12. A star of mass M and radius R is moving with velocity v through a cloud of particles of density ρ. If all the particles that collide with the star are trapped by it, show that the mass of the star will increase at a rate

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \pi\rho v\left(R^2 + \frac{2GMR}{v^2}\right).$$

Given that $M = 10^{31}$ kg and $R = 10^8$ km, find how the effective cross-sectional area compares with the geometric cross-section πR^2 for velocities of $1000\,\mathrm{km\,s^{-1}}$, $100\,\mathrm{km\,s^{-1}}$ and $10\,\mathrm{km\,s^{-1}}$.

13. *Find the polar equation of the orbit of an isotropic harmonic oscillator by solving the differential equation (4.25), and verify that it is an ellipse with centre at the origin. (*Hint*: Change to the variable $v = u^2$.) Check also that the period is given correctly by $\tau = 2mA/J$.
14. Discuss qualitatively the orbits of a particle under a repulsive force with potential energy function $V = \frac{1}{2}kr^2$ where k is *negative*, using the effective potential energy function U. How would the orbit equation, as found in Problem 13, differ in this case? What shape is the orbit?
15. *If the Earth's orbit is divided in two by the *latus rectum*, show that the difference in time spent in the two halves, in years, is

$$\frac{2}{\pi}\left(e\sqrt{1-e^2} + \arcsin e\right),$$

and hence for small e about twice as large as the difference computed in the example in §4.4. (*Hint*: Use Cartesian co-ordinates to evaluate the required area. The identity $\pi/2 - \arcsin\sqrt{1-e^2} = \arcsin e$ may be useful.)
16. A spacecraft is to travel from Earth to Jupiter along an elliptical orbit that just touches each of the planetary orbits (*i.e.*, the orbit of the asteroid in Problem 4). Use the results of Problems 3, 4 and 5 to find the relative velocity of the spacecraft with respect to the Earth just after launching and that with respect to Jupiter when it nears that planet, neglecting in each case the gravitational attraction of the planet. Where in its orbit must Jupiter be at the time of launch, relative to the Earth? Where will the Earth be when it arrives?

[This semi-elliptical trajectory is known as a *Hohmann transfer* and it is energy-efficient for interplanetary travel using high-thrust rockets in that a discrete boost is required at the beginning and at the end of the journey in order respectively to leave Earth and to arrive at the target planet. A similarly careful choice of timing for initiating a return to Earth is necessary, so that there is an inevitable minimum time that must be spent in the region of (here) Jupiter before this can be done. It is an interesting and important exercise to compare timings for round trips of planetary exploration using this form of transfer. For Mars, an obvious target in the short term, the time required is about 32 Earth months with about 15 months spent at Mars. There are of course other ways of effecting transfer, but at rather greater cost, either in fuel or in time spent at the destination!]

17. *Suppose that the asteroid of Problems 4 and 5 approaches the Earth with an impact parameter of $5R_{\mathrm{E}}$, where R_{E} = Earth's radius, moving in the same plane and overtaking it. (This is an improbably close encounter for a large asteroid; however the spectacular impact of comet Shoemaker–Levy 9 with Jupiter in July 1994 should prevent us from being too complacent about this threat!) Find the distance of closest approach and the angle through which the asteroid is scattered, in the frame of reference in which the Earth is at rest. (Assume that the asteroid is small enough to have negligible effect on the Earth's orbit.) What is its new velocity v relative to the Sun? Show that the semi-major axis of its new orbit is $a_{\mathrm{E}}v_{\mathrm{E}}^2/(2v_{\mathrm{E}}^2 - v^2)$, where a_{E} and v_{E} are the Earth's orbital radius and orbital velocity. Find the asteroid's new orbital period.
18. *Show that the position of a planet in its elliptical orbit can be expressed, using a frame with x-axis in the direction of *perihelion* (point of closest approach to the Sun), in terms of an angular parameter ψ by $x = a(\cos\psi - e)$, $y = b\sin\psi$. (See Problem B.1. In the literature, ψ is sometimes called the *eccentric anomaly*, while the polar angle θ is the *true anomaly*.) Show that $r = a(1 - e\cos\psi)$, and that the time from perihelion is given by $t = (\tau/2\pi)(\psi - e\sin\psi)$ (Kepler's equation).
19. Use the parametrization of Problem 18 to calculate the time-averaged values of the kinetic and potential energies T and V over a complete period. Hence verify the *virial theorem*, $\langle V\rangle_{\mathrm{av}} = -2\langle T\rangle_{\mathrm{av}}$.
20. *Find a parametrization similar to that of Problem 18 for a hyperbolic orbit, using hyperbolic functions.
21. On reaching the vicinity of Jupiter, the spacecraft in Problem 16 is swung around the planet by its gravitational attraction — a 'sling-shot' manoeuvre. Consider this encounter in the frame of reference in which Jupiter is at rest. What is the magnitude and direction of the spacecraft's velocity before scattering? What is its magnitude after scattering? If the scattering angle in this frame is 90°, what must be the impact parameter? What is the distance of closest approach to the planet, in terms of Jupiter radii? ($M_{\mathrm{J}} = 318M_{\mathrm{E}}$, $R_{\mathrm{J}} = 11.0R_{\mathrm{E}}$.)
22. *If the manoeuvre in Problem 21 is in the orbital plane, so that the final velocity of the spacecraft relative to Jupiter is radially away from the Sun, what is its velocity in magnitude and direction relative to the Sun? Use the radial energy equation to determine the spacecraft's

farthest distance from the Sun (its *aphelion* distance) in astronomical units. Find also its new orbital period. When it returns, what will be its perihelion distance?

23. An alternative to the manoeuvre described in Problem 22 is for the spacecraft to be scattered out of the orbital plane. Assume that relative to Jupiter its velocity after scattering is directed normal to the orbital plane. What is its velocity relative to the Sun? What will be its aphelion distance and orbital period? How far from the orbital plane will it reach? (*Hint*: Immediately after scattering, the radial component of its velocity is zero. This is therefore the perihelion point of the new orbit. The farthest point from the original orbital plane will occur when it is at one end of the semi-minor axis of the orbit.)
24. *A ballistic rocket (one that moves freely under gravity after its initial launch) is fired from the surface of the Earth with velocity $v < \sqrt{Rg}$ at an angle α to the vertical. (Ignore the Earth's rotation.) Find the equation of its orbit. Express the range $2R\theta$ (measured along the Earth's surface) in terms of the parameters l and a, and hence show that to maximize the range, we should choose α so that $l = 2a - R$. (*Hint*: A sketch may help.) Deduce that the maximum range is $2R\theta$ where $\sin\theta = v^2/(2Rg - v^2)$. Given that the maximum range is 3600 nautical miles, find the launch velocity and the angle at which the rocket should be launched. (*Note*: 1 nautical mile = 1 minute of arc over the Earth's surface.)
25. Discuss the possible types of orbit for a particle moving under a central inverse-cube-law force, described by the potential energy function $V = k/2r^2$. For the repulsive case ($k > 0$), show that the orbit equation is $r\cos n(\theta - \theta_0) = b$, where n, b and θ_0 are constants. Show that for the attractive case ($k < 0$), the nature of the orbit depends on the signs of $J^2 + mk$ and of E. Find the equation of the orbit for each possible type. (Include the cases where one of these parameters vanishes.)
26. Show that the scattering angle for particles of mass m and initial velocity v scattered by a repulsive inverse-cube-law force is $\pi - \pi/n$ (see Problem 25). Hence find the differential cross-section.
27. *The potential energy of a particle of mass m is $V(r) = k/r + c/3r^3$, where $k < 0$ and c is a small constant. (The gravitational potential energy in the equatorial plane of the Earth has approximately this form, because of its flattened shape — see Chapter 6.) Find the angular velocity ω in a circular orbit of radius a, and the angular frequency ω' of small radial oscillations about this circular orbit. Hence show that a

nearly circular orbit is approximately an ellipse whose axes precess at an angular velocity $\Omega \approx (c/|k|a^2)\omega$.

28. A beam of particles strikes a wall containing 2×10^{29} atoms per m^3. Each atom behaves like a sphere of radius $3 \times 10^{-15}\,\mathrm{m}$. Find the thickness of wall that exactly half the particles will penetrate without scattering. What thickness would be needed to stop all but one particle in 10^6?
29. An α-particle of energy $4\,\mathrm{keV}$ ($1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$) is scattered by an aluminium atom through an angle of $90°$. Calculate the distance of closest approach to the nucleus. (Atomic number of α-particle $= 2$, atomic number of Al $= 13$, $e = 1.6 \times 10^{-19}\,\mathrm{C}$.) A beam of such particles with a flux of $3 \times 10^8\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ strikes a target containing $50\,\mathrm{mg}$ of aluminium. A detector of cross-sectional area $400\,\mathrm{mm}^2$ is placed $0.6\,\mathrm{m}$ from the target in a direction at right angles to the beam direction. Find the rate of detection of α-particles. (Atomic mass of Al $= 27\,\mathrm{u}$; $1\,\mathrm{u} = 1.66 \times 10^{-27}\,\mathrm{kg}$.)
30. *It was shown in §3.4 that Kepler's second law of planetary motion implies that the force is central. Show that his first law — that the orbit is an ellipse with the Sun at a focus — implies the inverse square law. (*Hint*: By differentiating the orbit equation $l/r = 1 + e\cos\theta$, and using (3.26), find $\dot{r}$ and $\ddot{r}$ in terms of r and θ. Hence calculate the radial acceleration.)
31. Show that Kepler's third law, $\tau \propto a^{3/2}$, implies that the force on a planet is proportional to its mass.
    [This law was originally expressed by Kepler as $\tau \propto \bar{r}^{3/2}$, where $\bar{r}$ is a 'mean value' of r. For an ellipse, the mean over *angle* θ is in fact b; the mean over *time* is actually $a(1 + \frac{1}{2}e^2)$; it is the mean over *arc length* — or the median — which is given by a! Of course, for most planets in our Solar System these values are almost equal.]