Stability of periodic solutions using the Floquet Theory

PEF 5737 - Nonlinear dynamics and stability

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- To investigate the stability of periodic orbits using Floquet theory;
- References
    1. Mailybaev, A. A. (2019) Introduction to the theory of parametric resonance.
    2. Nayfeh, A. & Balachandran, B. (1995), Applied nonlinear dynamics - analytical, computational and experimental methods;
    3. Wiggins, S (1990). Introduction to applied nonlinear dynamical systems and chaos.

$$\left\{\begin{array}{c}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_n\end{array}\right\} = \left\{\begin{array}{c}f_1(x_1,x_2,x_3,\ldots,\boldsymbol{M})\\ f_2(x_1,x_2,x_3,\ldots,\boldsymbol{M})\\ f_3(x_1,x_2,x_3,\ldots,\boldsymbol{M})\\ \vdots\\ f_n(x_1,x_2,x_3,\ldots,\boldsymbol{M})\end{array}\right\} \rightarrow \dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x},\boldsymbol{M}) \qquad (1)$$

$\boldsymbol{M}$ being a m-dimensional vector with the parameters of the mathematical model.

$\boldsymbol{X_0} = \boldsymbol{X_0}(t)$ is a periodic solution of Eq. 1 with minimal period T at $\boldsymbol{M} = \boldsymbol{M}_0$. Let $\boldsymbol{y} = \boldsymbol{y}(t)$ a disturbance to be superimposed to $\boldsymbol{X_0}$ such that $\boldsymbol{X_0} + \boldsymbol{y} = \boldsymbol{x}(t) = \boldsymbol{x}$. Assuming $\boldsymbol{F}$ $(\mathcal{C}^2)$, Eq. 1 can be linearized around $\boldsymbol{X_0}$, leading to

$$\dot{\boldsymbol{y}} \approx \boldsymbol{A}(t, \boldsymbol{M_0})\boldsymbol{y}, \boldsymbol{A}(t, \boldsymbol{M_0}) = \boldsymbol{D_X}\boldsymbol{F}(\boldsymbol{X_0}, \boldsymbol{M_0}) \tag{2}$$

- Equation 2 is linear $\rightarrow$ It has n linearly independent (LI) solutions (fundamental set of solutions) $\boldsymbol{y}_i(t)$, $i, = 1, 2, \ldots, n$;
- We gather these solutions as the columns of a fundamental matrix solution
  $\boldsymbol{Y}(t) = [\boldsymbol{y}_1(t) \quad \boldsymbol{y}_2(t) \quad \boldsymbol{y}_3(t) \ldots \boldsymbol{y}_n(t)] \rightarrow \dot{\boldsymbol{Y}} = \boldsymbol{A}(t, \boldsymbol{M_0})\boldsymbol{Y}$;
- Change of variables $\tau = t + T \rightarrow$ Eq. 2 becomes

$$\frac{d\boldsymbol{Y}}{d\tau} = \boldsymbol{A}(\tau - T, \boldsymbol{M_0})\boldsymbol{Y} = \boldsymbol{A}(\tau, \boldsymbol{M_0})\boldsymbol{Y} \tag{3}$$

- We conclude that $\boldsymbol{Y}(t+T) = [\boldsymbol{y}_1(t+T) \quad \boldsymbol{y}_2(t+T) \quad \boldsymbol{y}_3(t+T) \ldots \boldsymbol{y}_n(t+T)]$ is another fundamental matrix solution.

- As Eq. 2 has n LI solutions, $\boldsymbol{y}_i(t+T)$ is a linear combination of $\boldsymbol{y}_1(t), \boldsymbol{y}_2(t) \ldots \boldsymbol{y}_n(t)$;
- In a more compact form $\boldsymbol{Y}(t+T) = \boldsymbol{Y}(t)\boldsymbol{\Phi}$, $\boldsymbol{\Phi}$ unknown (up to now). $\boldsymbol{\Phi}$ has dimension $n \times n$, depends on the choice of $\boldsymbol{Y}(t)$ and can be seen as a map from the initial condition to T;
- If $\boldsymbol{Y}(0) = \boldsymbol{I}$, then $\boldsymbol{\Phi} = \boldsymbol{Y}(T)$. $\boldsymbol{\Phi}$ is the monodromy matrix.

- Let $\boldsymbol{P}$ be a non-singular constant matrix of order $n \times n$ and we define $\boldsymbol{Y}(t) = \boldsymbol{V}(t)\boldsymbol{P}^{-1}$;
-

$$\boldsymbol{Y}(t+T) = \boldsymbol{Y}(t)\boldsymbol{\Phi} \leftrightarrow \boldsymbol{V}(t+T)\boldsymbol{P}^{-1} = \boldsymbol{V}(t)\boldsymbol{P}^{-1}\boldsymbol{\Phi} \leftrightarrow$$
$$\boldsymbol{V}(t+T) = \boldsymbol{V}(t)\underbrace{\boldsymbol{P}^{-1}\boldsymbol{\Phi}\boldsymbol{P}}_{\boldsymbol{J}} \tag{4}$$

- We can conveniently choose $\boldsymbol{J}$...

- In this case, we choose $\boldsymbol{P} = [\boldsymbol{p_1} \quad \boldsymbol{p_2} \dots \boldsymbol{p_n}]$, $\boldsymbol{p_m}$ being the eigenvectors of $\mathbf{\Phi}$ $\rightarrow$ Canonical Jordan form
- $$\begin{aligned} \boldsymbol{J} &= \boldsymbol{P}^{-1}\mathbf{\Phi}\boldsymbol{P} = \boldsymbol{P}^{-1}\mathbf{\Phi}[\boldsymbol{p_1} \quad \boldsymbol{p_2} \dots \boldsymbol{p_n}] = \\ &= \boldsymbol{P}^{-1}[\mathbf{\Phi}\boldsymbol{p_1} \quad \mathbf{\Phi}\boldsymbol{p_2} \dots \mathbf{\Phi}\boldsymbol{p_n}] = \boldsymbol{P}^{-1}[\rho_1\boldsymbol{p_1} \quad \rho_2\boldsymbol{p_2} \dots \rho_n\boldsymbol{p_n}] = \\ &= \boldsymbol{P}^{-1}\boldsymbol{P}\boldsymbol{D} = \boldsymbol{D} = \begin{bmatrix} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_n \end{bmatrix} \end{aligned} \tag{5}$$
- Each ρ_m is named Floquet or characteristic multiplier $\rightarrow$ gives a measure of local divergence or convergence of the orbit in a particular direction.

From Eqs. 5 and 4:

$$\boldsymbol{v}_m(t+T) = \rho_m \boldsymbol{v}_m(t), m = 1, 2, \ldots, n \leftrightarrow$$
$$\leftrightarrow \boldsymbol{v}_m(t+NT) = \rho_m^N \boldsymbol{v}_m(t), m = 1, 2, \ldots, n \text{ and N integer} \quad (6)$$

- Notice that $N \to \infty$ means $t \to \infty$. Eq. 6 indicates that the stability of the periodic orbits can be assessed by the eigenvalues of the monodromy matrix;
- $\boldsymbol{v}_m(t) \to 0$ if $|\rho_m| < 1$;
- $\boldsymbol{v}_m(t) \to \infty$ if $|\rho_m| > 1$.

- In this case, the behavior of the system depends on the algebraic and geometric multiplicities (a_m and g_m, respectively) of the Floquet multipliers.
- If $a_m = g_m > 1$, we can use the approach aforementioned;
- Now, we focus on the case with $g_m < a_m$.

- A Jordan chain can be defined as:

$$\begin{aligned}
\mathbf{\Phi} \boldsymbol{u}_1 &= \rho \boldsymbol{u}_1 \\
\mathbf{\Phi} \boldsymbol{u}_2 &= \rho \boldsymbol{u}_2 + \boldsymbol{p}_1 \\
\mathbf{\Phi} \boldsymbol{u}_3 &= \rho \boldsymbol{u}_3 + \boldsymbol{p}_2 \\
&\vdots \\
\mathbf{\Phi} \boldsymbol{u}_l &= \rho \boldsymbol{u}_l + \boldsymbol{p}_{l-1} \\
\mathbf{\Phi} \boldsymbol{u}_{l+1} &\neq \rho \boldsymbol{u}_{l+1} + \boldsymbol{p}_l
\end{aligned} \tag{7}$$

  $\boldsymbol{u}_1$ being the eigenvector and the $\boldsymbol{u}_2 \ldots, \boldsymbol{u}_l$ the generalized eigenvectors.

- The Jordan chain can be written as $\mathbf{\Phi P} = \boldsymbol{PJ}$, with $\boldsymbol{J} = \begin{bmatrix} \rho & 1 & \ldots & 0 \\ 0 & \rho & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho \end{bmatrix}$ being a Jordan block. The Jordan canonical form is composed of a number of Jordan blocks, according to g_m.

- Taking the derivative of Eq. 1

$$\ddot{\boldsymbol{x}} = \boldsymbol{D_X F}(\boldsymbol{x}, \boldsymbol{M})\dot{\boldsymbol{x}} \tag{8}$$

- If $\boldsymbol{x}$ is a solution of Eq. 1, $\dot{\boldsymbol{x}}$ is solution of both Eqs. 8 and 2;
- Provided $\boldsymbol{X_0}(t)$ is solution of Eq. 1, $\dot{\boldsymbol{X}}_{\boldsymbol{0}}(t)$ is solution of Eq. 2 and has period T. Hence $\dot{\boldsymbol{X}}_{\boldsymbol{0}}(t) = \dot{\boldsymbol{X}}_{\boldsymbol{0}}(t+T)$;
- We write $\dot{\boldsymbol{X}}_{\boldsymbol{0}}(t)$ as a linear combination of $\boldsymbol{y_1}(t), \boldsymbol{y_2}(t), \ldots, \boldsymbol{y_n}(t)$ as:

$$\dot{\boldsymbol{X}}_{\boldsymbol{0}}(t) = \boldsymbol{Y}(t)\alpha \tag{9}$$

  α being a vector of constants.
- From Eq. 9: $\dot{\boldsymbol{X}}_{\boldsymbol{0}}(0) = \boldsymbol{Y}(0)\alpha$ and $\dot{\boldsymbol{X}}_{\boldsymbol{0}}(T) = \boldsymbol{Y}(T)\alpha$.

- Provided $\boldsymbol{X_0}(t)$ has period T, recalling that $\boldsymbol{Y}(0) = \boldsymbol{I}$ and using the definition of the monodromy matrix;

$$\boldsymbol{\Phi}\boldsymbol{\alpha} = \boldsymbol{\alpha} \tag{10}$$

- From Eq. 10, one can notice that 1 is an eigenvalue of the monodromy matrix;
- **Definition (autonomous systems)**: A periodic solution of 1 is named as hyperbolic if only one Floquet multiplier is located in the unit circle;
- **Definition (autonomous systems)**: A periodic solution of 1 is named as non-hyperbolic if more than one Floquet multiplier are located in the unit circle;
- The Hartman-Grobaman's theorem is valid for hyperbolic periodic orbits.

- A hyperbolic period solution is asymptotically stable if no Floquet multiplier is outside the unit circle $\rightarrow$ periodic attractor or stable limit cycle;
- A hyperbolic period solution is unstable stable if at least on Floquet multiplier is outside the unit circle $\rightarrow$ periodic repellor or unstable limit cycle;

- Non-autonomous system of first-order differential equations

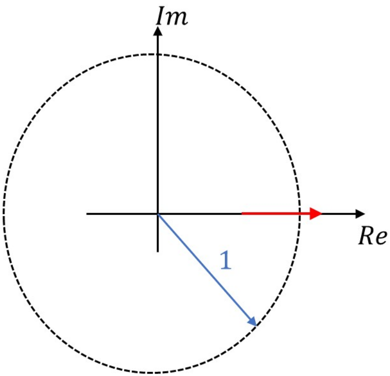
$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{M}, t) \tag{11}$$

- Similarly to what was carried out for the autonomous systems, we study the stability of the T-periodic solution $\boldsymbol{X_0} = \boldsymbol{X_0}(t)$ of Eq. 11 with $\boldsymbol{M} = \boldsymbol{M_0}$;
- We superimpose the perturbation $\boldsymbol{z} = \boldsymbol{z}(t)$ to $\boldsymbol{X_0}$, obtaining $\boldsymbol{x}(t) = \boldsymbol{x} = X_0 + z$;
- Expanding Eq. 11 in Taylor series and neglecting higher-order terms, one obtains:

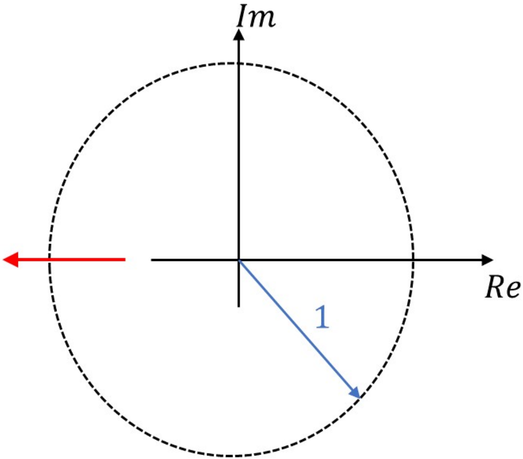
$$\dot{\boldsymbol{z}} = \boldsymbol{A}(t, \boldsymbol{M_0})\boldsymbol{z} \tag{12}$$

- Practically all the discussions already made for the autonomous systems remain valid for non-autonomous systems;
- Notice, however, that 1 is no longer an eigenvalue of the monodromy matrix;
- A periodic solution of 1 is named as hyperbolic if no Floquet multiplier is located in the unit circle;
- A periodic solution of 1 is named as non-hyperbolic if at least one Floquet multiplier is located in the unit circle;
- If all ρ_m are located within the unit circle $\rightarrow$ Periodic solution is a stable limit-cycle;
- If some ρ_m is located outside the unit circle $\rightarrow$ Periodic solution is a unstable.
- If all ρ_m are located outside the unit$\rightarrow$ Periodic solution is an unstable limit cycle;
- If some but not all ρ_m are located outside the unit circle$\rightarrow$ Periodic solution is a saddle;
- The Hartman-Grobaman's theorem is valid for hyperbolic periodic orbits.

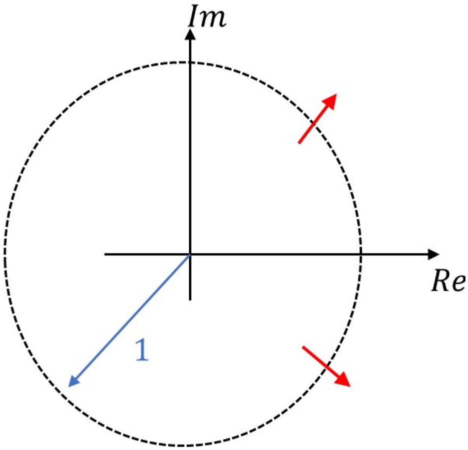
- We are interested in investigating with the stability of a periodic orbit when the vector gathering the parameters of the mathematical model ($\boldsymbol{M}$) is varied;
- We define the codimension of a bifurcation as the minimum number of independent control parameters that must be varied in order to observe the bifurcation;
- For periodic orbits, the bifurcations are observed when the Floquet multipliers cross the unity circle. Three cases, below illustrated may occur.



Floquet multiplier crosses the unity circle through +1.



Floquet multiplier crosses the unity circle through -1.



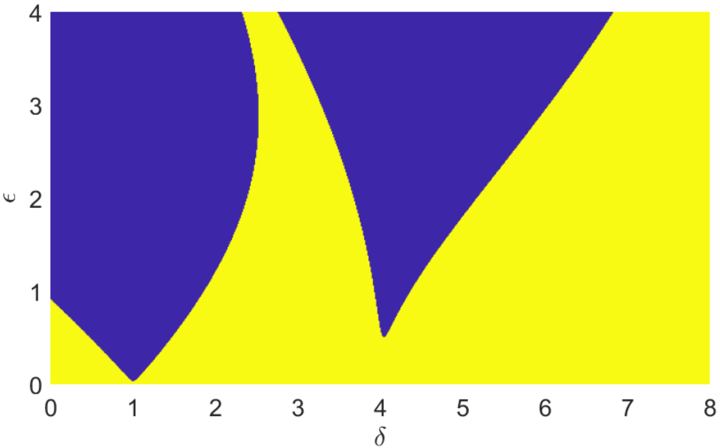
Floquet multiplier crosses the unity circle as complex conjugates.

- We recall that $\boldsymbol{v}_m(t+T) = \rho_m \boldsymbol{v}_m(t), m = 1, 2, \ldots, n$
- if $\rho_m = -1$ (period-doubling bifurcation)
  $\boldsymbol{v}_m(t+T) = -\boldsymbol{v}_m(t) \rightarrow \boldsymbol{v}_m(t)$ has period $2T$.

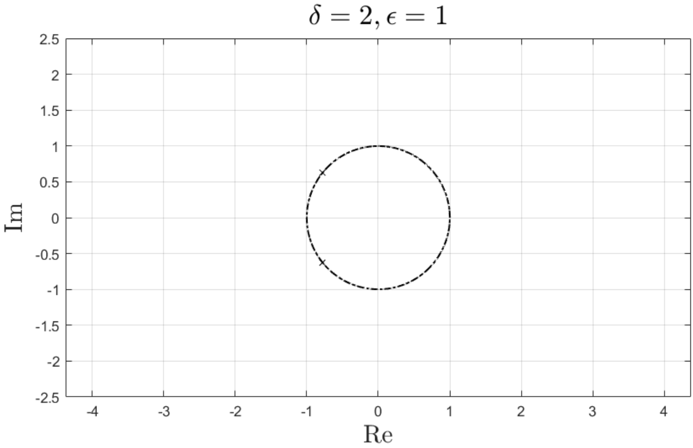
- $$\ddot{u} + (\delta + 2\epsilon \cos 2\tau)u = 0 \tag{13}$$
- We rewrite Eq. 13 as a system of first-order differential equations by defining $x_1 = u$ and $x_2 = \dot{u}$:

$$\dot{x}_1 = x_2 \tag{14}$$
$$\dot{x}_2 = -(\delta + 2\epsilon \cos 2\tau)x_1 \tag{15}$$

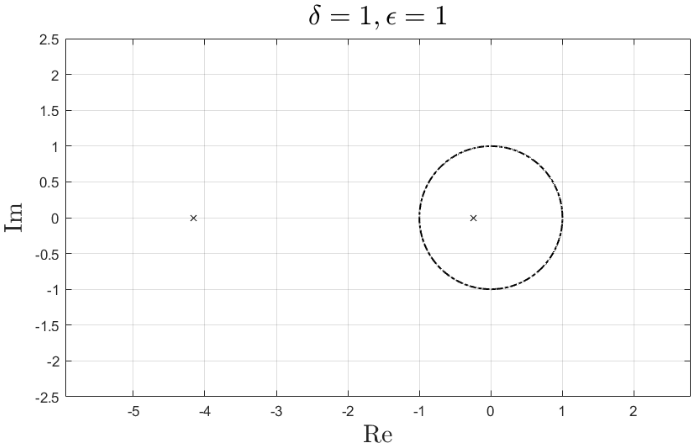
- The period of the non-autonomous linear system is $T = \pi$. We investigate the stability of the periodic solution as a function of the parameters of the plane of control parameters $(\delta; \epsilon)$ (Strutt's diagram).
- We compute the monodromy matrix by numerically integrating Eqs. 14 and 15 from 0 to T for different pairs $(\delta; \epsilon)$. If at least one Floquet multiplier has modulus larger than 1.05, we associate the blue color. If contrary, we associate the yellow color.
- Notice that the Floquet theory indicates 1 as the threshold for the stability. In this example, we adopt 1.05 for dealing with the errors intrinsically related with the numerical methods.

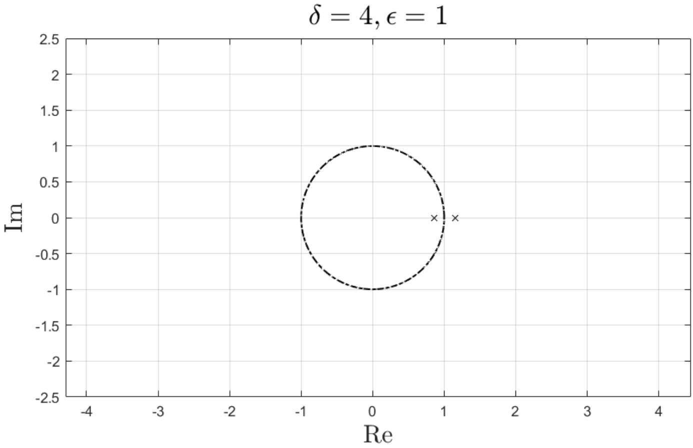


For the undamped Mathieu's equation, the instability tongue arise from $\epsilon = 0$. The adopted discretization (600×600) needs to be improved if this is an important aspect in the analysis. In a standard notebook (i7, 10th gen, 8Gb RAM), the Strutt's diagram was obtained in 6.6 minutes.



$\delta = 2, \epsilon = 1$
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
Im
-4
-3
-2
-1
0
1
2
3
4
Re


$\delta = 1, \epsilon = 1$
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
Im
-5
-4
-3
-2
-1
0
1
2
Re


$\delta = 4, \epsilon = 1$
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
Im
-4
-3
-2
-1
0
1
2
3
4
Re