frame, together with the physical assertion that such frames exist, while the second and third laws contain definitions of mass and force. These laws, supplemented by the laws of force, such as the law of universal gravitation, provide the equations from which we can determine the motion of any dynamical system.

Problems

Note. Here and in later chapters, starred problems are somewhat harder.

1. An object A moving with velocity $\boldsymbol{v}$ collides with a stationary object B. After the collision, A is moving with velocity $\frac{1}{2}\boldsymbol{v}$ and B with velocity $\frac{3}{2}\boldsymbol{v}$. Find the ratio of their masses. If, instead of bouncing apart, the two bodies stuck together after the collision, with what velocity would they then move?
2. The two components of a double star are observed to move in circles of radii r_1 and r_2. What is the ratio of their masses? (*Hint*: Write down their accelerations in terms of the angular velocity of rotation, ω.)
3. Consider a system of three particles, each of mass m, whose motion is described by (1.9). If particles 2 and 3, even though not rigidly bound together, are regarded as forming a composite body of mass $2m$ located at the mid-point $\boldsymbol{r} = \frac{1}{2}(\boldsymbol{r}_2+\boldsymbol{r}_3)$, find the equations describing the motion of the two-body system comprising particle 1 and the composite body (2+3). What is the force on the composite body due to particle 1? Show that the equations agree with (1.7). When the masses are *un*equal, what is the correct definition of the position of the composite $(2+3)$ that will make (1.7) still hold?
4. Find the distance r between two protons at which the electrostatic repulsion between them will equal the gravitational attraction of the Earth on one of them. (Proton charge $= 1.6 \times 10^{-19}\,\mathrm{C}$, proton mass $= 1.7 \times 10^{-27}\,\mathrm{kg}$.)
5. Consider a transformation to a relatively uniformly moving frame of reference, where each position vector $\boldsymbol{r}_i$ is replaced by $\boldsymbol{r}'_i = \boldsymbol{r}_i - \boldsymbol{v}t$. (Here $\boldsymbol{v}$ is a constant, the relative velocity of the two frames.) How does a relative position vector $\boldsymbol{r}_{ij}$ transform? How do momenta and forces transform? Show explicitly that if equations (1.1) to (1.4) hold in the original frame, then they also hold in the new one.
6. A body of mass 50 kg is suspended by two light, inextensible cables of

lengths 15 m and 20 m from rigid supports placed 25 m apart on the same level. Find the tensions in the cables. (Note that by convention 'light' means 'of negligible mass'. Take $g = 10\,\mathrm{m\,s^{-2}}$. This and the following two problems are applications of vector addition.)

7. *An aircraft is to fly to a destination 800 km due north of its starting point. Its airspeed is $800\,\mathrm{km\,h^{-1}}$. The wind is from the east at a speed of $30\,\mathrm{m\,s^{-1}}$. On what compass heading should the pilot fly? How long will the flight take? If the wind speed increases to $50\,\mathrm{m\,s^{-1}}$, and the wind backs to the north-east, but no allowance is made for this change, how far from its destination will the aircraft be at its expected arrival time, and in what direction?
8. *The two front legs of a tripod are each 1.4 m long, with feet 0.8 m apart. The third leg is 1.5 m long, and its foot is 1.5 m directly behind the mid-point of the line joining the other two. Find the height of the tripod, and the vectors representing the positions of its three feet relative to the top. (*Hint*: Choose a convenient origin and axes and write down the lengths of the legs in terms of the position vector of the top.) Given that the tripod carries a weight of mass 2 kg, find the forces in the legs, assuming they are purely compressional (*i.e.*, along the direction of the leg) and that the legs themselves have negligible weight. (Take $g = 10\,\mathrm{m\,s^{-2}}$.)
9. *Discuss the possibility of using force rather than mass as the basic quantity, taking for example a standard weight (at a given latitude) as the unit of force. How should one then define and measure the mass of a body?
10. The first estimate of Newton's constant was made by the astronomer Nevil Maskelyne in 1774 by measuring the angle between the directions of the apparent plumb-line vertical on opposite sides of the Scottish mountain Schiehallion (height 1081 m, chosen for its regular conical shape). Find a rough estimate of the angle through which a plumb line is deviated by the gravitational attraction of the mountain, by modelling the mountain as a sphere of radius 500 m and density $2.7 \times 10^3\,\mathrm{kg\,m^{-3}}$, and assuming that its gravitational effect is the same as though the total mass were concentrated at its centre. (This latter assumption will be justified, for a spherical object, in Chapter 6.)

only part of a larger system, and subject to external forces, there may be some transfer of energy and momentum from the system to its surroundings (or *vice versa*). For a single particle moving on a straight line, the total (kinetic plus potential) energy is conserved if the force is a function only of x. If the force is velocity-dependent, however, there will be some loss of energy to the surroundings.

The importance of the harmonic oscillator, treated in detail in this chapter, lies in the fact that any system with one degree of freedom behaves like a harmonic oscillator when sufficiently near to a point of stable equilibrium. The methods we have discussed for solving the oscillator equation can therefore be applied to a great variety of problems. In particular, the phenomenon of resonance can occur in any system subjected to periodic forces. It occurs when there is a natural frequency of oscillation close to the forcing frequency, and leads, if the damping is small, to oscillations of very large amplitude.

Problems

1. A harmonic oscillator of angular frequency $2\,\mathrm{s}^{-1}$ is initially at $x = -3\,\mathrm{m}$, with $\dot{x} = 8\,\mathrm{m\,s}^{-1}$. Write the solution in each of the three forms (2.19), (2.20) and (2.23). Find the first time at which $x = 0$, and the first at which $\dot{x} = 0$. (A sketch of the solution may help.)
2. A weight of mass m is hung from the end of a spring which provides a restoring force equal to k times its extension. The weight is released from rest with the spring unextended. Find its position as a function of time, assuming negligible damping.
3. When a mass is suspended from a spring, the equilibrium length is increased by 50 mm. The mass is then given a blow which starts it moving vertically at $200\,\mathrm{mm\,s}^{-1}$. Find the period and amplitude of the resulting oscillations, assuming negligible damping.
4. A pendulum of period $2\,\mathrm{s}$ is released from rest at an inclination of $5°$ to the (downward) vertical. What is its angular velocity when it reaches the vertical? When it first returns to its starting point, it is given an impulsive blow *towards* the vertical that increases the amplitude of swing to $10°$. Find its subsequent angular position as a function of time.
5. Write down the potential energy function corresponding to the force $-GMm/x^2$ experienced by a particle of mass m at a distance x from a

planet of mass M ($\gg m$). The particle is released from rest at a distance a from the centre of the planet, whose radius is R, and falls under gravity. What is its speed when it strikes the surface? Evaluate this speed for a particle falling to the Earth from $a = 2R$. (Use $R = 6400$ km, and $GM/R^2 = g = 10\,\mathrm{m\,s^{-2}}$.)

6. A particle of mass m moves under a force $F = -cx^3$, where c is a positive constant. Find the potential energy function. If the particle starts from rest at $x = -a$, what is its velocity when it reaches $x = 0$? Where in the subsequent motion does it instantaneously come to rest?
7. A particle of mass m moves (in the region $x > 0$) under a force $F = -kx+c/x$, where k and c are positive constants. Find the corresponding potential energy function. Determine the position of equilibrium, and the frequency of small oscillations about it.
8. A particle of mass m has the potential energy function $V(x) = mk|x|$, where k is a positive constant. What is the force when $x > 0$, and when $x < 0$? Sketch the function V and describe the motion. If the particle starts from rest at $x = -a$, find the time it takes to reach $x = a$.
9. A particle of mass m moves under a conservative force with potential energy function given by

$$V(x) = \begin{cases} \frac{1}{2}k(a^2 - x^2) & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

where a and k are constants, and $a > 0$. What is the force on the particle? Sketch the function V, for both cases $k > 0$ and $k < 0$, and describe the possible types of motion.

10. The particle of Problem 9 with $k = -m\omega^2 < 0$ is initially in the region $x < -a$, moving to the right with velocity v. When it emerges into the region $x > a$, will it do so earlier or later than if it were moving freely under no force? Find an expression for the time difference. (To do the required integral, use a substitution of the form $x = \text{constant} \times \sin\theta$.)
11. The potential energy function of a particle of mass m is $V = -\frac{1}{2}c(x^2 - a^2)^2$, where c and a are positive constants. Sketch this function, and describe the possible types of motion in the three cases (a) $E > 0$, (b) $E < -\frac{1}{2}ca^4$, and (c) $-\frac{1}{2}ca^4 < E < 0$.
12. *The potential energy function of a particle of mass m is $V = cx/(x^2 + a^2)$, where c and a are positive constants. Sketch V as a function of x. Find the position of stable equilibrium, and the period of small oscillations about it. Given that the particle starts from this point

with velocity v, find the ranges of values of v for which it (a) oscillates, (b) escapes to $-\infty$, and (c) escapes to $+\infty$.

13. *A particle falling under gravity is subject to a retarding force proportional to its velocity. Find its position as a function of time, if it starts from rest, and show that it will eventually reach a terminal velocity. [The equation of motion can be integrated once to give, with a suitable choice of origin and definition of γ (differing from (2.28) by a factor of 2), $\dot{z} + \gamma z = -gt$. To integrate again, use an integrating factor, *i.e.* a function $f(t)$ such that when the equation is multiplied by $f(t)$ the left-hand side becomes an exact derivative, in fact the derivative of zf. The final stage requires an integration by parts.]
14. The terminal speed of the particle in Problem 13 is $50\,\mathrm{m\,s^{-1}}$. Find the time it takes to reach a speed of $40\,\mathrm{m\,s^{-1}}$, and the distance it has fallen in that time. (Take $g = 10\,\mathrm{m\,s^{-2}}$.)
15. *A particle moves vertically under gravity and a retarding force proportional to the *square* of its velocity. (This is more appropriate than a linear relation for larger particles — see Problem 8.10.) If v is its upward or downward speed, show that $\dot{v} = \mp g - kv^2$, respectively, where k is a constant. If the particle is moving upwards, show that its position at time t is given by $z = z_0 + (1/k)\ln\cos[\sqrt{gk}\,(t_0 - t)]$, where z_0 and t_0 are integration constants. If its initial velocity at $t = 0$ is u, find the time at which it comes instantaneously to rest, and its height then. [Note that ln always denotes the natural logarithm: $\ln x \equiv \log_{\mathrm{e}} x$. You may find the identity $\ln\cos x = -\frac{1}{2}\ln(1 + \tan^2 x)$ useful.]
16. *Show that if the particle of the previous question falls from rest its speed after a time t is given by $v = \sqrt{g/k}\,\tanh(\sqrt{gk}\,t)$. What is its limiting speed? How long does it take to hit the ground if dropped from height h?
17. The pendulum described in §2.1 is released from rest at an angle θ_0 to the downward vertical. Find its angular velocity as a function of θ, and the period of small oscillations about the position of stable equilibrium. Write down the solution for θ as a function of time, assuming that θ_0 is small.
18. *For the pendulum described in §2.1, find the equation of motion for small displacements from the position of *un*stable equilibrium, $\theta = \pi$. Show that if it is released from rest at a small angle to the upward vertical, then the time taken for the angular displacement to increase by a factor of 10 will be approximately $\sqrt{l/g}\ln 20$. Evaluate this time for

a pendulum of period 2 s, and find the angular velocity of the pendulum when it reaches the downward vertical.

19. *The particle of Problem 11 starts from rest at $x = -a$, and is given a small push to start it moving to the right. What is its velocity when it reaches the point x? Given that $t = 0$ is the instant when it reaches $x = 0$, find its position as a function of time. (Assume that the push has negligible effect on its energy.)
20. *A particle of mass m moves in the region $x > 0$ under the force $F = -m\omega^2(x - a^4/x^3)$, where ω and a are constants. Sketch the potential energy function. Find the position of equilibrium, and the period of small oscillations about it. The particle starts from this point with velocity v. Find the limiting values of x in the subsequent motion. Show that the period of oscillation is independent of v. (To do the integration, transform to the variable $y = x^2$, then add a constant to 'complete the square', and finally use a trigonometric substitution.)
21. Repeat the calculation of Problem 2 assuming that the system is critically damped. Given that the final position of equilibrium is 0.4 m below the point of release, find how close to the equilibrium position the particle is after 1 s.
22. *A pendulum whose period in a vacuum is 1 s is placed in a resistive medium. Its amplitude on each swing is observed to be half that on the previous swing. What is its new period? Given that the pendulum bob is of mass 0.1 kg, and is subjected to a periodic force of amplitude 0.02 N and period 1 s, find the angular amplitude of the resulting forced oscillation. Compare your answer with the angular deviation that would be induced by a constant force of this magnitude in a pendulum at rest.
23. Write down the solution to the oscillator equation for the case $\omega_0 > \gamma$ if the oscillator starts from $x = 0$ with velocity v. Show that, as ω_0 is reduced to the critical value γ, the solution tends to the corresponding solution for the critically damped oscillator.
24. *Solve the problem of an oscillator under a simple periodic force (turned on at $t = 0$) by the Green's function method, and verify that it reproduces the solution of §2.6. [Assume that the damping is less than critical. To do the integral, write $\sin\omega t$ as $(\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t})/2\mathrm{i}$.]
25. *For an oscillator under a periodic force $F(t) = F_1 \cos\omega_1 t$, calculate the power, the rate at which the force does work. (Neglect the transient.) Show that the average power is $P = m\gamma\omega_1^2 a_1^2$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. Show that the power P is a maximum, as a function

of ω_1, at $\omega_1 = \omega_0$, and find the values of ω_1 for which it has half its maximum value.

26. *Find the average value $\bar{E}$ of the total energy of an oscillator under a periodic force. If W is the work done against friction in one period, show that when $\omega_1 = \omega_0$ the ratio $W/\bar{E}$ is related to the quality factor Q by $W/\bar{E} = 2\pi/Q$.
27. Three perfectly elastic bodies of masses 5 kg, 1 kg, 5 kg are arranged in that order on a straight line, and are free to move along it. Initially, the middle one is moving with velocity $27\,\mathrm{m\,s^{-1}}$, and the others are at rest. Find how many collisions take place in the subsequent motion, and verify that the final value of the kinetic energy is equal to the initial value.
28. A ball is dropped from height h and bounces. The coefficient of restitution at each bounce is e. Find the velocity immediately after the first bounce, and immediately after the nth bounce. Show that the ball finally comes to rest after a time

$$\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}.$$

29. *An oscillator with free period τ is critically damped and subjected to a force with the 'saw-tooth' form

$$F(t) = c(t - n\tau) \qquad \text{for} \qquad (n - \tfrac{1}{2})\tau < t \leq (n + \tfrac{1}{2})\tau,$$

for each integer n (c is a constant). Find the ratios of the amplitudes a_n of oscillation at the angular frequencies $2\pi n/\tau$.

30. *A particle moving under a conservative force oscillates between x_1 and x_2. Show that the period of oscillation is

$$\tau = 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2[V(x_2) - V(x)]}}\,\mathrm{d}x.$$

In particular, if $V = \frac{1}{2}m\omega_0^2(x^2 - bx^4)$, show that the period for oscillations of amplitude a is

$$\tau = \frac{2}{\omega_0}\int_{-a}^{a} \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}\sqrt{1 - b(a^2 + x^2)}}.$$

Using the binomial theorem to expand in powers of b, and the substitution $x = a\sin\theta$, show that for small amplitude the period is approximately $\tau \approx 2\pi(1 + \frac{3}{4}ba^2)/\omega_0$.

31. Use the result of the preceding question to obtain an estimate of the correction factor to the period of a simple pendulum when its angular amplitude is 30°. (Write $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$.)
32. *Find the Green's function of an oscillator in the case $\gamma > \omega_0$. Use it to solve the problem of an oscillator that is initially in equilibrium, and is subjected from $t = 0$ to a force increasing linearly with time, $F = ct$.