MAP 2210 – Aplicações de Álgebra Linear

1º Semestre - 2020

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Objetivos

Formação básica de álgebra linear aplicada a problemas numéricos. Resolução de problemas em microcomputadores usando linguagens e/ou software adequados fora do horário de aula.

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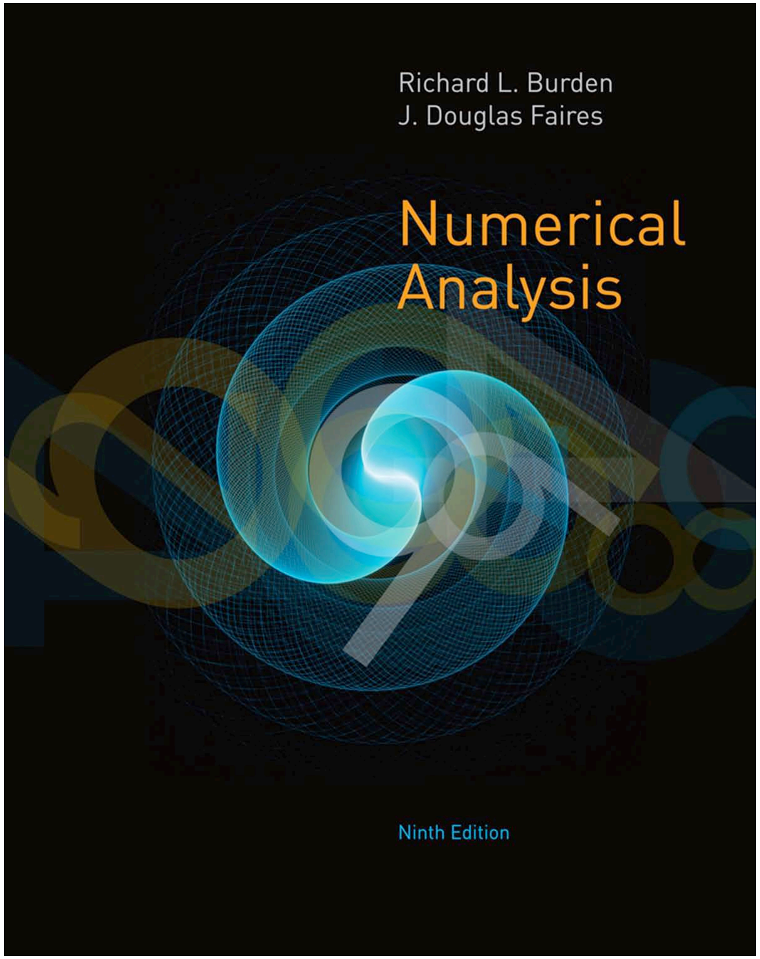

Numerical Analysis

NINTH EDITION

Richard L. Burden
Youngstown State University

J. Douglas Faires
Youngstown State University

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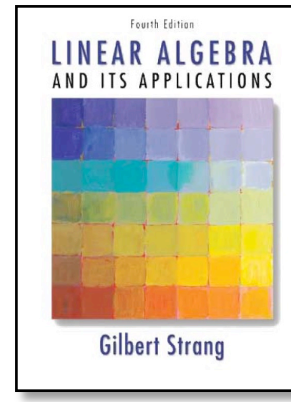

Fourth Edition
LINEAR ALGEBRA
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Gilbert Strang

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6.3 Linear Algebra and Matrix Inversion

Definition 6.1 An $\boldsymbol{n \times m}$ ($\boldsymbol{n}$ by $\boldsymbol{m}$) **matrix** is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array. ■

Definition 6.2 Two matrices A and B are **equal** if they have the same number of rows and columns, say $n \times m$, and if $a_{ij} = b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. ■

Definition 6.3 If A and B are both $n \times m$ matrices, then the **sum** of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. ■

Definition 6.4 If A is an $n \times m$ matrix and λ is a real number, then the **scalar multiplication** of λ and A, denoted λA, is the $n \times m$ matrix whose entries are λa_{ij}, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. ■

We have the following general properties for matrix addition and scalar multiplication. These properties are sufficient to classify the set of all $n \times m$ matrices with real entries as a **vector space** over the field of real numbers.

- We let O denote a matrix all of whose entries are 0 and $-A$ denote the matrix whose entries are $-a_{ij}$.

Theorem 6.5 Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$, **(ii)** $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$, **(iv)** $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$, **(vi)** $(\lambda + \mu)A = \lambda A + \mu A$,

(vii) $\lambda(\mu A) = (\lambda\mu)A$, **(viii)** $1A = A$.

All these properties follow from similar results concerning the real numbers. ■

Matrix-Vector Products

The product of matrices can also be defined in certain instances. We will first consider the product of an $n \times m$ matrix and a $m \times 1$ column vector.

Definition 6.6 Let A be an $n \times m$ matrix and $\mathbf{b}$ an m-dimensional column vector. The **matrix-vector product** of A and $\mathbf{b}$, denoted $A\mathbf{b}$, is an n-dimensional column vector given by

$$A\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i}b_i \\ \sum_{i=1}^{m} a_{2i}b_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni}b_i \end{bmatrix}.$$

For this product to be defined the number of columns of the matrix A must match the number of rows of the vector $\mathbf{b}$, and the result is another column vector with the number of rows matching the number of rows in the matrix.

Matrix-Matrix Products

We can use this matrix-vector multiplication to define general matrix-matrix multiplication.

Definition 6.7 Let A be an $n \times m$ matrix and B an $m \times p$ matrix. The **matrix product** of A and B, denoted AB, is an $n \times p$ matrix C whose entries c_{ij} are

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj},$$

for each $i = 1, 2, \cdots n$, and $j = 1, 2, \cdots, p$. ■

The computation of c_{ij} can be viewed as the multiplication of the entries of the ith row of A with corresponding entries in the jth column of B, followed by a summation; that is,

$$[a_{i1}, a_{i2}, \cdots, a_{im}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{bmatrix} = c_{ij},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}.$$

This explains why the number of columns of A must equal the number of rows of B for the product AB to be defined.

O produto de matrizes é comutativo ?

Square Matrices

Definition 6.9

(i) A **square** matrix has the same number of rows as columns.

(ii) A **diagonal** matrix $D = [d_{ij}]$ is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

(iii) The **identity matrix of order** n, $I_n = [\delta_{ij}]$, is a diagonal matrix whose diagonal entries are all 1s. When the size of I_n is clear, this matrix is generally written simply as I. ■

Definition 6.10

An **upper-triangular** $n \times n$ matrix $U = [u_{ij}]$ has, for each $j = 1, 2, \cdots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \cdots, n;$$

and a **lower-triangular** matrix $L = [l_{ij}]$ has, for each $j = 1, 2, \cdots, n$, the entries

$$l_{ij} = 0, \quad \text{for each } i = 1, 2, \cdots, j-1.$$

■

Inverse Matrices

Definition 6.11

An $n \times n$ matrix A is said to be **nonsingular** (or *invertible*) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular** (or *noninvertible*). ■

Theorem 6.12 For any nonsingular $n \times n$ matrix A:

(i) A^{-1} is unique.

(ii) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

(iii) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$. ■

Como encontrar a inversa ?

To find a method of computing A^{-1} assuming A is nonsingular, let us look again at matrix multiplication. Let B_j be the jth column of the $n \times n$ matrix B,

$$B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

If $AB = C$, then the jth column of C is given by the product

$$\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{1k}b_{kj} \\ \sum_{k=1}^{n} a_{2k}b_{kj} \\ \vdots \\ \sum_{k=1}^{n} a_{nk}b_{kj} \end{bmatrix}.$$

Suppose that A^{-1} exists and that $A^{-1} = B = (b_{ij})$. Then $AB = I$ and

$$AB_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{where the value 1 appears in the } j\text{th row.}$$

To find B we need to solve n linear systems in which the jth column of the inverse is the solution of the linear system with right-hand side the jth column of I. The next illustration demonstrates this method.

Illustration

determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix},$$

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$$A^{-1} = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} & -\frac{1}{9} \\ \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

As we saw in the illustration, in order to compute A^{-1} it is convenient to set up a larger augmented matrix,

$$[\, A \;\vdots\; I \,].$$

Upon performing the elimination in accordance with Algorithm 6.1, we obtain an augmented matrix of the form

$$[\, U \;\vdots\; Y \,],$$

where U is an upper-triangular matrix and Y is the matrix obtained by performing the same operations on the identity I that were performed to take A into U.

Gaussian elimination with backward substitution requires

$$\frac{4}{3}n^3 - \frac{1}{3}n \text{ multiplications/divisions} \quad \text{and} \quad \frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{n}{6} \text{ additions/subtractions.}$$

to solve the n linear systems (see Exercise 8(a)).

Transpose of a Matrix

Definition 6.13

The **transpose** of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$, where for each i, the ith column of A^t is the same as the ith row of A. A square matrix A is called **symmetric** if $A = A^t$. ■

Theorem 6.14

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$,

(ii) $(A + B)^t = A^t + B^t$,

(iii) $(AB)^t = B^t A^t$,

(iv) if A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$. ■

6.4 The Determinant of a Matrix

The *determinant* of a matrix provides existence and uniqueness results for linear systems having the same number of equations and unknowns. We will denote the determinant of a square matrix A by $\det A$, but it is also common to use the notation $|A|$.

Definition 6.15

Suppose that A is a square matrix.

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) If A is an $n \times n$ matrix, with $n > 1$ the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(iii) The **cofactor** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iv) The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

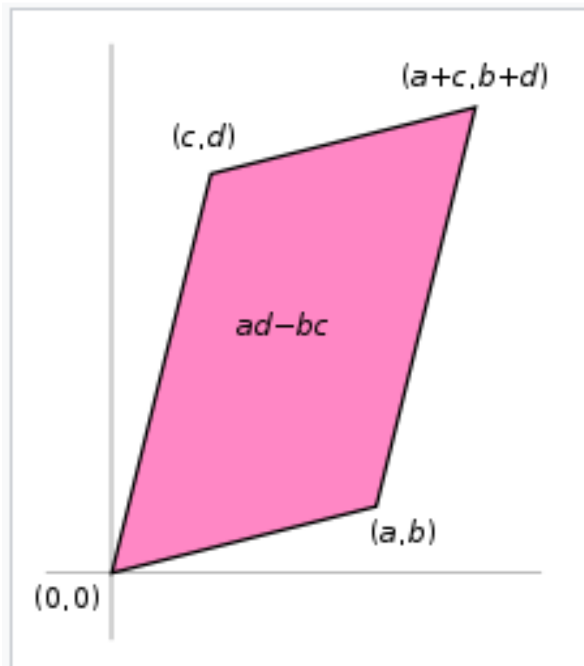
$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \cdots, n,$$

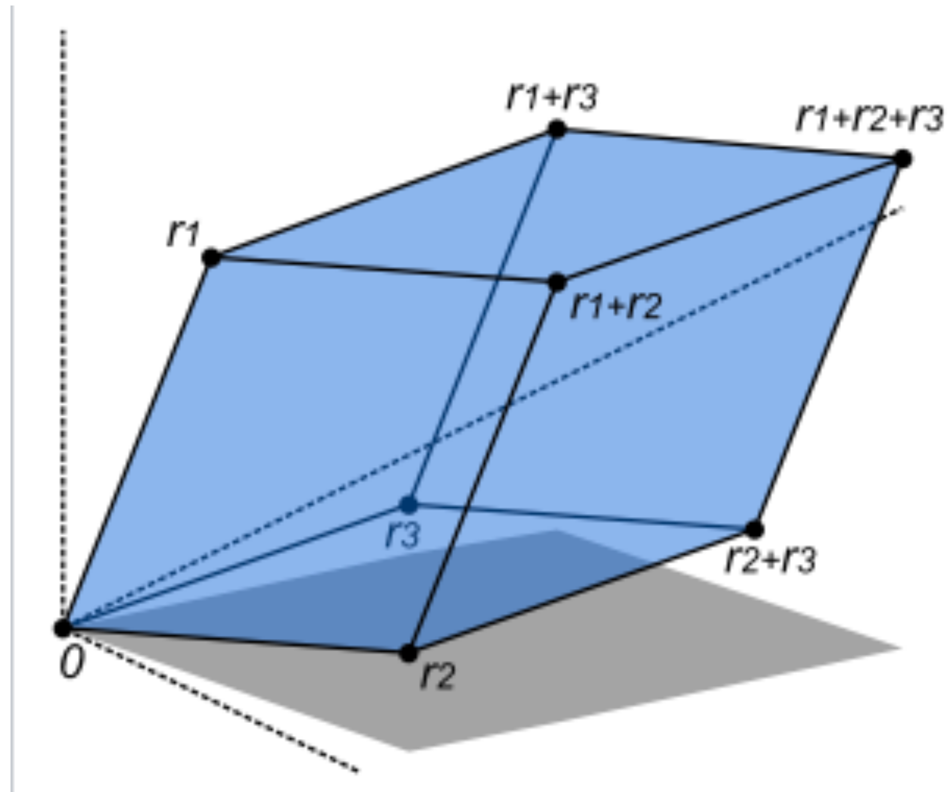
or by

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \cdots, n. \qquad \blacksquare$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{aligned} |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} \\ &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh. \end{aligned}$$





$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{aligned} |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \begin{vmatrix} \Box & \Box & \Box \\ \Box & e & f \\ \Box & h & i \end{vmatrix} - b \begin{vmatrix} \Box & \Box & \Box \\ d & \Box & f \\ g & \Box & i \end{vmatrix} + c \begin{vmatrix} \Box & \Box & \Box \\ d & e & \Box \\ g & h & \Box \end{vmatrix} \\ &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh. \end{aligned}$$

Example 2

Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

Grupo A: Use a linha 2

Grupo B: Use a coluna 3

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Theorem 6.16

Suppose A is an $n \times n$ matrix:

(i) If any row or column of A has only zero entries, then $\det A = 0$.

(ii) If A has two rows or two columns the same, then $\det A = 0$.

(iii) If $\tilde{A}$ is obtained from A by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then $\det \tilde{A} = -\det A$.

(iv) If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \rightarrow (E_i)$, then $\det \tilde{A} = \lambda \det A$.

(v) If $\tilde{A}$ is obtained from A by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then $\det \tilde{A} = \det A$.

(vi) If B is also an $n \times n$ matrix, then $\det AB = \det A \det B$.

(vii) $\det A^t = \det A$.

(viii) When A^{-1} exists, $\det A^{-1} = (\det A)^{-1}$.

(ix) If A is an upper triangular, lower triangular, or diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$. ■

It can be shown (see Exercise 9) that to calculate the determinant of a general $n \times n$ matrix by this definition requires $O(n!)$ multiplications/divisions and additions/subtractions.

Example 2

Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$$

Usando as regras do Teorema 6.16

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Chapter

4 Determinants

4.4 APPLICATIONS OF DETERMINANTS

This section follows through on four major applications: *inverse of A, solving $Ax = b$, volumes of boxes*, and *pivots*. They are among the key computations in linear algebra (done by elimination). Determinants give formulas for the answers.

1. Computation of A^{-1}. The 2 by 2 case shows how cofactors go into A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}.$$

We are dividing by the determinant, and A is invertible exactly when $\det A$ is nonzero. The number $C_{11} = d$ is the cofactor of a. The number $C_{12} = -c$ is the cofactor of b (note the minus sign). That number C_{12} goes in row 2, column 1!

The row a, b times the column C_{11}, C_{12} produces $ad - bc$. This is the cofactor expansion of $\det A$. That is the clue we need: $\boldsymbol{A^{-1}}$ **divides the cofactors by det** $\boldsymbol{A}$.

Cofactor matrix
***C* is transposed**

$$A^{-1} = \frac{C^{\mathrm{T}}}{\det A} \quad \text{means} \quad (A^{-1})_{ij} = \frac{C_{ji}}{\det A}. \tag{1}$$

Our goal is to verify this formula for A^{-1}. We have to see why $AC^{\mathrm{T}} = (\det A)I$:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \det A \end{bmatrix}. \tag{2}$$

With cofactors $C_{11}, \ldots, C_{1n}$ in the first *column* and not the first row, they multiply $a_{11}, \ldots, a_{1n}$ and give the diagonal entry $\det A$. Every row of A multiplies its cofactors (*the cofactor expansion*) to give the same answer $\det A$ on the diagonal.

2. The Solution of $Ax = b$. The multiplication $x = A^{-1}b$ is just $C^{\mathrm{T}}b$ divided by $\det A$. There is a famous way in which to write the answer $(x_1, \ldots, x_n)$:

4C *Cramer's rule*: The jth component of $x = A^{-1}b$ is the ratio

$$x_j = \frac{\det B_j}{\det A}, \quad \text{where} \quad B_j = \begin{bmatrix} a_{11} & a_{12} & \boldsymbol{b_1} & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \boldsymbol{b_n} & a_{nn} \end{bmatrix} \text{ has } b \text{ in column } j. \qquad (4)$$

Proof Expand $\det B_j$ in cofactors of its jth column (which is b). Since the cofactors ignore that column, $\det B_j$ is exactly the jth component in the product $C^{\mathrm{T}}b$:

$$\det B_j = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}.$$

Dividing this by $\det A$ gives x_j. Each component of x is a *ratio of two determinants*. That fact might have been recognized from Gaussian elimination, but it never was. ■

The key result relating nonsingularity, Gaussian elimination, linear systems, and determinants is that the following statements are equivalent.

Theorem 6.17

The following statements are equivalent for any $n \times n$ matrix A:

(i) The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(ii) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$.

(iii) The matrix A is nonsingular; that is, A^{-1} exists.

(iv) $\det A \neq 0$.

(v) Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any n-dimensional column vector $\mathbf{b}$. ■

7. Given the two 4×4 linear systems having the same coefficient matrix:

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 6, \\ x_1 \quad - x_3 + x_4 &= 4, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= -2, \\ -x_2 + x_3 - x_4 &= 5; \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 1, \\ x_1 - x_3 + x_4 &= 1, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= 2, \\ -x_2 + x_3 - x_4 &= -1. \end{aligned}$$

a. Solve the linear systems by applying Gaussian elimination to the augmented matrix

$$\left[\begin{array}{rrrr:rr} 1 & -1 & 2 & -1 & 6 & 1 \\ 1 & 0 & -1 & 1 & 4 & 1 \\ 2 & 1 & 3 & -4 & -2 & 2 \\ 0 & -1 & 1 & -1 & 5 & -1 \end{array}\right].$$

E calcule o determinante da matriz A

Fim...

AULA 03