& Structural Impact
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Dynamic structural analysis:

Response history

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Problem:

- given a known load (in time and space) obtain the structure response (in time and space)
- or, obtain
  – displacement,
  – velocity
  – acceleration

  of a loaded structure

Source: Concepts and applications of finite element analysis, RD Cook, DS Malkus, ME Plesha, RJ Witt

Response history

Modal methods:
It is necessary to solve an eigen-problem

Ritz vectors:
More efficient than modal methods

Response spectra

Component mode synthesis (substructuring)

Usually no good
for non-linear response

Response history

$$\{D\}_{n+1} = f(\{D\}_n, \{D\}_{n+1}, \{\ddot{D}\}_{n+1}, \{\dot{D}\}_n, \{\ddot{D}\}_n)$$

Implicit direct integration

Direct integration methods

algorithms

Explicit methods

$$\{D\}_{n+1} = f(\{D\}_n, \{\dot{D}\}_n, \{\ddot{D}\}_n, \{D\}_{n-1})$$

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n$$

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + [K]\{D\}_n = \{R^{ext}\}_n$$

- Number of multiplications per time step implicit/explicit:
    - 2 for 1D
    - 15-150 2D
    - 4000 3D
- Because in implicit, matrix becomes less narrowly banded
- Implicit requires more storage

Explicit methods

- Conditionally stable (a critical time step must not be exceeded to avoid instability)
- Matrices can be made diagonal (uncoupling)
- Low cost per time step but many steps
- Wave propagation problem:
    - Blast and impact loads
    - High modes are important
    - Response spans over small time interval

Implicit methods

- Unconditionally stable (calculation remains stable regardless of time step but accuracy may suffer)
- Matrices cannot be made diagonal (coupling)
- High cost per time step but few steps
- Structural dynamics problem:
    - Response dominated by lower modes, eg structure vibration, earthquacke
    - Response spans over many fundamental periods

Explicit direct integration

$-\Delta t \quad \Delta t$

$n-1 \quad n \quad n+1$

$$\{D\}_{n+1} = \{D\}_n + \Delta t\{\dot{D}\}_n + \frac{\Delta t^2}{2}\{\ddot{D}\}_n + \ldots \qquad (1)$$

$$\{D\}_{n-1} = \{D\}_n - \Delta t\{\dot{D}\}_n + \frac{\Delta t^2}{2}\{\ddot{D}\}_n - \ldots \qquad (2)$$

1 - 2 $$\{\dot{D}\}_n = \frac{1}{2\Delta t}\left(\{D\}_{n+1} - \{D\}_{n-1}\right) \rightarrow \boxed{\{D\}_{n+1} = \{D\}_{n-1} + 2\Delta t\{\dot{D}\}_n}$$

1 + 2 $$\boxed{\{\ddot{D}\}_n = \frac{1}{\Delta t^2}\left(\{D\}_{n+1} - 2\{D\}_n + \{D\}_{n-1}\right)}$$

$$\boxed{[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n}$$

$$\boxed{\left[\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\,\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n - \{\mathbf{R}^{\text{int}}\}_n + \frac{2}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_n - \left[\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\,\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n-1}}$$

The Taylor series is given by $f(a+t) = f(a) + (\mathrm{d}f(a)/\mathrm{d}x)t/1! + (\mathrm{d}^2 f(a)/\mathrm{d}x^2)t^2/2! + \ldots$ and the present method is called central difference, in counter–distinction to the half–step central difference. See also R.D. Cook, D.S. Malkus, M.E. Plesha and R.J. Witt, Concepts and Applications of Finite Element Analysis, John Wiley, 2002.

Now, if we expand in Taylor series the displacement $\boldsymbol{u}$ we obtain

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \Delta t \dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2} \ddot{\boldsymbol{u}}_n + \ldots$$

and

$$\boldsymbol{u}_{n-1} = \boldsymbol{u}_n - \Delta t \dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2} \ddot{\boldsymbol{u}}_n + \ldots,$$

from which

$$\dot{\boldsymbol{u}} = \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_{n-1}}{2\Delta t} \qquad \text{and} \qquad \ddot{\boldsymbol{u}}_n = \frac{\boldsymbol{u}_{n+1} + \boldsymbol{u}_{n-1} - 2\boldsymbol{u}_n}{\Delta t^2},$$

allowing the equation of motion

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{R}^{ext}$$

to be written as

$$\boldsymbol{u}_{n+1} = \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{C}}{2\Delta t} \right)^{-1} \left\{ \boldsymbol{R}_n^{ext} - \boldsymbol{K}\boldsymbol{u}_n + \frac{2\boldsymbol{M}}{\Delta t^2}\boldsymbol{u}_n - \left(\frac{\boldsymbol{M}}{\Delta t^2} - \frac{\boldsymbol{C}}{2\Delta t} \right) \boldsymbol{u}_{n-1} \right\}.$$

$$\frac{1}{2\ \Delta t}[\mathbf{C}]\left(\{\mathbf{D}\}_{n-1}-\{\mathbf{D}\}_{n+1}\right)\approx\frac{1}{\Delta t}[\mathbf{C}]\left(\{\mathbf{D}\}_{n-1}-\{\mathbf{D}\}_{n}\right)$$

Use this in the bottom eq. slide 6 to obtain:

$$\boxed{\frac{1}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n-\{\mathbf{R}^{\text{int}}\}_n+\left[\frac{2}{\Delta t^2}\mathbf{M}-\frac{1}{\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_n-\left[\frac{1}{\Delta t^2}\mathbf{M}-\frac{1}{\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n-1}}$$

THIS IS EXPLICIT: TO BE IMPLEMENTED IN A PROGRAM

SEE EXERCISE I

Remarks on explicit integration

- [M] being diagonal uncouples the system [only if [C] is diagonal]
- The critical time step in the previous equation is

$$\Delta t \le \frac{2}{\omega_{\max}}\left(\sqrt{1-\xi^2}-\xi\right)$$

$$\xi = \text{damping ratio}$$

$$\omega_{\max} = \text{largest possible "calculated" frequency}$$

- Internal forces can be calculated from either

It is not necessary to form [K]

$$\{R^{\text{int}}\}_n = [K]\{D\}_n$$

$$\{R_{\text{int}}\}_n = \sum_{i=1}^{N_{els}} \left(\{r_{\text{int}}\}_n\right)_i \quad \{r_{\text{int}}\}_n = \int [B]^T \{\sigma\}_n dV$$

Minimize n. of integration points to 1: be carefull with stress calculation though!

Half-step central differences

$$\frac{1}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n - \{\mathbf{R}^{\text{int}}\}_n + \frac{1}{\Delta t^2}[\mathbf{M}]\left(\{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_{n-1/2}\right) - [\mathbf{C}]\{\dot{\mathbf{D}}\}_{n-1/2}$$

$$\{\dot{\mathbf{D}}\}_{n-1/2} = \frac{1}{\Delta t}\left(\{\mathbf{D}\}_n - \{\mathbf{D}\}_{n-1}\right) \quad \text{and} \quad \{\dot{\mathbf{D}}\}_{n+1/2} = \frac{1}{\Delta t}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n\right)$$

$$\{\ddot{\mathbf{D}}\}_n = \frac{1}{\Delta t}\left(\{\dot{\mathbf{D}}\}_{n+1/2} - \{\dot{\mathbf{D}}\}_{n-1/2}\right) = \frac{1}{\Delta t^2}\left(\{\mathbf{D}\}_{n+1} - 2\{\mathbf{D}\}_n + \{\mathbf{D}\}_{n-1}\right)$$

$$\{\mathbf{D}\}_{n+1} = \{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_{n+1/2}$$

$$\{\dot{\mathbf{D}}\}_{n+1/2} = \{\dot{\mathbf{D}}\}_{n-1/2} + \Delta t\{\ddot{\mathbf{D}}\}_n$$

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\}_n + [\mathbf{C}]\{\dot{\mathbf{D}}\}_{n-1/2} + \{\mathbf{R}^{\text{int}}\}_n = \{\mathbf{R}^{\text{ext}}\}_n$$

To start the processes

$$\{\ddot{D}\}_0 = \frac{1}{\Delta t/2}\left(\{\dot{D}\}_0 - \{\dot{D}\}_{-1/2}\right) \rightarrow \{\dot{D}\}_{-1/2} = \{\dot{D}\}_0 - \frac{\Delta t}{2}\{\ddot{D}\}_0$$

$$\{\ddot{D}\}_0 = [M]^{-1}\left(\{R^{ext}\}_0 - [K]\{D\}_0 - [C]\{\dot{D}\}_0\right)$$

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n$$

$$\{D\}_{-1} = \{D\}_0 - \Delta t\{\dot{D}\}_0 + \frac{\Delta t^2}{2}\{\ddot{D}\}_0$$

$$\boxed{\boldsymbol{u}_{n-1} = \boldsymbol{u}_n - \Delta t\dot{\boldsymbol{u}}_n + \frac{\Delta t^2}{2}\ddot{\boldsymbol{u}}_n + \ldots,}$$

Critical time step

$$\Delta t_{cr} = \frac{L_{mesh}}{c} = \frac{L_{mesh}}{\sqrt{E/\rho}} \qquad C_n = \frac{\Delta t_{actual}}{\Delta t_{cr}}$$

- consistent mass matrix gives higher frequencies than lumped mass matrix
- lumped mass increases Δt
- higher order elements have higher frquencies than lower order ones. Use the latter

Exercise I

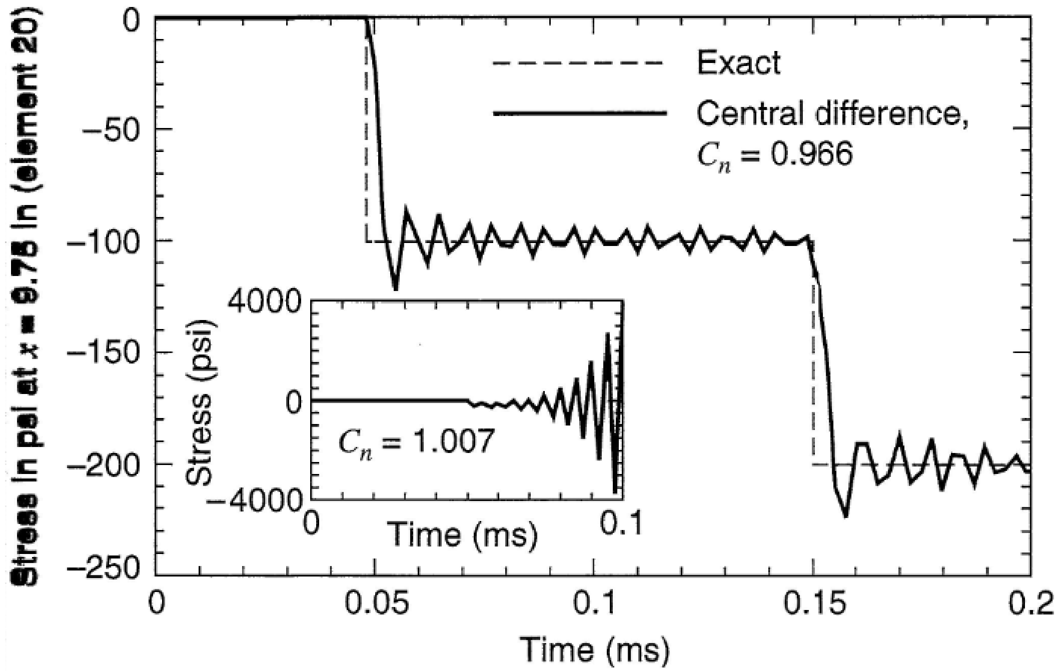


Figure 11.12-2. Axial stress versus time for a 40-element model of the bar in Fig. 11.12-1. Central difference solution with $\Delta t = 2.400(10^{-6})$ s ($C_n = 0.966$). Inset shows instability that results from too large a Δt ($\Delta t = 2.500(10^{-6})$ s, for which $C_n = 1.007$).

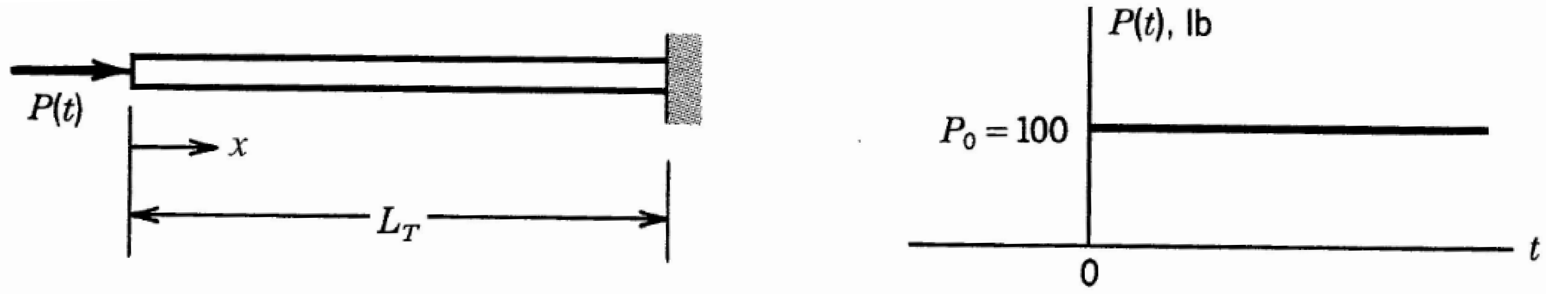
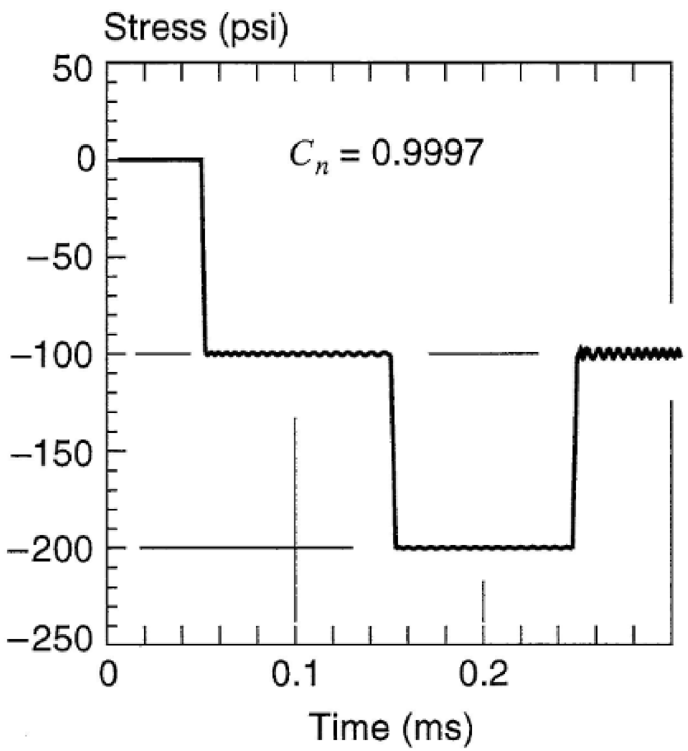
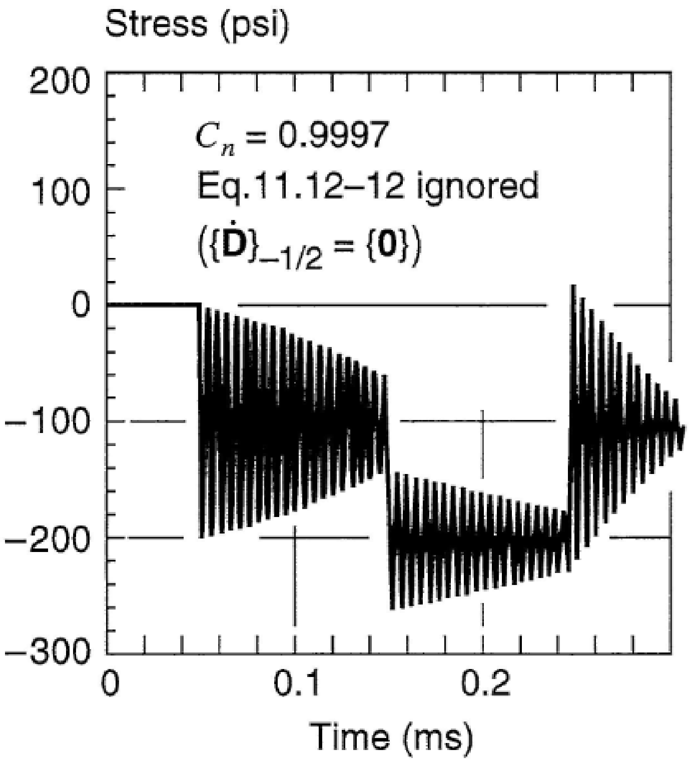


Figure 11.12-1. One-dimensional uniform bar with instantaneous axial tip loading. $A = 1.0$ in^2, $E = 30(10^6)$ psi, $\rho = 7.4(10^{-4})$ lb-s^2/in^4, $L_T = 20$ in. Load $P_0 = 100$ lb is applied at $t = 0$.



(a)



(b)

Figure 11.12-3. Central difference solutions for axial stress versus time for a 40-element model of the bar in Fig. 11.12-1. $\Delta t = 2.483(10^{-6})$ s ($C_n = 0.9997$), proper and improper initial conditions.

Implicit direct integration (Newmark family methods)

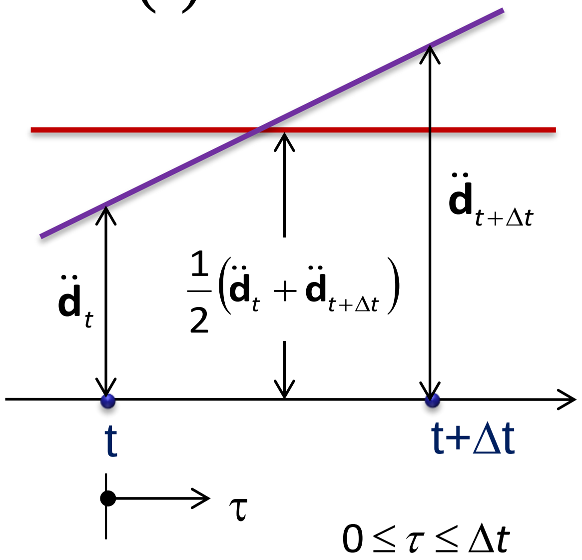
- Most of these methods are *unconditionable stable:* large time keeps the solution stable, although may compromise accuracy

$$0 \leq \tau \leq \Delta t$$

$$\Delta t = t_{n+1} - t_n$$

$$u = u(t)$$

Integração com aceleração cte.



Average acceleration method

$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$\dot{u}(\tau) = \dot{u}_n + \frac{\tau}{2}(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u(\tau) = u_n + \tau\dot{u}_n + \frac{\tau^2}{4}(\ddot{u}_{n+1} + \ddot{u}_n)$$

Linear acceleration method

$$\ddot{u}(\tau) = \ddot{u}_n + \frac{\tau}{\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

$$\dot{u}(\tau) = \dot{u}_n + \tau\ddot{u}_n + \frac{\tau^2}{2\,\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

$$u(\tau) = u_n + \tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + \frac{\tau^3}{6\,\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

Integração com aceleração linear

IMPLICIT

At step n+1 ($\tau=\Delta t$)

Average Acceleration (Eqs. 11.13-1):

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u_{n+1} = u_n + \Delta t\,\dot{u}_n + \frac{1}{4}\Delta t^2(\ddot{u}_{n+1} + \ddot{u}_n)$$

Linear Acceleration (Eqs. 11.13-2):

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u_{n+1} = u_n + \Delta t\,\dot{u}_n + \Delta t^2\left(\frac{1}{6}\ddot{u}_{n+1} + \frac{1}{3}\ddot{u}_n\right)$$

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t(\dot{u}_{n+1} + \dot{u}_n)$$

Newmark relations

$$\{\dot{\mathbf{D}}\}_{n+1} = \{\dot{\mathbf{D}}\}_n + \Delta t\left[\gamma\{\ddot{\mathbf{D}}\}_{n+1} + (1-\gamma)\{\ddot{\mathbf{D}}\}_n\right] \qquad 1$$

$$\{\mathbf{D}\}_{n+1} = \{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_n + \frac{1}{2}\Delta t^2\left[2\beta\{\ddot{\mathbf{D}}\}_{n+1} + (1-2\beta)\{\ddot{\mathbf{D}}\}_n\right] \qquad 2$$

TABLE 11.13-1. STABILITY AND ACCURACY OF SELECTED IMPLICIT DIRECT INTEGRATION METHODS.

Version [or references]	γ	β	Stability condition	Error in $\{\mathbf{D}\}$ for $\xi = \mathbf{0}$
		Newmark Methods		
Average acceleration	$\frac{1}{2}$	$\frac{1}{4}$	Unconditional	$O(\Delta t^2)$
Linear acceleration	$\frac{1}{2}$	$\frac{1}{6}$	$\Omega_{crit} = 3.464$ if $\xi = 0$	$O(\Delta t^2)$
Fox-Goodwin	$\frac{1}{2}$	$\frac{1}{12}$	$\Omega_{crit} = 2.449$ if $\xi = 0$	$O(\Delta t^4)$
Algorithmically damped	$\geq \frac{1}{2}$	$\geq \frac{1}{4}(\gamma + \frac{1}{2})^2$	Unconditional	$O(\Delta t)$
	Hilber-Hughes-Taylor (α-method), $-\frac{1}{3} \leq \alpha \leq 0$			
[2.13,11.55]	$\frac{1}{2}(1 - 2\alpha)$	$\frac{1}{4}(1 - \alpha)^2$	Unconditional	$O(\Delta t^2)$

$$\{\ddot{\mathbf{D}}\}_{n+1} = \frac{1}{\beta\,\Delta t^2}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n - \Delta t\,\{\dot{\mathbf{D}}\}_n\right) - \left(\frac{1}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n$$

De 2

$$\{\dot{\mathbf{D}}\}_{n+1} = \frac{\gamma}{\beta\,\Delta t}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n\right) - \left(\frac{\gamma}{\beta} - 1\right)\{\dot{\mathbf{D}}\}_n - \Delta t\left(\frac{\gamma}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n$$

De 1

$$[M]\{\ddot{D}\}_{n+1} + [C]\{\dot{D}\}_{n+1} + [K]\{D\}_{n+1} = \{R^{ext}\}_{n+1}$$

$$\{\ddot{D}\}_0 = [M]^{-1}\left(\{R^{ext}\}_0 - [K]\{D\}_0 - [C]\{\dot{D}\}_0\right)$$

$$[\mathbf{K}^{\text{eff}}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_{n+1} + [\mathbf{M}]\left\{\frac{1}{\beta\,\Delta t^2}\{\mathbf{D}\}_n + \frac{1}{\beta\,\Delta t}\{\dot{\mathbf{D}}\}_n + \left(\frac{1}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n\right\}$$

$$+ [\mathbf{C}]\left\{\frac{\gamma}{\beta\,\Delta t}\{\mathbf{D}\}_n + \left(\frac{\gamma}{\beta} - 1\right)\{\dot{\mathbf{D}}\}_n + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n\right\}$$

$$[\mathbf{K}^{\text{eff}}] = \frac{1}{\beta\,\Delta t^2}[\mathbf{M}] + \frac{\gamma}{\beta\,\Delta t}[\mathbf{C}] + [\mathbf{K}]$$

Implicit: to implement

Frequência Natural

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n$$

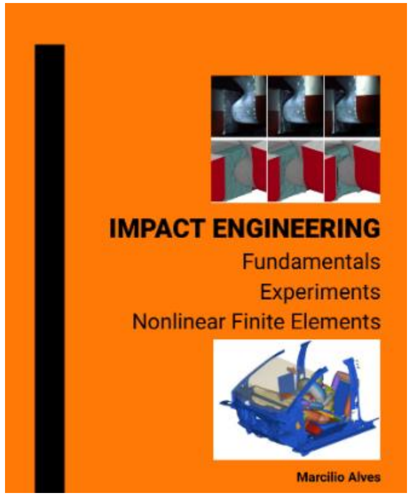
$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + [K]\{D\}_n = \{R^{ext}\}_n$$

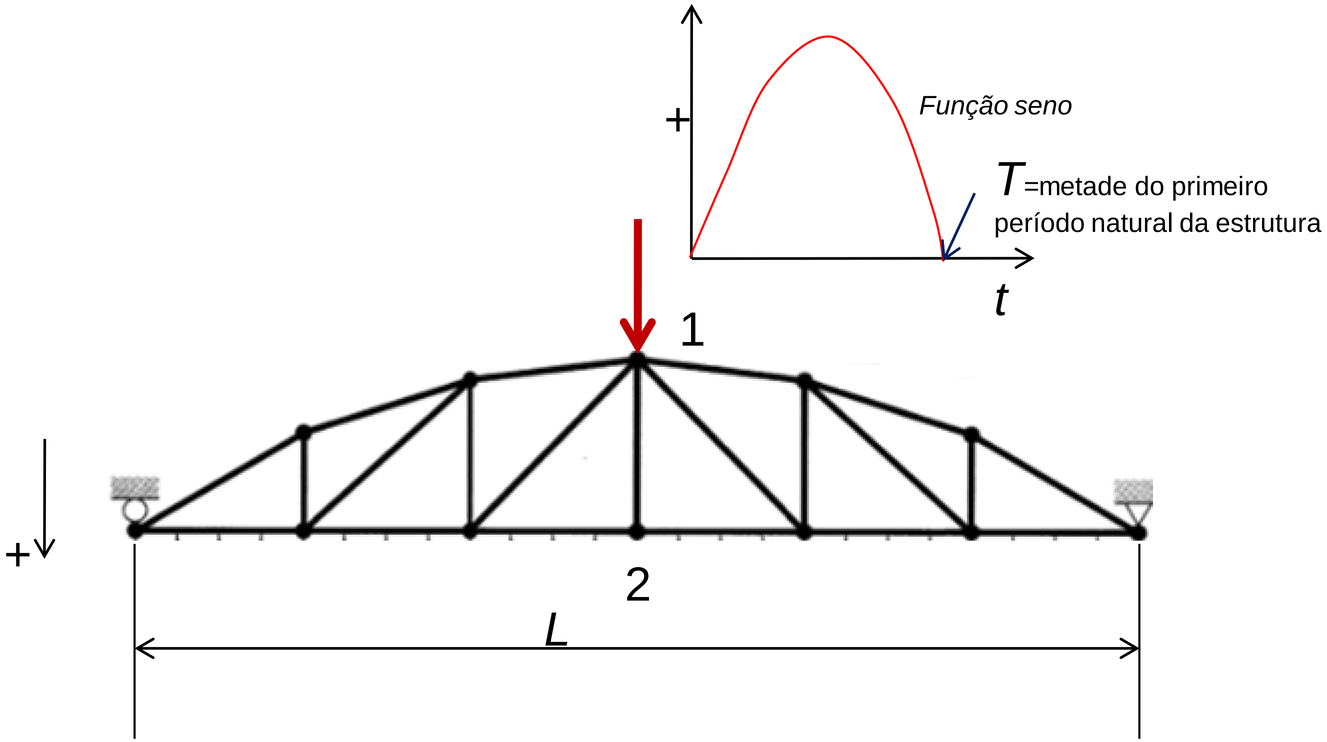
Como calcular *T*?

$$D = X\sin(\omega t)$$

Exercise II

1. Atribua dimensões e material realistas à ponte de pedestres abaixo e calcule suas frequências naturais e modos de vibrar. Em seguida, adicione o carregamento indicado ao nó 1 e calcule a amplitude da força para que o deslocamento do nó 2 seja L/10. Use os métodos implícitos e explícitos. Plote a resposta do nó 2 no intervalo [0..4*T*].





Entrega de Exercícios I e II via e-disc até 22/03 22hs